Automata Theory - Quiz I (Solutions)

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Problems

1. Professor Sikorski claims to have an inductive proof for the following hypothesis: $S(n) : n = n + 1, \ \forall n \geq 1$. His proof is as follows: Assume that $S(k)$ is true. Therefore, $k = k + 1$. Now, consider $S(k + 1)$. We need to show that $k + 1 = k + 2$. From, the inductive hypothesis, $k = k + 1$. So we can substitute $k + 1$ for k in the hypothesis of $S(k+1)$, which means that we have to show $(k+1)+1 = k+2$. But this is trivially true, so $S(k+1)$ holds. Since, we have established that $S(k) \rightarrow S(k + 1)$, it follows that $S(n)$ is true, i.e., $n = n + 1, \ \forall n \geq 1$. Can you spot the flaw in the Professor's argument?

   Solution: The Professor did not prove the base case. Indeed, $1 \neq 2$ and hence, the principle of mathematical induction does not apply. $\Box$

2. Design a DFA to accept the following language:
   $L = \{w \mid w \in \{0, 1\}^*$ and w when interpreted as a number is not divisible by 3. $\}$

   Solution: We make the following observations:

   (i) Any binary string, when interpreted as a number is either exactly divisible by 3, or $1 \ mod \ 3$, or $2 \ mod \ 3$.

   (ii) When a 0 is added to the right of a binary string, its value is doubled. Likewise, when a 1 is added to the right of a binary string, the value of the new number, is the sum of 1 and twice the value of the old number. (Work out a few examples and convince yourself, that this is true!)

   (iii) If a number which is divisible by 3 is doubled, the resultant number stays divisible by 3; on the other hand if 1 is added after the doubling, then the resultant number is $1 \ mod \ 3$.

   (iv) If a number which is $1 \ mod \ 3$ is doubled, the resultant number becomes $2 \ mod \ 3$ and if 1 is added after the doubling, the resultant number is exactly divisible by 3.

   (v) If a number is $2 \ mod \ 3$, then doubling it, results in a number which is $1 \ mod \ 3$ and adding 1 to this number, gives a number which is $2 \ mod \ 3$.

   (vi) If a given DFA A, accepts a language L, then we can design a DFA to accept the complement of L, merely by switching the final and non-final states of A.

   Figure 1 represents a DFA that accepts the language consisting of binary strings, which when interpreted as a number are not divisible by 3.

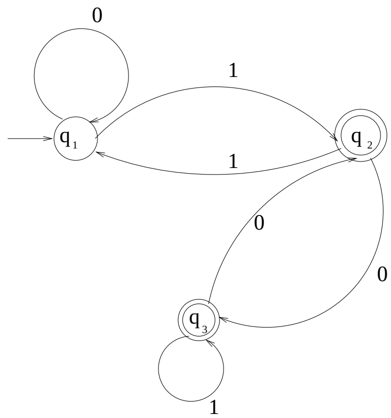

Figure 1: DFA accepting binary strings, which when interpreted as numbers are not divisible by 3.

□

3. Informally, describe the language accepted by the following DFA.

	0	1
$\rightarrow\ *A$	B	A
$*B$	C	A
C	C	C

Solution: The above DFA accepts all and only strings that do not contain two consecutive $0's$. □

4. Convert the following NFA to a DFA.

	0	1
$\rightarrow p$	$\{p, q\}$	$\{p\}$
q	$\{r\}$	$\{r\}$
r	$\{s\}$	ϕ
$*s$	$\{s\}$	$\{s\}$

Solution: We apply the subset construction algorithm discussed in class and described in Section 2.3.5 of [HMU01]. Accordingly, the constructed DFA has 16 states, viz., ϕ, $\{p\}$ and so on. Step 3 of the subset construction algorithm permits us to construct the transition table for the DFA. Finally, we remove unreachable states to get the following DFA:

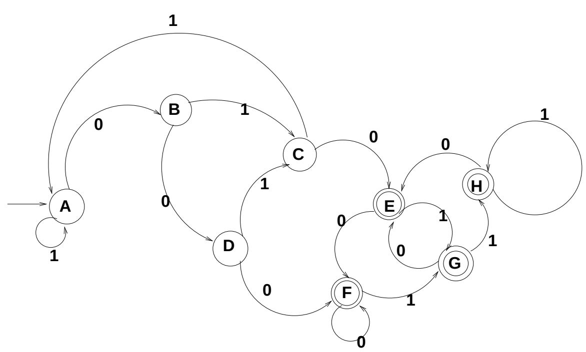

Figure 2: NFA to DFA Conversion

where $A = \{p\}$, $B = \{p, q\}$, $C = \{p, r\}$, $D = \{p.q.r\}$, $E = \{p, q, s\}$, $F = \{p, q, r, s\}$, $G = \{p, r, s\}$ and $H = \{p, s\}$. $\Box$

5. Formally argue that the NFA in Figure 3 accepts the language L, where,
$L = \{w \mid w \in \{a,\ b\}^* \text{ and } x \text{ consists of } 0 \text{ or more } a's, \text{ followed by a } b\ \}$.

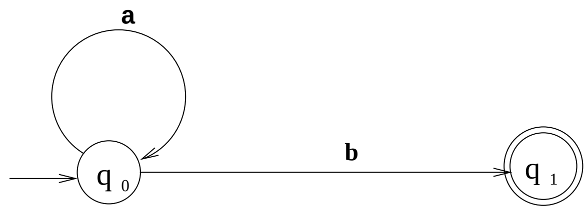

Figure 3: NFA accepting 0 or more $a's$, followed by a b

Solution: We use the notation a^* to represent a string that is constituted of zero or more $a's$ only. Note that the alphabet for this NFA is $\Sigma = \{a, b\}$.

Observe that we need to prove the following claim.

Claim 1.1 *$\hat{\delta}(q_0, w) = q_1$, if and only if $w = a^* b$.*

In order to prove Claim 1.1, it is helpful to prove the following lemma first.

Lemma 1.1 *$\hat{\delta}(q_0, w) = q_0$, if and only if $w = a^*$*

Proof:
If: We need to show that if $w = a^*$, then $\hat{\delta}(q_0, w) = q_0$.

BASIS: Let $|w| = 0$; this implies that $w = \epsilon$. For any state q in any NFA A, we know that $\hat{\delta}(q, \epsilon) = q$. Accordingly, $\hat{\delta}(q_0, \epsilon) = q_0$, which proves the base case.

INDUCTIVE STEP: Assume that $\hat{\delta}(q_0, w) = q_0$, if $|w| \le n$. Let $|w| = n + 1$, where $w = x \cdot a$ and $|x| = n$. From the inductive hypothesis, we know that $\hat{\delta}(q_0, x) = q_0$, whereas, from the transition diagram, we know that $\delta(q_0, a) = q_0$. Now observe that from the definition of $\hat{\delta}$,

$$\begin{aligned} \hat{\delta}(q_0, w) &= \hat{\delta}(q_0, x \cdot a) \\ &= \delta(\hat{\delta}(q_0, x), a) \\ &= \delta(q_0, a) \ \ \text{(from the inductive hypothesis)} \\ &= q_0. \end{aligned}$$

Only-If: We need to show that if $\hat{\delta}(q_0, w) = q_0$, then $w = a^*$.

Once again, we use induction on the length of w.

BASIS: Let $|w| = 0$, which implies that $w = \epsilon$. Clearly, $\hat{\delta}(q_0, \epsilon) = q_0$ and $\epsilon \in a^*$. It follows that the base case is proven.

INDUCTIVE STEP: Assume that if $\hat{\delta}(q_0, w) = q_0$ and $|w| \le n$, then $w = a^*$. Let $|w| = n + 1$ and $w = x \cdot c$, where $|x| = n$ and $c = \{a, b\}$. Now $\hat{\delta}(q_0, x)$ is one of $\{\phi,\ q_0,\ q_1\}$, where ϕ denotes the "dead state". If $\hat{\delta}(q_0, x)$ is either ϕ or q_1, then as per the transition diagram, $\hat{\delta}(q_0, x \cdot c) = \hat{\delta}(q_0, w) = \phi$. Hence, the hypothesis of the statement "If $\hat{\delta}(q_0, w) = q_0$, then $w = a^*$" is **false**, making the statement **true**.

Now, consider the case in which $\hat{\delta}(q_0, x) = q_0$. From the inductive hypothesis, we know that $x \in a^*$. Now $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, x \cdot c) = q_0$. Therefore, we have, $\delta(\hat{\delta}(q_0, x), c) = q_0$. From the above discussion, it follows that $\delta(q_0, c) = q_0$. The transition diagram indicates that c must be a. Accordingly, $w = a^* \cdot a = a^*$, thereby proving the inductive step. $\Box$

We are now ready to prove Claim 1.1.

Proof (of Claim 1.1):
If: We are required to show that if $w = a^*b$, then $\hat{\delta}(q_0, w) = q_1$. Observe that

$$\begin{aligned}\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, a^* \cdot b) \\ &= \delta(\hat{\delta}(q_0, a^*), b) \\ &= \delta(q_0, b) \ \ \text{(from Lemma 1.1)} \\ &= q_1 \ \ \text{(from the transition diagram)}\end{aligned}$$

Only-If: We are required to show that if $\hat{\delta}(q_0, w) = q_1$, then $w = a^*b$.

We use induction on the length of w.

Let $|w| = 0$; it follows that $w = \epsilon$.

BASIS: Since $\hat{\delta}(q_0, \epsilon) = q_0 \neq q_1$, the hypothesis is **false**, making the statement **true**, thereby proving the base case.

INDUCTIVE STEP: Assume that if $\hat{\delta}(q_0, w) = q_1$ and $|w| \leq n$, then $w = a^*b$. Now consider a string $w = x \cdot c$, where $|w| = n + 1$, $|x| = n$ and $c \in \{a, b\}$. We consider the following cases:

(i) $\hat{\delta}(q_0, x) = q_0$ - From Lemma 1.1, we know that $x = a^*$. Hence, if $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, x \cdot c) = q_1$, then it must be the case that $c = b$, as per the transition diagram. In other words, $w = a^*b$, proving the inductive step.

(ii) $\hat{\delta}(q_0, x) = q_1$ - In this case, $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, x \cdot c) = \phi$, regardless of whether c is a or b, as per the transition diagram. Since the hypothesis is **false**, the statement is **true** and the inductive step is proven.

(iii) $\hat{\delta}(q_0, x) = \phi$ - This case is identical to the case above.

□

□

References

[HMU01] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *“Introduction to Automata Theory, Language, and Computation”*. Addison–Wesley, 2nd edition edition, 2001.