Fractional Knapsack

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1 Statement of Problem

In the Fractional Knapsack problem, you are given n objects $O = \{o_1, o_2, \ldots, o_n\}$ with respective weights $W = \{w_1, w_2, \ldots, w_n\}$ and respective profits $P = \{p_1, p_2, \ldots, p_n\}$. The goal is to pack these objects into a knapsack of capacity M, such that the profit of the objects in the knapsack is maximized, while the weight constraint is not violated. You may choose a fraction of an object, if you so decide; if $\alpha_i,\ 0 \leq \alpha_i \leq 1$ of object o_i is chosen, then the profit contribution of this object is $\alpha_i \cdot o_i$ and its weight contribution is $\alpha_i \cdot w_i$. Design a greedy algorithm for this problem and argue its correctness.

2 Solution

The solution technique consists of the following steps:

(i) Order the objects by profit per unit weight, so that $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \frac{p_n}{w_n}$.

(ii) Process the objects from o_1 to o_n. Pack as much as possible of o_1 in the knapsack. If the knapsack is full stop; otherwise, o_1 is included as a whole and there is weight capacity left over. Then pack as much as possible of o_2 in the knapsack and so on.

Let $X =< x_1, x_2, \ldots, x_n >$ denote the greedy solution vector, where $x_i,\ 0 \leq x_i \leq 1$ is the fraction of o_i that is included in the knapsack. As per the description of the greedy algorithm, 0 or more of the $x_i s$ will be 1, followed by a fractional quantity, followed by $0s$. Let j be the first index such that $x_j \neq 1$. Then $x_i = 1,\ i = 1, 2, \ldots, j-1$ and $x_i = 0,\ i = j+1, j+2, \ldots, n$. Let $Y =< y_1, y_2, \ldots, y_n >$ denote an arbitrary optimal solution vector. We will show that Y can be gradually transformed into X, without decreasing profitability, while maintaining feasibility.

We assume that $\sum_{i=1}^{n} w_i \cdot y_i = M$, since otherwise, we could pack more (of) objects into the knapsack, thereby proving that Y is sub-optimal. From the mechanics of the greedy algorithm, either $\sum_{i=1}^{n} w_i \cdot x_i = M$ or $X =< 1, 1, \ldots, 1 >$. In the latter case, X must be optimal, so there is nothing to be proved.

Let k be the first index, where $x_k \neq y_k$. It must be the case that $x_k > y_k$. If $k < j$, then $x_k = 1$ and $x_k \neq y_k$ implies that $y_k < x_k$. If $k \geq j$ and $y_k > x_k$, then $\sum_{i=1}^{n} w_i \cdot y_i > M$, and knapsack feasibility is violated.

Now increase y_k till it becomes x_k, while decreasing some or all of the $y_i s, i = k+1, \ldots, n$, so that the total weight in the knapsack stays the same. Let $Z =< z_1, z_2, \ldots, z_n >$ denote the new solution. Observe that $w_k \cdot (z_k - y_k) = \sum_{i=k+1}^{n} w_i \cdot (y_i - z_i)$, in order to maintain feasibility.

Now,

$$\begin{aligned}
\sum_{i=1}^{n} p_i \cdot z_i &= \sum_{i=1}^{n} p_i \cdot y_i + p_k \cdot (z_k - y_k) - \sum_{i=k+1}^{n} p_i \cdot (y_i - z_i) \\
&= \sum_{i=1}^{n} p_i \cdot y_i + p_k \cdot (z_k - y_k) \cdot \frac{w_k}{w_k} - \sum_{i=k+1}^{n} p_i \cdot (y_i - z_i) \cdot \frac{w_i}{w_i}
\end{aligned}$$

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\begin{aligned}
&\geq \sum_{i=1}^{n} p_i \cdot y_i + \frac{p_k}{w_k} \cdot (z_k - y_k) \cdot w_k - \sum_{i=k+1}^{n} \frac{p_k}{w_k} \cdot (y_i - z_i) \cdot w_i \\
&= \sum_{i=1}^{n} p_i \cdot y_i + \frac{p_k}{w_k} \cdot [(z_k - y_k) \cdot w_k - \sum_{i=k+1}^{n} w_i \cdot (z_i - y_i)] \\
&= \sum_{i=1}^{n} p_i \cdot y_i
\end{aligned}
$$

Thus, Z is one step closer to X than Y is; arguing in this fashion, we can gradually transform Y into X, while maintaining feasibility and not decreasing profitability. This proves that the greedy solution is optimal.