Analysis of Algorithms - Scrimmage I (Solutions)

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1 Problems

1. Prove that

$$\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}.$$

Solution: We will be using the first principle of mathematical induction. Let $P(n)$ denote the proposition

$$\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}.$$

BASIS: At $n = 1$, the LHS is $1^2 = 1$. The RHS is $\frac{1 \cdot 2 \cdot 3}{6} = 1$. Since the LHS and RHS are identical, the basis is proven.

INDUCTIVE STEP: Assume that $P(k)$ is true, for some $k \geq 1$. We thus have

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2k+1)}{6}.$$

We need to show that $P(k+1)$ is true.

Observe that the LHS of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k \cdot (k+1) \cdot (2k+1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}[k \cdot (2k+1) + 6 \cdot (k+1)] \\
&= \frac{k+1}{6}[2k^2 + k + 6 \cdot k + 6] \\
&= \frac{k+1}{6}[2k^2 + 7 \cdot k + 6] \\
&= \frac{k+1}{6}[2k^2 + 4 \cdot k + 3 \cdot k + 6] \\
&= \frac{k+1}{6}[2k \cdot (k+2) + 3 \cdot (k+2)]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k+1}{6}[(k+2)\cdot(2k+3)] \\
&= \frac{(k+1)\cdot(k+2)\cdot(2(k+1)+3)}{6} \\
&= \text{RHS of P(k + 1)}
\end{aligned}
$$

Since $P(k) \rightarrow P(k+1)$, we can apply the first principle of mathematical induction to conclude that $P(n)$ is true for all n. $\square$

2. Given $a > 0$ and $0 < r < 1$, argue that

$$\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot (1 - r^{n+1})}{1 - r}$$

Solution: We will be using the first principle of mathematical induction. Let $P(n)$ denote the proposition

$$\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot (1 - r^{n+1})}{1 - r}$$

BASIS: At $n = 0$, the LHS is $a \cdot r^0 = a$. The RHS is $\frac{a\cdot(1-r)^{0+1}}{1-r} = a$. Since the LHS and RHS are identical, the basis is proven.

INDUCTIVE STEP: Assume that $P(k)$ is true, for some $k \geq 1$. We thus have

$$\sum_{i=0}^{k} a \cdot r^i = \frac{a \cdot (1 - r^{k+1})}{1 - r}$$

We need to show that $P(k+1)$ is true.

Observe that the LHS of $P(k+1)$ is

$$
\begin{aligned}
\sum_{i=1}^{k+1} a \cdot r^i &= \sum_{i=1}^{k} a \cdot r^i + a \cdot r^{k+1} \\
&= \frac{a \cdot (1 - r^{k+1})}{1 - r} + a \cdot r^{k+1}, \text{ using the inductive hypothesis} \\
&= \frac{a - a \cdot r^{k+1} + a \cdot r^{k+1} - a \cdot r^{k+2}}{1 - r} \\
&= \frac{a - a \cdot r^{k+2}}{1 - r} \\
&= \frac{a \cdot (1 - r^{k+2})}{1 - r} \\
&= \frac{a \cdot (1 - r^{(k+1)+1})}{1 - r} \\
&= \text{RHS of P(k + 1)}
\end{aligned}
$$

Since $P(k) \rightarrow P(k+1)$, we can apply the first principle of mathematical induction to conclude that $P(n)$ is true for all n.

An alternative proof without using induction follows.

Let

$$S_n = \sum_{i=0}^{n} a \cdot r^i.$$

It follows that

$$r \cdot S_n = \sum_{i=0}^{n} a \cdot r^{i+1}.$$

Hence,

$$\begin{aligned} S_n - r \cdot S_n &= \sum_{i=0}^{n} (a \cdot r^i - a \cdot r^{i+1}) \\ &= a \cdot r^0 - a \cdot r^{n+1} \end{aligned}$$

It therefore follows that

$$\begin{aligned} (1-r) \cdot S_n &= a \cdot r^0 - a \cdot r^{n+1} \\ &= a - a \cdot r^{n+1} \\ &= a \cdot (1 - r^{n+1}) \end{aligned}$$

and hence $S_n = \frac{a \cdot (1-r^{n+1})}{1-r}$. □

3. Let T denote a proper binary tree. Show that the maximum number of nodes in level i is 2^i.

   Solution: We use structural induction on the levels of the tree. Observe that at level 0, there can be at most one node, viz., the root of the tree. Thus at level 0, the maximum number of nodes is 2^0.

   Assume that the maximum number of nodes in level k is 2^k.

   Now consider level $k+1$ of the tree. Each node at this level has a parent node in level k. But each node at level k can have at most 2 children. Thus, the maximum number of nodes in level $k+1$ is $2 \cdot 2^k = 2^{k+1}$. From the first principle of mathematical induction, it follows that the maximum number of nodes in level i of a proper binary tree is 2^i. □

4. Let T denote a proper binary tree with n nodes and height h. Argue that $h \leq \frac{n-1}{2}$.

   Solution: At each level of a proper binary tree, other than level 0, there have to be a minimum of two nodes, while at level 0, there is precisely one node. Thus, the total number of nodes n is at least $2h+1$. It follows that $h \leq \frac{n-1}{2}$. □

5. Argue using induction that the exact solution to the recurrence relation:

$$\begin{aligned} T(0) &= 1 \\ T(n) &= 2 \cdot T(n-1), \; n \geq 1 \end{aligned}$$

   is $G(n) = 2^n$.

   Solution:

   BASIS: At $n = 0$, $T(0) = 1$, while $G(0) = 2^0 = 1$. Since $T(0)$ and $G(0)$ are identical, the basis is proven.

   INDUCTIVE STEP: Assume that $T(k) = G(k)$ for some $k \geq 1$.

   Observe that

$$\begin{aligned} T(k+1) &= 2 \cdot T(k), \text{ by definition} \\ &= 2 \cdot G(k), \text{ using the inductive hypothesis} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \\ &= G(k+1) \end{aligned}$$

   We thus see that $(T(k) = G(k)) \Rightarrow (T(k+1) \Rightarrow G(k+1))$ and hence by applying the first principle of mathematical induction, we can conclude that $T(n) = G(n)$ for all n. □

6. Argue that $2^n \in \Omega(5^{\log n})$

Solution: Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{5^{\log n}} &= \frac{n \cdot \log 2}{\log n \cdot \log 5} \\
&= c_1 \frac{n}{\log n}, \text{ for some constant } c_1 \\
&= c_1 \frac{1}{\frac{1}{n}}, \text{ using L'Hospital's rule} \\
&\to \infty
\end{aligned}$$

It follows that $2^n \in \Omega(5^{\log n})$. $\square$