Analysis of Algorithms - Scrimmage II (Solutions)

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1 Problems

1. Show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so is $f(n) + g(n)$.

   Solution: As per the definition of a monotonically increasing function, we know that $(a < b) \Rightarrow f(a) < f(b)$; likewise, $(a < b) \Rightarrow g(a) < g(b)$. Let $h(n) = f(n)+g(n)$. Observe that $h(a) = f(a)+g(a)$ and $h(b) = f(b)+g(b)$. Therefore, if $a < b$, $f(a) + g(a) < f(b) + g(b)$; in other words, $(a < b) \Rightarrow h(a) < h(b)$; thereby proving that $h(n)$ is a monotonically increasing function as well. $\Box$

2. Prove that $n! \in \Omega(2^n)$.

   Solution: Observe that

$$\begin{aligned} \lim_{n\to\infty} \frac{n!}{2^n} &= \lim_{n\to\infty} \frac{\log n!}{\log 2^n} \\ &\geq \lim_{n\to\infty} \frac{c_1 n \cdot \log n}{n} \text{ as per Quiz I} \\ &\to \infty \end{aligned}$$

   The key fact that we have used is $\log n! \in \Omega(n \cdot \log n)$. It follows that $n! \in \Omega(2^n)$. Can you think of another way of proving this result? $\Box$

3. Prove that $f(n) + O(f(n)) \in O(f(n))$.

   Solution: Let $g(n)$ be any function such that $g(n) \in O(f(n))$. It follows that $g(n) \leq c_1 \cdot f(n)$, for some $c_1 > 0$. Therefore, for any function $g(n) \in O(f(n))$, there exists some constant c_1, such that $f(n)+g(n) \leq f(n)+c_1 \cdot f(n) \leq c_2 \cdot f(n)$. It follows that $f(n) + O(f(n)) \in O(f(n))$. $\Box$

4. Prove that

$$1^2 + 3^2 + \ldots (2n-1)^2 = \frac{n \cdot (2n-1) \cdot (2n+1)}{3}.$$

   Solution: Observe that,

$$1^2 + 3^2 + \ldots (2n-1)^2 = \sum_{i=1}^{n} (2i-1)^2$$

   Let $P(n)$ denote the proposition that $\sum_{i=1}^{n} (2i-1)^2 = \frac{n \cdot (2n-1) \cdot (2n+1)}{3}$, for all positive integers n.
   BASIS: At $n = 1$, the LHS is $1^2 = 1$, while the RHS is $\frac{1 \cdot 2 \cdot 3}{3} = 1$. Since the LHS and RHS are identical, the basis is proven.

INDUCTIVE STEP: Assume that $P(k)$ is true, for some $k \geq 1$, i.e.,

$$\sum_{i=1}^{k}(2i-1)^2 = \frac{k \cdot (2k-1) \cdot (2k+1)}{3}$$

We need to show that $P(k+1)$ is true.

Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1)^2 &= \sum_{i=1}^{k}(2i-1)^2 + (2(k+1)-1)^2 \\
&= \sum_{i=1}^{k}(2i-1)^2 + (2k+1)^2 \\
&= \frac{k \cdot (2k-1) \cdot (2k+1)}{3} + (2k+1)^2, \text{ using the inductive hypothesis} \\
&= (2k+1) \cdot [\frac{k \cdot (2k-1)}{3} + (2k+1)] \\
&= (2k+1) \cdot \frac{2k^2 - k + 6k + 3}{3} \\
&= (2k+1) \cdot \frac{2k^2 + 5k + 3}{3} \\
&= (2k+1) \cdot \frac{2k^2 + 2k + 3k + 3}{3} \\
&= (2k+1) \cdot \frac{2k \cdot (k+1) + 3 \cdot (k+1)}{3} \\
&= (2k+1) \cdot \frac{(2k+3) \cdot (k+1)}{3} \\
&= \frac{(k+1) \cdot (2(k+1)-1) \cdot (2(k+1)+1)}{3} \\
&= \text{RHS of P(k+1)}
\end{aligned}$$

We have thus shown that $P(k) \Rightarrow P(k+1)$ and by applying the first principle of mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$.

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