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ORIGINAL PAPER

Inverse integer programming

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Abstract We consider the integer programming version of inverse optimization. Using superadditive duality, we provide a polyhedral description of the set of inverse-feasible objectives. We then describe two algorithmic approaches for solving the inverse integer programming problem.

Keywords Inverse optimization · Integer programming · Superadditive duality · Polyhedral theory

1 Introduction

We consider inverse *integer* programming, where an integer vector x^0, is given, as well as a constraint matrix, right-hand side and a target objective. The goal is to find a vector d that minimizes the weighted norm from a target objective d^0 such that x^0 is optimal for the pure integer program defined by the objective d. Algorithms for inverse linear programming have been developed and refined in [1] and [8]. The inverse counterparts of various combinatorial optimization problems have been described, including shortest paths, spanning trees, and minimum cost flows. Ahuja and Orlin [1] showed that, under mild conditions, the inverse version of a polynomially solvable optimization problem under the L_1 and L_∞ norms are polynomially solvable. Less is known about inverse integer programming. Huang [6] showed that the inverse knapsack problem and the general inverse integer programming problem with a fixed number of rows can be solved in pseudo-polynomial time. See the recent extensive survey of inverse combinatorial optimization by Heuberger [5] for more details.

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2 Preliminaries

Let $D = \{d \in \mathbb{R}^n_+ | Gd \leq g\}$ and for any $d \in D$, define the parameterized integer program $IP(d)$:

$$\begin{aligned}
&\max \sum_{j \in J} d_j x_j \\
&\text{subject to} \\
&\quad \sum_{j \in J} a_{ij} x_j \leq b_i, \quad \forall i \in I, \qquad (1)\\
&\quad x_j \in \mathbb{Z}_+, \quad \forall j \in J;
\end{aligned}$$

where $m = |I|$ and $n = |J|$. We assume that $b \in \mathbb{Z}^m_{++}$, that is, the components of b are strictly positive integers. Let $B = \otimes_{i=1}^m [0, b_i]$ and define the lattice $\hat{B} = B \cap \mathbb{Z}^m$. We assume that $a_j \in \hat{B}, \forall\, j \in J$. The *value function* of $IP(d)$ is the function $z^d : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$z^d(\beta) = \max\{d^T x : x \in S(\beta)\}, \qquad (2)$$

where, for all $\beta \in \mathbb{R}^m$,

$$S(\beta) = \left\{x \in \mathbb{Z}^n_+ : Ax \leq \beta\right\}.$$

$z^d(\beta)$ is finite for all $\beta \in \mathbb{R}^m_+$, and is nondecreasing and superadditive over $\mathbb{R}^m$ [7], that is, $\beta_1 \geq \beta_2 \Rightarrow z^d(\beta_1) \geq z^d(\beta_2)$ and $z^d(\beta_1) + z^d(\beta_2) \leq z^d(\beta_1 + \beta_2)$.

The *superadditive dual problem* of $IP(d)$, $SDP(d)$, is given by:

$$\begin{aligned}
&\min F^d(b) \\
&\text{subject to} \\
&\quad F^d(a_j) \geq d_j \quad \forall j \in J, \qquad (3a)\\
&\quad F^d : \mathbb{R}^m \rightarrow \mathbb{R} \text{ nondecreasing and superadditive}, \qquad (3b)\\
&\quad F(\mathbf{0}) = 0. \qquad (3c)
\end{aligned}$$

Strong duality holds, so that if x^0 is an optimal solution to $IP(d)$, and F^d is an optimal solution to $SDP(d)$, $F^d(b) = d^T x^0$ [7]. In particular, the value function z^d solves $SDP(d)$.

Theorem 1 is an integer programming version of complementary slackness [7].

Theorem 1 *For any $d \in D$, if x^0 is an optimal solution to $IP(d)$ and F^d is an optimal solution to $SDP(d)$, then*

$$F^d(Ax) = d^T x \quad \text{and} \quad F^d(Ax) + F^d(b - Ax) = F^d(b) \qquad (4)$$

for all $x \in \mathbb{Z}^n_+$ such that $x \leq x^0$.

3 Characterizing inverse-feasible objectives

Let $x^0 \in \mathbb{Z}^n_+$ satisfy $Ax^0 \leq b$. A vector $d \in D$ is *inverse-feasible* if x^0 is an optimal solution to $IP(d)$. We will characterize the set of inverse-feasible objectives $D^* \subseteq \mathbb{R}^n_+$. We introduce variables $\phi(\beta)$ for all $\beta \in \hat{B}$, which represent the superadditive dual $F^d(\cdot)$.

Consider the following:

$$d_j - \phi(a_j) \leq 0 \quad \forall j \in J, \tag{5a}$$
$$d^T x^0 - \phi(b) = 0 \tag{5b}$$
$$\phi(\beta_1) + \phi(\beta_2) - \phi(\beta_1 + \beta_2) \leq 0 \quad \forall \beta_1, \beta_2, \beta_1 + \beta_2 \in \hat{B}, \tag{5c}$$
$$\phi(\beta_2) - \phi(\beta_1) \leq 0 \quad \forall \beta_1, \beta_2 \in \hat{B}, \beta_1 \geq \beta_2, \tag{5d}$$
$$\phi(\mathbf{0}) = 0, \tag{5e}$$
$$\phi(\beta) \in \mathbb{R}_+ \quad \forall \beta \in \hat{B}, \tag{5f}$$
$$d \in D. \tag{5g}$$

Theorem 2 *Consider the polyhedron*

$$P = \left\{ \phi \in \mathbb{R}^{|\hat{B}|}_+, d \in D \,\middle|\, (5a), (5b), (5c), (5d), (5e), (5f) \right\}. \tag{6}$$

The set of inverse-feasible objectives is given by $D^* = \text{proj}_D(P)$.

Proof $D^* \subseteq \text{proj}_D(P)$: Consider an inverse-feasible objective $d \in D^*$, and the corresponding optimal value function $\phi^d : \hat{B} \to \mathbb{R}_+$. As ϕ^d is nondecreasing and superadditive, and strong duality holds, it is readily seen that $(\phi^d, d) \in P$, so $d \in \text{proj}_D(P)$.

$\text{proj}_D(P) \subseteq D^*$: Consider a vector $d \in \text{proj}_D(P)$. By the definition of projection, there must exist a vector ϕ^d such that $(\phi^d, d) \in P$. We extend ϕ^d to a vector $F^d : \mathbb{R}^m \to \mathbb{R}$ which we then show is feasible for $SDP(d)$. Define

$$F^d(\beta) = \begin{cases} \phi^d(\lfloor \beta \rfloor) & \text{if } \beta \in B, \\ d^T x^0 \left(\sum_{i=1}^m \beta_i\right) & \text{if } \beta \in \mathbb{R}^m_+ \backslash B, \\ -\infty & \text{if } \beta \notin \mathbb{R}^m_+. \end{cases} \tag{7}$$

The following lemma provides an upper bound on F^d for $\beta \in B$.

Lemma 1 *For any* $\beta \in B$, $F^d(\beta) \leq d^T x^0 (\sum_{i=1}^m \beta_i)$.

Proof of Lemma 1 If $\sum_{i=1}^m \beta_i \geq 1$, the result follows from (5b,d) and the fact that B is a hyperrectangle whose largest element is b. Otherwise, $F^d(\beta) = \phi^d(\lfloor \beta \rfloor) = \phi^d(\mathbf{0}) = 0$. □

We now show that F^d is feasible for $SDP(d)$. $a_j \in \hat{B}$ implies that $F^d(a_j) \geq d_j$ for $j \in J$ since ϕ^d satisfies (5a). Similarly, $F^d(\mathbf{0}) = 0$. Because ϕ^d is nondecreasing, $F^d(\beta)$ is nondecreasing over B, and is clearly nondecreasing over $\mathbb{R}^m \backslash B$. Suppose $\beta_1 \in B$, $\beta_2 \in \mathbb{R}^m_+ \backslash B$. Notice that $F^d(\beta_1) = \phi^d(\lfloor \beta_1 \rfloor) \leq d^T x^0$ by (5b,d) and the fact that $\beta_1 \leq b$. Since $\beta_2 \notin B$, there must be a dimension i for which $(\beta_2)_i > b_i \geq 1$, so $\sum_{i=1}^{m} (\beta_2)_i > 1$. Therefore, $F^d(\beta_1) \leq d^T x^0 < d^T x^0 \sum_{i=1}^{m} (\beta_2)_i = F^d(\beta_2)$.

To show that F^d is superadditive over $\mathbb{R}^m$, consider any $\beta_1, \beta_2 \in \mathbb{R}^m$. There are five cases:

1. If $\beta_1, \beta_2, \beta_1 + \beta_2 \in B$, $\lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor \leq \lfloor \beta_1 + \beta_2 \rfloor$, so the superadditivity of F^d follows from the superadditivity of ϕ^d.
2. If either β_1 or β_2 have a negative component, then $F^d(\beta_1) + F^d(\beta_2) = -\infty$, so superadditivity holds.
3. If $\beta_1, \beta_2 \in B$, and $\beta_1 + \beta_2 \in \mathbb{R}_+^m \backslash B$, superadditivity follows from the bounds given in Lemma 1 applied to β_1 and β_2.
4. If $\beta_1 \in B$, $\beta_2 \in \mathbb{R}_+^m \backslash B$, it follows that $\beta_1 + \beta_2 \in \mathbb{R}_+^m \backslash B$, so superadditivity follows from the bound given in Lemma 1 applied to β_1.
5. If $\beta_1, \beta_2 \in \mathbb{R}_+^m \backslash B$, $F^d(\beta_1) + F^d(\beta_2) = F^d(\beta_1 + \beta_2)$.

Therefore, F^d is feasible for $SDP(d)$. The fact that x^0 is optimal to $IP(d)$ follows from the fact that $SDP(d)$ is a strong dual of $IP(d)$, and Equation (5b). Therefore, $d \in D^*$. □

The set of inverse-feasible objectives for one inverse integer programming problem is a subset of the set of inverse-feasible objectives for a related set of inverse integer programming problems. For any $\hat{x} \in \mathbb{Z}_+^n$ and any $\hat{b} \in \mathbb{Z}_+^m$, we let $\Delta^*(\hat{x}, \hat{b})$ denote the set of inverse-feasible vectors to IP given right-hand side $\hat{b}$ with respect to primal vector $\hat{x}$. That is, $d \in \Delta^*(\hat{x}, \hat{b})$ if and only if $\hat{x}$ is an optimal solution to $\max \left\{ d^T x | Ax \leq \hat{b}, x \in \mathbb{Z}_+^n \right\}$. Note that $D^* = \Delta^*(x^0, b)$. Consider

$$\phi(\hat{b}) - d^T \hat{x} = 0. \tag{8}$$

For any $\hat{x} \in \mathbb{Z}_+^n$ and any $\hat{b} \in \mathbb{Z}_+^m$, we define

$$Q(\hat{x}, \hat{b}) = \left\{ \phi \in \mathbb{R}_+^{|\hat{B}|}, d \in D \,\middle|\, (5a), (5c), (5d), (5e), (5f), (8) \right\}.$$

Proposition 1 *For any $\hat{x} \in \mathbb{Z}_+^n$ and any $\hat{b} \in \mathbb{Z}_+^m$, $\Delta^*(\hat{x}, \hat{b}) = \text{proj}_D\big(Q(\hat{x}, \hat{b})\big)$.*

The proof of Proposition 1 is similar to that of Theorem 2 and is omitted.

Theorem 3 *Let $x^0 \in \mathbb{Z}_+^n$ and suppose $Ax^0 \leq b$. Then $D^* \subseteq \Delta^*(\hat{x}, A\hat{x})$ and $D^* \subseteq \Delta^*(x^0 - \hat{x}, b - A\hat{x})$ for all $\hat{x} \in \mathbb{Z}_+^n$, $\hat{x} \leq x^0$.*

Proof Notice that $a_j \in \hat{B}$ implies that $A\hat{x} \in \hat{B}$ and $b - A\hat{x} \in \hat{B}$. Choose $d \in D^*$. By Theorem 2, there exists ϕ^d such that $(d, \phi^d) \in P$. The only difference between P and $Q(\hat{x}, A\hat{x})$ is that Equality (5b) is replaced by (8). By Theorem 1, $\phi^d(A\hat{x}) = d^T \hat{x}$, so (8) holds. Therefore, $(d, \phi^d) \in Q(\hat{x}, \hat{b})$ so by Proposition 1, $d \in \Delta^*(\hat{x}, A\hat{x})$. A similar proof holds for $D^* \subseteq \Delta^*(x^0 - \hat{x}, b - A\hat{x})$. □

Note that in general $D^* \neq \Delta^*(\hat{x}, A\hat{x})$ because $\mathbf{0} \leq x^0$ and $\Delta^*(\mathbf{0}, A\mathbf{0}) = D$.

4 Solving inverse integer programs

Let $d^0 \in D$ be a known target vector, and let $v \in \mathbb{R}^n_+$ be a set of weights. The objective of the inverse integer programming problem under the L_1 norm is to find an inverse-feasible d that solves

$$\min_{d \in D^*} \left\{ \sum_{j \in J} v_j |d_j^0 - d_j| \right\}.$$

Cai et al. [2] showed that the inverse center location problem is $\mathcal{NP}$-hard. As the center location problem can be formulated as an integer program [3], it follows that the inverse integer programming problem is also $\mathcal{NP}$-hard. We provide two algorithmic approaches for solving the inverse integer programming problem.

The first approach is to find the integer hull of problem (1) and apply inverse *linear* programming to it. Finding this integer hull is challenging. Assuming that the size of the m inequalities in (1) is no more than ρ, Hartmann [4, Claim 4.2.1] provided an algorithm that performs $\mathcal{O}(2^{2^{n+1}}(8n^2 m\rho)^{2n})$ operations on numbers of polynomial size in m and ρ.

For the second approach, we introduce nonnegative variables α and γ, and using Theorem 2, we formulate the inverse integer programming problem as the following *linear* program.

$$\min \textstyle\sum_{j \in J} v_j(\alpha_j + \gamma_j)$$
subject to

$$d_j^0 + \alpha_j - \gamma_j - \phi(a_j) \le 0 \quad \forall j \in J, \tag{9a}$$

$$(d^0 + \alpha - \gamma)^T x^0 - \phi(b) = 0, \tag{9b}$$

$$\phi(\beta_1) + \phi(\beta_2) - \phi(\beta_1 + \beta_2) \le 0 \quad \forall \beta_1, \beta_2, \beta_1 + \beta_2 \in \hat{B}, \tag{9c}$$

$$\phi(\beta_2) - \phi(\beta_1) \le 0 \quad \forall \beta_1, \beta_2 \in \hat{B}, \beta_1 \ge \beta_2, \tag{9d}$$

$$\phi(\mathbf{0}) = 0, \tag{9e}$$

$$G(d^0 + \alpha - \gamma) \le g, \tag{9f}$$

$$\phi(\beta) \in \mathbb{R}_+ \quad \forall \beta \in \hat{B}, \tag{9g}$$

$$\alpha, \gamma \in \mathbb{R}^n_+. \tag{9h}$$

In the case where $D = \mathbb{R}^n_+$, constraints (9f) are eliminated. The case of the L_∞ norm is handled with a reformulation similar to that of Ahuja and Orlin [1]. Although (9) is a linear program, its size is exponentially large. If B is an m-dimensional hypercube $B = \bigotimes_{i=1}^m [0, \hat{b}]$, then it is easily seen that the linear program (9) has $(\hat{b}+1)^m + 2n$ variables and $n + 3 + 2\big[(\frac{(\hat{b}+1)(\hat{b}+2)}{2})^m - (\hat{b}+1)^m\big]$ constraints.

Obviously, neither of these approaches are practical for large instances, as they grow exponentially. However, the worst-case performance of the integer hull algorithm appears to grow much more rapidly than that of the linear programming formulation. Assuming that B is a hypercube, Table 1 shows the magnitude of this upper bound

Table 1 Values of $2^{2^{n+1}}(8n^2m\phi)^{2n}$ as well as the size of the linear program (9) for small values of m, n, ρ and $\hat{b}$, assuming that B is a hypercube. Recall that Hartmann's [4] algorithm for finding the integer hull of a polyhedron requires $\mathcal{O}(2^{2^{n+1}}(8n^2m\rho)^{2n})$ operations

m	n	ρ	$\hat{b}$	$2^{2^{n+1}}(8n^2m\rho)^{2n}$	Variables in (9)	Constraints in (9)
2	3	4	4	2.4×10^{21}	31	406
4	5	5	5	1.9×10^{55}	1,306	386,378
3	8	8	8	3.6×10^{219}	745	180,803
5	10	10	9	$\approx 1.0 \times 10^{703}$	100,020	1.0×10^9

and the size of the linear program (9) for various values of m, n, ρ and b. While the size of the linear program (9) grows rapidly with b and m, the worst-case running time of Hartmann's algorithm is astronomical even for the smallest of instances. Furthermore, Table 1 provides the exact size of the linear program (9), whereas the bound for Hartmann's algorithm ignores constants.

The fact that (9) has many more constraints than variables suggests a cutting-plane algorithm that starts with all the ϕ variables and adds constraints (9c,d) as needed. Such an algorithm would be limited to small instances, as the size of $|\hat{B}|$ grows exponentially.

Consider the following constraints:

$$\phi(Ax) - \sum_{j \in J} x_j(\alpha_j - \gamma_j) = \sum_{j \in J} d_j^0 x_j, \quad \forall x \in \mathbb{Z}_+^n, \ x \leq x^0, \tag{10}$$

and

$$\phi(Ax) + \phi(b - Ax) - \sum_{j \in J} x_j^0(\alpha_j - \gamma_j) = \sum_{j \in J} d_j^0 x_j^0, \quad \forall x \in \mathbb{Z}_+^n, \ x \leq x^0. \tag{11}$$

By Theorem 1, for any $d \in D^*$, (ϕ^d, d) satisfy (10) and (11). Given some subset of constraints (9c,d), the incorporation of these constraints may provide stronger relaxations.

5 Conclusions

We provide a polyhedral description of the inverse-feasible objectives for a pure integer program. The more general question of characterizing the inverse-feasible objectives for inverse mixed-integer programs remains.

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