



Calculating some inverse linear programming problems

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Abstract

In this paper we consider some inverse LP problems in which we need to adjust the cost coefficients of a given LP problem as less as possible so that a known feasible solution becomes the optimal one. A method for solving general inverse LP problem including upper and lower bound constraints is suggested which is based on the optimality conditions for LP problems. It is found that when the method is applied to inverse minimum cost flow problem or inverse assignment problem, we are able to obtain strongly polynomial algorithms.

Keywords: Inverse problem; Linear programming; Minimum cost flow; Assignment problem; Residual network; Strongly polynomial complexity

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1. Introduction

Recently there are several papers discussing some inverse combinatorial optimization problems. In these problems a feasible solution x^0 is given which is not an optimal solution under the current objective function, and it is required to revise the cost coefficients in the objective function as less as possible so that x^0 becomes an optimal solution.

Main application of these problems, as explained in [2, 5], is that in some cases although an optimization model is established, the costs (or other amounts such as weights or utilities) associated with the decision variables in the model are very difficult to determine accurately. If by experience or by conducting some experiments, the optimal solution in certain particular cases is known, we wish to adjust the data in the objective function by using this information.

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In [2], Burton and Toint considered the computations of an inverse shortest path problem. As they use l_2 norm to measure closeness between two vectors, they transform the inverse problem into a quadratic programming problem to solve. In [7], Zhang et al. use l_1 norm and deal with the inverse shortest path problem as a special LP problem. The combinatorial structure of the feasible region for inverse shortest path problem is exposed in [5]. Some inverse minimum spanning tree problems are studied in [4, 6].

In this paper we are going to propose a method for solving general inverse LP problems. The method is based on the optimality conditions for LP problems. As some particular examples, we shall apply the method to two classes of problems: inverse minimum cost flow problem and inverse assignment problem. It will be seen that when this method is used, the calculation of the first problem is reduced to a minimum cost circulation problem, whereas the second one can be solved by calculating the original assignment problem or a minimum cost circulation problem depending on different additional requirements. In all these cases we are able to obtain strongly polynomial algorithms.

This paper is organized as follows. In Section 2, we first prove equivalence of the general inverse LP problem to other two related models, from which a method for calculating inverse LP problems is derived. Section 3 is devoted to the application of the method to solve inverse minimum cost flow problems. In Section 4 we use the method to calculate inverse assignment problems. If some weights in the assignment model are not allowed to adjust, we call such problems restricted inverse assignment problems, and a special case of which is solved in Section 5.

2. General inverse LP problems

Given a general LP problem

$$(LP) \quad \min \{cx \mid Ax = b, x \geq 0\},$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is a $m \times n$ matrix. Let x^0 be a feasible solution. We need to change the vector c as less as possible and let x^0 become an optimal solution of the adjusted (LP). If we define

$$\mathcal{F}(x^0) = \{\tilde{c} \in \mathbb{R}^n \mid \min \{\tilde{c}x \mid Ax = b, x \geq 0\} = \tilde{c}x^0\}, \quad (2.1)$$

then the inverse problem of (LP) can be expressed as

$$\min \{\|c - \tilde{c}\| \mid \tilde{c} \in \mathcal{F}(x^0)\}. \quad (2.2)$$

Throughout this paper we take the l_1 norm for $\|c - \tilde{c}\|$, i.e., $\|c - \tilde{c}\| = \sum_j |c_j - \tilde{c}_j|$.

Lemma 2.1. *Let x^0 be a feasible solution of problem (LP). Then x^0 is an optimal solution of the problem if and only if there exists $\pi \in \mathbb{R}^m$ such that for all $j = 1, \dots, n$,*

(a) $\pi P_j \leq c_j$, and

(b) $x_j^0 > 0 \Rightarrow \pi P_j = c_j$,

where P_j are the columns of A .

From this lemma we know that if x^0 is a feasible, but not optimal solution, then for any vector $\pi \in \mathbb{R}^m$, there must be a j such that either condition (a), or condition (b) fails for the index j .

Lemma 2.2. For any $\pi \in \mathbb{R}^m$, a vector c^π can be constructed which belongs to the set $\mathcal{F}(x^0)$.

Proof. A given π divides the columns of A into three groups:

$$A^+ = \{P_j | \pi P_j > c_j\}, \quad A^0 = \{P_j | \pi P_j = c_j\}, \quad A^- = \{P_j | \pi P_j < c_j\}.$$

We then define

$$c_j^\pi = \begin{cases} c_j, & P_j \in A^0; \text{ or } P_j \in A^- \text{ and } x_j^0 = 0, \\ c_j - \theta_j, & P_j \in A^- \text{ and } x_j^0 > 0, \text{ where } \theta_j = c_j - \pi P_j, \\ c_j + \theta_j, & P_j \in A^+, \text{ where } \theta_j = \pi P_j - c_j. \end{cases} \quad (2.3)$$

Clearly, c_j^π has the following properties:

$$\pi P_j < c_j^\pi, \quad \text{if } P_j \in A^- \text{ and } x_j^0 = 0, \quad (2.4)$$

$$\pi P_j = c_j^\pi, \quad \text{otherwise}; \quad (2.5)$$

from which it is easy to verify that $c_j = c_j^\pi$ ($j = 1, \dots, n$) satisfy the conditions (a) and (b) in Lemma 2.1. Therefore, vector $c^\pi = (c_1^\pi, \dots, c_n^\pi) \in \mathcal{F}(x^0)$. \square

We denote by $\mathcal{F}'(x_0)$ the set of vectors c^π defined by (2.3) for all $\pi \in \mathbb{R}^m$. As we proved in Lemma 2.2,

$$\mathcal{F}'(x_0) \subseteq \mathcal{F}(x^0). \quad (2.6)$$

Theorem 2.3. $\min\{\|c^\pi - c\| | c^\pi \in \mathcal{F}'(x^0)\} = \min\{\|\tilde{c} - c\| | \tilde{c} \in \mathcal{F}(x^0)\}.$

Proof. Due to (2.6), we only need to show that for every $\tilde{c} \in \mathcal{F}(x^0)$, there exists a $c^\pi \in \mathcal{F}'(x^0)$ such that $\|\tilde{c} - c\| \geq \|c^\pi - c\|$.

As $\tilde{c} \in \mathcal{F}(x^0)$, x^0 is an optimal solution of the LP problem $\min\{\tilde{c}x | Ax = b, x \geq 0\}$. So, by Lemma 2.1, there is a vector $\pi \in \mathbb{R}^m$ satisfying

(a) $\pi P_j \leq \tilde{c}_j$, and

(b) $x_j^0 > 0 \Rightarrow \pi P_j = \tilde{c}_j$

for $j = 1, \dots, n$. We now define c^π by the formula (2.3) in Lemma 2.2 and show that for each j , $|\tilde{c}_j - c_j| \geq |c_j^\pi - c_j|$.

In fact if $\pi P_j \geq c_j$, then as $c_j^\pi = \pi P_j$, $\tilde{c}_j \geq \pi P_j = c_j^\pi \geq c_j$. Thus, $\tilde{c}_j - c_j \geq c_j^\pi - c_j \geq 0$. On the other hand, if $\pi P_j < c_j$, there are two possible cases: if $x_j^0 = 0$, then $c_j^\pi = c_j$, and of course $0 = |c_j^\pi - c_j| \leq |\tilde{c}_j - c_j|$; if $x_j^0 > 0$, then $c_j^\pi = \pi P_j = \tilde{c}_j$ and hence $|\tilde{c}_j - c_j| = |c_j^\pi - c_j|$.

From the above analysis we see that for each $j = 1, \dots, n$, $|\tilde{c}_j - c_j| \geq |c_j^\pi - c_j|$, and therefore, $\|\tilde{c} - c\| \geq \|c^\pi - c\|$. \square

We now consider the inverse LP problem from another viewpoint. For any $\pi \in \mathbb{R}^m$, we try to make necessary adjustment for c_i by θ_j so that $c_j + \theta_j$ satisfies the optimality conditions in Lemma

2.1, that is,

if $x_j^0 > 0$, then find θ_j such that $\pi P_j = c_j + \theta_j$;
 if $x_j^0 = 0$, then find $\theta_j \geq 0$ such that $\pi P_j \leq c_j + \theta_j$.

Note that in the second case any adjustment of $\theta_j < 0$ is unnecessary as $\theta_j = 0$ already meets the optimality condition and we want to make as less adjustment as possible. So, for each j we can define

$$\begin{aligned} D_j^\pi &= \left\{ c_j + \theta_j \mid \theta_j \text{ satisfies: } \begin{array}{ll} \pi P_j = c_j + \theta_j, & \text{if } x_j^0 > 0 \\ \pi P_j \leq c_j + \theta_j \text{ and } \theta_j \geq 0, & \text{if } x_j^0 = 0 \end{array} \right\} \\ &= \left\{ \hat{c}_j \mid \begin{array}{ll} \pi P_j = \hat{c}_j, & \text{if } x_j^0 > 0 \\ \pi P_j \leq \hat{c}_j \text{ and } \hat{c}_j \geq c_j, & \text{if } x_j^0 = 0 \end{array} \right\}. \end{aligned}$$

It is easy to check that the c_j^π defined by (2.3) meets the above conditions on \hat{c}_j , i.e.,

$$c_j^\pi \in D_j^\pi. \quad (2.7)$$

Now if we let

$$D^\pi = \{(\hat{c}_1, \dots, \hat{c}_n) \mid \hat{c}_j \in D_j^\pi, j = 1, \dots, n\}$$

and

$$D(x^0) = \bigcup_{\pi \in \mathbb{R}^m} D^\pi,$$

then (2.5) implies

$$\mathcal{F}'(x^0) \subseteq D(x^0) \subseteq \mathcal{F}(x^0)$$

and therefore, by Theorem 2.3 we know that

$$\min \{ \|\hat{c} - c\| \mid \hat{c} \in D(x^0) \} = \min \{ \|\tilde{c} - c\| \mid \tilde{c} \in \mathcal{F}(x^0) \}.$$

In other words, we have proved the following result:

Theorem 2.4. *The inverse LP problem (2.2) is equivalent to*

$$\begin{aligned} \min \quad & \|\theta\| \\ \text{s.t.} \quad & \pi P_j - \theta_j = c_j, \quad j \in \bar{J}, \\ & \pi P_j - \theta_j \leq c_j, \quad j \in J, \\ & \theta_j \geq 0, \quad j \in J, \end{aligned} \quad (2.8)$$

where $J = \{j \mid x_j^0 = 0\}$, $\bar{J} = \{j \mid x_j^0 > 0\}$.

We can extend the above result to the inverse LP problems where the variables have both lower and upper bounds. Let x^0 be a feasible solution of the LP problem

$$\begin{aligned} \text{(BLP)} \quad & \min \quad cx \\ & \text{s.t.} \quad Ax = b, \quad l \leq x \leq u, \end{aligned}$$

where $l < u$, and let $\mathcal{F}(x^0)$ be the set of cost vectors \tilde{c} under which x^0 becomes an optimal solution of problem (BLP). Then the inverse problem $\min \{\|\tilde{c} - c\| \mid \tilde{c} \in \mathcal{F}(x^0)\}$ is equivalent to the following problem:

$$\begin{aligned} \min \quad & \|\theta\| \\ \text{s.t.} \quad & \pi P_j \leq \theta_j + c_j, \quad j \in J^-, \\ & \pi P_j = c_j - \theta_j, \quad j \in J^0, \\ & \pi P_j \geq c_j - \theta_j, \quad j \in J^+, \\ & \theta_j \geq 0, \quad j \in J^- \cup J^+, \end{aligned} \tag{2.9}$$

where P_j is the j th column of matrix A , and

$$J^- = \{j \mid x_j^0 = l_j\}, \quad J^0 = \{j \mid l_j < x_j^0 < u_j\}, \quad J^+ = \{j \mid x_j^0 = u_j\}.$$

Suppose (π^*, θ^*) is an optimal solution of problem (2.9), then the solution for the inverse (BLP) problem is

$$\tilde{c}_j^* = \theta_j^* + c_j, \quad j \in J^-; \quad \text{and} \quad \tilde{c}_j^* = c_j - \theta_j^*, \quad j \in J^0 \cup J^+.$$

A special case of problem (BLP) is $l = 0$ and $u = \mathbf{1}$. ($\mathbf{1}$ is the all-1 vector.) If in addition x^0 is a 0–1 feasible solution, then (2.9) becomes

$$\begin{aligned} \min \quad & \|\theta\| \\ \text{s.t.} \quad & \pi P_j - \theta_j \leq c_j, \quad j \in J^-, \\ & \pi P_j + \theta_j \geq c_j, \quad j \in J^+, \\ & \theta_j \geq 0, \quad j \in J^- \cup J^+. \end{aligned} \tag{2.10}$$

3. Inverse minimum cost flow problem

If problem (BLP) is a minimum cost flow problem

$$\begin{aligned} \min \quad & \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij} \\ \text{s.t.} \quad & - \sum_{j \in f(i)} x_{ij} + \sum_{j \in t(i)} x_{ji} = b_i, \quad i \in \mathcal{V}, \\ & l_{ij} \leq x_{ij} \leq u_{ij}, \quad (i,j) \in \mathcal{A}, \end{aligned} \tag{3.1}$$

where \mathcal{A} and \mathcal{V} are respectively the sets of arcs and nodes in the concerned graph; (i, j) stands for an arc from i to j and x_{ij} is the flow on the arc; $f(i)$ is the set of nodes to which there are arcs from the node i , and $t(i)$ is the set of nodes from which there are arcs to node i , then according to (2.10), its inverse problem can be expressed as

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} \theta_{ij} + \sum_{(i,j) \in J^+} \alpha_{ij} + \sum_{(i,j) \in J^0} |\beta_{ij}| \\ \text{s.t.} \quad & -\pi_i + \pi_j - \theta_{ij} \leq c_{ij}, (i,j) \in J^-, \\ & -\pi_i + \pi_j + \beta_{ij} = c_{ij}, (i,j) \in J^0, \\ & -\pi_i + \pi_j + \alpha_{ij} \geq c_{ij}, (i,j) \in J^+, \\ & \theta_{ij}, \alpha_{ij} \geq 0, \end{aligned} \quad (3.2)$$

and the adjusted costs are $c_{ij} + \theta_{ij}$, $c_{ij} - \beta_{ij}$ and $c_{ij} - \alpha_{ij}$ for the arcs in J^- , J^0 and J^+ respectively.

Let $\beta_{ij} = \mu_{ij} - \gamma_{ij}$, $\mu_{ij}, \gamma_{ij} \geq 0$, then (3.2) becomes

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} \theta_{ij} + \sum_{(i,j) \in J^+} \alpha_{ij} + \sum_{(i,j) \in J^0} (\mu_{ij} + \gamma_{ij}) \\ \text{s.t.} \quad & -\pi_i + \pi_j - \theta_{ij} \leq c_{ij}, (i,j) \in J^-, \\ & -\pi_i + \pi_j + \mu_{ij} - \gamma_{ij} = c_{ij}, (i,j) \in J^0, \\ & -\pi_i + \pi_j + \alpha_{ij} \geq c_{ij}, (i,j) \in J^+, \\ & \theta_{ij}, \mu_{ij}, \gamma_{ij}, \alpha_{ij} \geq 0. \end{aligned} \quad (3.3)$$

If we let $Jf(i)$ denote the set of nodes in J to which there are arcs from node i , and $Jt(i)$ the set of nodes in J from which there are arcs to node i , the dual LP problem of (3.3) is

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} c_{ij} y_{ij} - \sum_{(i,j) \in J^0} c_{ij} y_{ij} - \sum_{(i,j) \in J^+} c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{j \in J^-f(i)} y_{ij} - \sum_{j \in J^-t(i)} y_{ji} - \sum_{j \in J^0f(i)} y_{ij} + \sum_{j \in J^0t(i)} y_{ji} - \sum_{j \in J^+f(i)} y_{ij} + \sum_{j \in J^+t(i)} y_{ji} = 0, \\ & 0 \leq y_{ij} \leq 1, (i,j) \in J^- \cup J^+, \\ & -1 \leq y_{ij} \leq 1, (i,j) \in J^0. \end{aligned} \quad (3.4)$$

Problem (3.4) is a minimum cost circulation problem.

When problem (3.1) is a minimum cost flow problem with unit capacity, i.e., all $l_{ij} = 0$, $u_{ij} = 1$, and if x^0 is a 0–1 solution, then its inverse problem has the form

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} \theta_{ij} + \sum_{(i,j) \in J^+} \alpha_{ij} \\ \text{s.t.} \quad & -\pi_i + \pi_j - \theta_{ij} \leq c_{ij}, (i,j) \in J^-, \\ & -\pi_i + \pi_j + \alpha_{ij} \geq c_{ij}, (i,j) \in J^+, \\ & \theta_{ij}, \alpha_{ij} \geq 0, \end{aligned} \quad (3.5)$$

where $J^- = \{(i, j) | x_{ij}^0 = 0\}$, $J^+ = \{(i, j) | x_{ij}^0 = 1\}$. The dual of the above LP problem can be expressed as

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} c_{ij} y_{ij} - \sum_{(i,j) \in J^+} c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{j \in J^- f(i)} y_{ij} - \sum_{j \in J^+ f(i)} y_{ij} - \sum_{j \in J^- t(i)} y_{ji} + \sum_{j \in J^+ t(i)} y_{ji} = 0, \\ & 0 \leq y_{ij} \leq 1, (i, j) \in \mathcal{A}. \end{aligned} \quad (3.6)$$

If for any arc in J^+ , we reverse its direction, i.e., swap $J^+ f(i)$ and $J^+ t(i)$, and change its original cost into the opposite value, but still maintain unit capacity, then it is obvious that problem (3.6) becomes a typical minimum cost circulation problem.

4. Inverse assignment problems

Let x^0 be a feasible solution of the assignment problem

$$\begin{aligned} (\text{AS}) \quad \min \quad & \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j \in f(i)} x_{ij} = 1, i = 1, \dots, n, \\ & \sum_{i \in t(j)} x_{ij} = 1, j = 1, \dots, n, \\ & 0 \leq x_{ij} \leq 1, (i, j) \in \mathcal{A} \end{aligned}$$

and assume that the components of x^0 are either 0 or 1. By (2.10) we know that the inverse problem of problem (AS) with respect to x^0 can be formulated as

$$\begin{aligned} \min \quad & \sum_{(i,j) \in \mathcal{A}} \theta_{ij} \\ \text{s.t.} \quad & u_i + v_j - \theta_{ij} \leq c_{ij}, (i, j) \in J^-, \\ & u_i + v_j + \theta_{ij} \geq c_{ij}, (i, j) \in J^+, \\ & \theta_{ij} \geq 0, (i, j) \in \mathcal{A}. \end{aligned} \quad (4.1)$$

Without loss of generality, we may assume that $J^+ = \{(1, 1), (2, 2), \dots, (n, n)\}$ and $J^- = \mathcal{A} \setminus J^+$ so that problem (4.1) can be written as

$$\begin{aligned} \min \quad & \sum_{(i,j) \in \mathcal{A}} \theta_{ij} \\ \text{s.t.} \quad & u_i + v_j - \theta_{ij} \leq c_{ij}, (i, j) \in J^-, \\ & u_i + v_i + \theta_{ii} = c_{ii}, i = 1, 2, \dots, n, \\ & \theta_{ij} \geq 0, (i, j) \in \mathcal{A}. \end{aligned} \quad (4.2)$$

Note that in the second set of constraints we changed “ \geq ” into “ $=$ ” as we can, if necessary, reduce the value of u_i letting the inequality become equality while maintaining the first set of constraints to be satisfied and not altering the objective value.

Lemma 4.1. *Problem (4.2) is equivalent to the following problem:*

$$\begin{aligned} \min \quad & \sum_{(i,j) \in \mathcal{A}} \theta_{ij} \\ \text{s.t.} \quad & v_j - v_i - (\theta_{ii} + \theta_{ij}) \leq d_{ij}, \quad (i,j) \in J^-, \\ & \theta_{ij} \geq 0, \quad (i,j) \in \mathcal{A}, \end{aligned} \quad (4.3)$$

where $d_{ij} = c_{ij} - c_{ii}$.

Proof. If (u, v, θ) is a feasible solution of problem (4.2), then it is easy to see that (v, θ) meets the constraints of problem (4.3). Conversely, if (v, θ) is feasible to problem (4.3), then by defining $u_i = c_{ii} - v_i - \theta_{ii}$, (u, v, θ) meets all constraints of problem (4.2). So, the two problems have the same objective function and feasible region. \square

Lemma 4.2. *Problems (4.2) and (4.3) have an optimal solution in which $\theta_{ij} = 0$ for all $(i,j) \in J^-$.*

Proof. Suppose $(\hat{v}, \hat{\theta})$ is an optimal solution of problem (4.3). Let

$$\delta_i^* = \max\{\hat{\theta}_{ii} + \hat{\theta}_{ij} \mid \text{all } j \text{ such that } (i,j) \in J^-\},$$

($i = 1, \dots, n$). Obviously, for each $(i,j) \in J^-$,

$$\hat{v}_j - \hat{v}_i - \delta_i^* \leq \hat{v}_j - \hat{v}_i - (\hat{\theta}_{ii} + \hat{\theta}_{ij}) \leq d_{ij},$$

which means that $\hat{v}, \tilde{\theta}_{ii} = \delta_i^*$ and $\tilde{\theta}_{ij} = 0$ ($i = 1, \dots, n; (i,j) \in J^-$) form a feasible solution of problem (4.3). Furthermore, as for this $(\hat{v}, \tilde{\theta})$,

$$\begin{aligned} \sum_{(i,j) \in \mathcal{A}} \tilde{\theta}_{ij} &= \sum_{i=1}^n \delta_i^* \\ &= \sum_{i=1}^n \max\{\hat{\theta}_{ii} + \hat{\theta}_{ij} \mid (i,j) \in J^-\} \\ &= \sum_{i=1}^n \hat{\theta}_{ii} + \sum_{i=1}^n \max\{\hat{\theta}_{ij} \mid (i,j) \in J^-\} \\ &\leq \sum_{i=1}^n \hat{\theta}_{ii} + \sum_{i=1}^n \sum_{\{j \mid (i,j) \in J^-\}} \hat{\theta}_{ij} \\ &= \sum_{(i,j) \in \mathcal{A}} \hat{\theta}_{ij}, \end{aligned}$$

we know that $(\hat{v}, \tilde{\theta})$ must be an optimal solution of problem (4.3), and in this solution all $\tilde{\theta}_{ij} = 0$ if $(i,j) \in J^-$. \square

Due to the above lemma, we can obtain an optimal solution of problem (4.2) by solving problem

$$\begin{aligned}
 \min \quad & \sum_{i=1}^n \theta_{ii} \\
 \text{s.t.} \quad & u_i + v_j \leq c_{ij}, \quad (i, j) \in J^-, \\
 & u_i + v_i + \theta_{ii} = c_{ii}, \quad i = 1, \dots, n, \\
 & \theta_{ii} \geq 0, \quad i = 1, \dots, n.
 \end{aligned} \tag{4.4}$$

If we substitute $\theta_{ii} = c_{ii} - (u_i + v_i)$ into the objective function, problem (4.4) can be reduced to

$$\begin{aligned}
 \max \quad & \sum_{i=1}^n (u_i + v_i) \\
 \text{s.t.} \quad & u_i + v_j \leq c_{ij}, \quad (i, j) \in \mathcal{A}.
 \end{aligned} \tag{4.5}$$

We summarize the above analysis in the following theorem:

Theorem 4.3. *The optimal solution of the inverse assignment problem can be obtained by solving problem (4.5).*

It is interesting that (4.5) is just the dual problem of the assignment problem (AS). So, an $O(n^2)$ algorithm for solving the inverse assignment problem can be stated as follows:

Step 1: Change the order of the nodes on one side of the bipartite graph corresponding to the assignment problem (AS) so that $J^+ = \{(1, 1), (2, 2), \dots, (n, n)\}$.

Step 2: Solve the assignment problem (AS) to obtain its dual optimal solution (u^*, v^*) .

Step 3:

$$\theta_{ij}^* = \begin{cases} c_{ii} - u_i^* - v_i^*, & \text{if } j = i, \\ 0, & \text{if } (i, j) \in J^- \end{cases}$$

is the minimum adjustment for vector c in the inverse assignment problem, i.e., the new cost vector is

$$c_{ij}^* = \begin{cases} u_i^* + v_i^*, & \text{if } j = i, \\ c_{ij}, & \text{if } (i, j) \in J^-. \end{cases}$$

We now use an example to explain the method.

Example 4.4. An assignment problem with the following cost matrix

$$\begin{bmatrix} 5 & 7 & 3 & \infty \\ 4 & 7 & 4 & 2 \\ \infty & 3 & 8 & 3 \\ 8 & 7 & 4 & 6 \end{bmatrix}$$

has a feasible solution x^0 in which $x_{ii}^0 = 1$, $i = 1, 2, 3, 4$; and $x_{ij}^0 = 0$, $i \neq j$. We first solve the assignment problem, obtaining its optimal solution $x_{11} = x_{24} = x_{32} = x_{43} = 1$, other $x_{ij} = 0$ and a dual optimal solution: $u_1^* = 3$, $u_2^* = 2$, $u_3^* = 3$, $u_4^* = 4$; $v_1^* = 2$, $v_2^* = v_3^* = v_4^* = 0$. So, by the formula in Step 3, $\theta_{22}^* = 5$, $\theta_{33}^* = 5$, $\theta_{44}^* = 2$, and all other $\theta_{ij}^* = 0$, i.e., the adjusted cost matrix is

$$\begin{bmatrix} 5 & 7 & 3 & \infty \\ 4 & 2 & 4 & 2 \\ \infty & 3 & 3 & 3 \\ 8 & 7 & 4 & 4 \end{bmatrix}$$

and the total adjustment of the cost coefficients is 12.

5. A restricted inverse assignment problem

It is seen from the previous section that for inverse assignment problem, the minimum adjustment can be realized by changing only the cost coefficients c_{ij} that correspond to the components $x_{ij}^0 = 1$ in the given solution x^0 . If some of these c_{ij} are not allowed to change, the solution of the inverse problem of course may be different. For brevity, we call this type of problems restricted inverse problems.

Restricted inverse problems may have some potential applications. For example, in a toll traffic network a route (path) P has not been used sufficiently meanwhile congestion occurs on other roads. In order to change the situation and to encourage more vehicles to use this route, the network manager can: (i) reduce the tolls for passing the roads (arcs) in P ; or (ii) increase the tolls for access to the roads not in P ; or (iii) do both, so that P becomes the cheapest route between its two endpoints. From profit making viewpoint, choice (ii) would be the best one to the network owner. As it is well known that a shortest path problem in a directed network can be transformed into an assignment problem (see [3] for example), choice (ii) is equivalent to an inverse assignment problem in which the weights of a given assignment pattern should be fixed.

In this section we discuss such a restricted inverse assignment problem. Again, let $J^+ = \{(i, j) | x_{ij}^0 = 1\}$; $J^- = \{(i, j) | x_{ij}^0 = 0\}$ and assume $J^+ = \{(1, 1), \dots, (n, n)\}$. It is requested that all c_{ii} cannot change. In other words, we consider the question: if only c_{ij} , $(i, j) \in J^-$, are permitted to change, what is the minimum total adjustment of c , and how to change these c_{ij} so that x^0 becomes an optimal solution of problem (AS)?

As now θ_{ii} must be zero, from problem (4.3) we see immediately that the restricted inverse problem is

$$\begin{aligned} \min \quad & \sum_{(i,j) \in J^-} \theta_{ij} \\ \text{s.t.} \quad & v_j - v_i - \theta_{ij} \leq d_{ij}, \quad (i, j) \in J^-, \\ & \theta_{ij} \geq 0, \quad (i, j) \in J^-. \end{aligned} \tag{5.1}$$

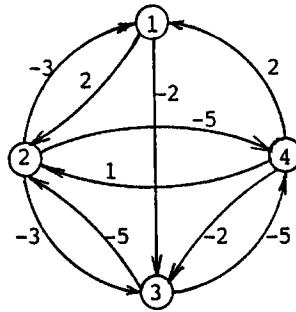


Fig. 1.

It is not difficult to obtain its dual LP problem:

$$\begin{aligned}
 \min \quad & \sum_{(i,j) \in J^-} d_{ij} y_{ij} \\
 \text{s.t.} \quad & \sum_{j \in f(i)} y_{ij} - \sum_{j \in t(i)} y_{ji} = 0, \quad i = 1, \dots, n, \\
 & 0 \leq y_{ij} \leq 1, \quad (i,j) \in J^-.
 \end{aligned} \tag{5.2}$$

Problem (5.2) is a minimum cost circulation problem with unit capacity, which has many quick methods to solve. A strongly polynomial algorithm for solving minimum cost circulation problems is the minimum mean cycle-cancelling algorithm, see [1] for example.

Let $V = \{1, 2, \dots, n\}$ and J^- and d_{ij} be defined as before, then $G(V, J^-, d)$ is the network corresponding to problem (5.2). For the assignment problem given in Example 4.4, the graph of problem (5.2) is shown in Fig. 1.

We now consider how to obtain the cost adjustment θ_{ij} associated with the optimal y_{ij} , i.e., after solving the dual problem (5.2), how to get the primal optimal solution. Suppose we use any version of cycle-cancelling algorithm to solve problem (5.2). Let y^* be the optimal basic solution, which must be a 0–1 vector. Let $\tilde{G}(y^*)$ be the residual network of G with respect to y^* . According to the algorithm, $\tilde{G}(y^*)$ must not contain any negative cost cycle. By the definition of residual network (see [1]), let $\tilde{\mathcal{A}} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \{(i, j) \mid (i, j) \in J^- \text{ and } y_{ij}^* = 0\},$$

$$\mathcal{A}_2 = \{(i, j) \mid (j, i) \in J^- \text{ and } y_{ji}^* = 1\},$$

and define

$$\tilde{d}_{ij} = \begin{cases} d_{ij}, & (i, j) \in \mathcal{A}_1, \\ -d_{ji}, & (i, j) \in \mathcal{A}_2, \end{cases}$$

then $\tilde{G}(y^*) = (V, \tilde{\mathcal{A}}, \tilde{d})$. We add another node s together with n arcs (s, i) into $\tilde{G}(y^*)$ and let $\tilde{d}_{si} = 0$ ($i = 1, \dots, n$). Denote the resulting augmented graph by $\tilde{G}^+(y^*)$.

As $\tilde{G}(y^*)$ has no negative cycle, so does $\tilde{G}^+(y^*)$, which means the shortest distance from s to each i is well defined under the length vector \tilde{d} . Say the shortest distance is π_i . So, for each arc $(i, j) \in \tilde{\mathcal{A}}$,

$\pi_j - \pi_i \leq \tilde{d}_{ij}$. Considering the arcs in \mathcal{A}_1 and \mathcal{A}_2 separately, we know that

$$(i, j) \in \mathcal{A}_1 \Rightarrow \pi_j - \pi_i \leq \tilde{d}_{ij} = d_{ij}, \quad (5.3)$$

$$(j, i) \in \mathcal{A}_2 \Rightarrow \pi_i - \pi_j \leq \tilde{d}_{ji} = -d_{ij} \\ \Rightarrow \pi_j - \pi_i \geq d_{ij}. \quad (5.4)$$

Based on the above result, for each $(i, j) \in J^-$, we define

$$\theta_{ij}^* = \begin{cases} 0, & \text{if } (i, j) \in \mathcal{A}_1, \\ \pi_j - \pi_i - d_{ij}, & \text{if } (j, i) \in \mathcal{A}_2. \end{cases} \quad (5.5)$$

Theorem 5.1. *The so defined π_i and θ_{ij}^* ($i = 1, \dots, n; (i, j) \in J^-$) must be an optimal solution of problem (5.1).*

Proof. Obviously (π, θ^*) is feasible as it always satisfies the constraints of problem (5.1). By the optimality condition for LP problems, (π, θ^*) is an optimal solution if it and y^* meet the complementary slackness condition, i.e., if

- (a) $\theta_{ij}^* > 0 \Rightarrow y_{ij}^* = 1$, and
- (b) $y_{ij}^* = 1 (> 0) \Rightarrow \pi_j - \pi_i - \theta_{ij}^* = d_{ij}$.

It is easy to verify that (a) and (b) are true. First, if $\theta_{ij}^* > 0$, then by (5.5), $(j, i) \in \mathcal{A}_2$ so that y_{ij}^* must be 1; second, if $y_{ij}^* = 1$, then $(j, i) \in \mathcal{A}_2$, and from (5.5) we see that $\pi_j - \pi_i - \theta_{ij}^* = d_{ij}$ holds. \square

We now summarize the method that we proposed in this section for solving the restricted inverse assignment problem:

Step 1: Solve the minimum cost flow problem (5.2) to obtain y^* and its corresponding residual network $\tilde{G}(y^*)$. (The residual network shall be generated automatically if a cycle-cancelling algorithm is used to solve problem (5.2).)

Step 2: Formulate the augmented network $\tilde{G}^+(y^*)$, from which find the shortest distance π_i from s to each $i \in V$.

Step 3: Use formula (5.5) to obtain θ_{ij}^* for $(i, j) \in J^-$.

Let us return to Example 4.4. The optimal solution of the minimum cost flow of Fig. 1 is

$$y_{13}^* = y_{21}^* = y_{24}^* = y_{32}^* = y_{34}^* = y_{42}^* = y_{43}^* = 1; \text{ other } y_{ij}^* = 0,$$

with a minimum objective value -21 . The associated graphs $\tilde{G}(y^*)$ and $\tilde{G}^+(y^*)$ are shown respectively in Figs. 2 and 3. Note that in Fig. 2 dotted curves represent the arcs in \mathcal{A}_1 whereas the solid arcs are in \mathcal{A}_2 .

From Fig. 3 we see that the shortest distances from s to $i = 1, 2, 3, 4$ are $\pi_1 = -1$, $\pi_2 = 0$, $\pi_3 = -3$, $\pi_4 = -1$. We then use (5.5) to determine θ_{ij}^* : $\theta_{21}^* = 2$, $\theta_{24}^* = 4$, $\theta_{32}^* = 8$, $\theta_{34}^* = 7$, and other $\theta_{ij}^* = 0$. Therefore, the adjusted cost matrix becomes

$$\begin{bmatrix} 5 & 7 & 3 & \infty \\ 6^* & 7 & 4 & 6^* \\ \infty & 11^* & 8 & 10^* \\ 8 & 7 & 4 & 6 \end{bmatrix}$$

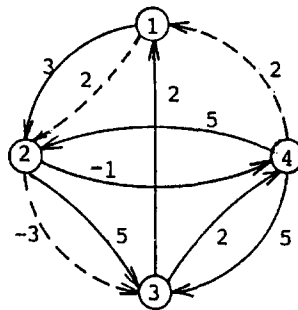


Fig. 2.

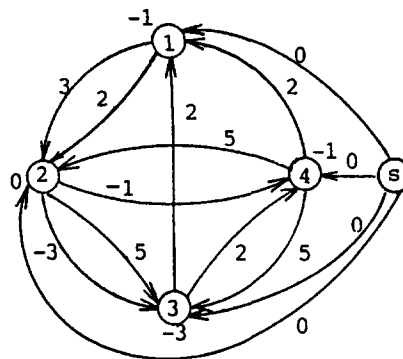


Fig. 3.

in which the costs with asterisk are increased from the original data by corresponding values of $\theta_{ij}^* > 0$.

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