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# A further study on inverse linear programming problems

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**Abstract**

In this paper we continue our previous study (Zhang and Liu, J. Comput. Appl. Math. 72 (1996) 261–273) on inverse linear programming problems which requires us to adjust the cost coefficients of a given LP problem as less as possible so that a known feasible solution becomes the optimal one. In particular, we consider the cases in which the given feasible solution and one optimal solution of the LP problem are 0–1 vectors which often occur in network programming and combinatorial optimization, and give very simple methods for solving this type of inverse LP problems. Besides, instead of the commonly used $l_1$ measure, we also consider the inverse LP problems under $l_\infty$ measure and propose solution methods. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

We know that in an optimization problem, all parameters of the model are given, and we need to find from among all feasible solutions an optimal solution for a specified objective function. In an inverse optimization problem, however, the situation is reversed and we need to adjust the values of the parameters in a model as little as possible (under $l_1, l_2$, or $l_\infty$ measure) such that a given feasible solution becomes an optimal solution under the new parameter values. See [4–6,9–12]. Sometimes the adjustment of various parameters cause different costs, and the objective is to use a minimum cost to change the given feasible solution into an optimal one.

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This type of problems has potential applications. For example, if a transportation planning authority is going to improve a road system and to use minimum cost to make a particular path between two towns the quickest path or the maximum capacity path between them (the capacity of a path is defined as the minimum capacity of the arcs on the path), then the problem is an inverse shortest path problem provided that the travelling times along the arcs of the network are regarded as the adjustable parameters, or an inverse maximum capacity path problem if we take the capacities of the arcs as the parameters to be adjusted.

Another application is inverse location problem. As we know, in a location problem a network is given and we need to find the best place to install a facility or to build a centre for the system. Here the measurement for the optimality can be either the min–max ($l_\infty$) or the min-sum ($l_1$) criterion. However, sometimes the facility has already been fixed, or the centre has been constructed, at certain place, and we can only consider how to improve the network system with the minimum cost so that this particular place indeed becomes the centre location under the $l_1$ or $l_\infty$ norm. This is just an inverse location problem.

Some inverse optimization problems are not so apparent. For example, in the DEA models (see [7]), for each decision making unit (DMU), all inputs and outputs are known and we need to determine the efficiency index of the DMU. In fact this efficiency index is calculated as the optimal value of a particular LP problem which takes all inputs/outputs as given parameters. However, some practitioners met the following problems: the DEA method has once been used to analyse the efficiencies of the DMUs in the system, but now for a particular DMU, the inputs are increased and they want to forecast the outputs, or the outputs have to be increased to certain level, and they want to estimate the required increment of the inputs for making the changes of the outputs, under the assumption that the efficiency index of the DMU remains the same. Mathematically, these two questions can be described as how to adjust a part of the parameters (either the outputs, or the inputs of the DMU) so that the optimal value of the associated LP model equals the given efficiency index. This is an inverse DEA problem. See [8] for the details.

There is also a kind of generalized inverse optimization problems in which no particular feasible solution is requested to become the optimal solution. Instead, we should adjust the parameter values as little as possible so that the optimal solution or the optimal value of the adjusted model meets some given requirement. For example, in [3] Burton et al. posed the problem of making minimum adjustment on the lengths of the arcs in a network so that the length of the shortest path between a pair of nodes will not exceed a given upper bound. Other relevant applications of this kind can be found, e.g. in [1,2].

Most authors use $l_1$ norm as the measure so that the study of inverse optimization can be carried out in the field of linear programming. Several methods have been given including column generation method and ellipsoid method for general inverse problems and some strongly polynomial methods for certain types of inverse optimization problems. In [11] we have considered some inverse LP problems. A general method for solving inverse LP problems was suggested which is based on the optimality conditions for LP problems. It has been found that when the method is applied to some problems, such as the inverse minimum cost flow problem, the method can achieve strongly polynomial complexity.

It is found that inverse LP problems can be further delved. Especially, if the given feasible solution is a 0–1 vector, and one optimal solution of the original LP problem has all components between 0 and 1, which often happen in network or combinatorial optimization problems, then the inverse

LP problems can be solved very easily. Also, the study of inverse LP problems under $l_\infty$ norm can bring about some new features even though these problems are still in the scope of linear programming.

This paper is organized as follows. In Section 2 we specify the type of inverse LP problems that we will discuss in the paper and formulate their inverse problems in a general way. Section 3 is devoted to study this type of inverse LP problems under $l_1$ norm, and Theorem 3.1 gives a simple method to solve such inverse problems. Then in Section 4 we extend the result of Section 3 to the inverse problems in which the decision variables of the LP problem subject to bound constraints and obtain Theorem 4.1 as the core result. Section 5 is arranged to consider inverse LP problems under $l_\infty$ norm, and Theorem 5.1 summarizes the method to solve this type of problems. Finally, in Section 6 we apply the methods and results obtained in the previous sections to solve some inverse network optimization problems as special examples.

Unless particularly specified, in this paper vectors are considered as column vectors.

## 2. Inverse LP problem

Given a linear program

$$
\begin{array}{lll}
\text{(LP)} & \text{Min} & c^{\mathrm{T}}x \\
 & \text{s.t.} & Ax = b, \\
 & & x \geqslant 0,
\end{array}
$$

where $A \in R^{m\times n}, b \in R^m$ and $c, x \in R^n$, and let $x^0$ be a feasible solution, we consider the problem of changing the cost vector $c$ as less as possible such that $x^0$ becomes an optimal solution of (LP) under the new cost vector $\bar{c}$. According to [11], this inverse problem can be formulated as

$$
\begin{array}{lll}
\text{(ILP)} & \text{Min} & \|\bar{c} - c\| \\
 & \text{s.t.} & \pi p_j \leqslant \bar{c}_j,\ j \in \underline{J}, \\
 & & \pi p_j = \bar{c}_j,\ j \in J,
\end{array}
$$

where $\underline{J} = \{j \mid x_j^0 = 0\}$, $J = \{j \mid x_j^0 > 0\}$, $p_j$ is the $j$-th column of $A, \pi$ is a row vector of dimension $m$, and $\|\cdot\|$ is a vector norm.

Let $\bar{c}_j = c_j + \theta_j - \alpha_j$, and $\theta_j, \alpha_j \geqslant 0$ for $j = 1, \ldots, n$, where $\theta_j$ and $\alpha_j$ are respectively the increment and decrement of $c_j$. Notice that in our model $\theta_j \alpha_j = 0$, i.e. $\theta_j$ and $\alpha_j$ can never be positive at the same time. Then problem (ILP) can be expressed as

$$
\begin{array}{ll}
\text{Min} & \|\theta + \alpha\| \\
\text{s.t.} & \pi p_j - \theta_j + \alpha_j \leqslant c_j,\ \ j \in \underline{J}, \\
 & \pi p_j - \theta_j + \alpha_j = c_j,\ \ j \in J, \\
 & \theta_j, \alpha_j \geqslant 0,\ \ j = 1, 2, \ldots, n,
\end{array}
\tag{2.1}
$$

Apparently, problem (2.1) is equivalent to

$$\begin{aligned} \text{Min}\quad & \|\theta+\alpha\| \\ \text{s.t.}\quad & \pi p_j-\theta_j \leqslant c_j,\ j\in \underline{J}, \\ & \pi p_j-\theta_j+\alpha_j=c_j,\ j\in J, \\ & \theta_j\geqslant 0,\ j\in \underline{J}\cup J, \\ & \alpha_j\geqslant 0,\ j\in J. \end{aligned} \tag{2.2}$$

Note that the second group of constraints in Eq. (2.2) can be expressed as

$$-\pi p_j+\theta_j-\alpha_j\geqslant -c_j,\qquad \pi p_j-\theta_j+\alpha_j\geqslant c_j, \tag{2.3}$$

which, under the condition $\theta_j,\alpha_j\geqslant 0$, imply

$$-\pi p_j+\theta_j\geqslant -c_j,\qquad \pi p_j+\alpha_j\geqslant c_j. \tag{2.4}$$

On the contrary, if Eq. (2.4) holds, and if $\theta_j>0$ then in the optimal solution $\alpha_j=0$ and

$$-\pi p_j+\theta_j=-c_j,$$

which ensure the condition (2.3). If $\alpha_j>0$, we again can derive Eq. (2.3). Therefore, problem (2.2) is equivalent to

$$\begin{aligned} \text{Min}\ & \|\theta+\alpha\| \\ \text{s.t.}\ & -\pi p_j+\theta_j\geqslant -c_j,\ j\in \underline{J}, \\ & -\pi p_j+\theta_j\geqslant -c_j,\ j\in J, \\ & \pi p_j+\alpha_j\geqslant c_j,\ j\in J, \\ & \theta_j\geqslant 0,\ j\in \underline{J}\cup J, \\ & \alpha_j\geqslant 0,\ j\in J. \end{aligned} \tag{2.5}$$

If the LP problem with bounded variables

$$\begin{aligned} \text{(BLP)}\quad \text{Min}\quad & c^{\mathrm{T}}x \\ \text{s.t.}\quad & Ax=b, \\ & 0\leqslant x\leqslant u \end{aligned}$$

is concerned, where $u$ is a given nonnegative vector, then for a given feasible solution $x^0$, the inverse problem can be formulated in a more symmetric form:

$$\begin{aligned} \text{(IBLP)}\quad \text{Min}\quad & \|\theta+\alpha\| \\ \text{s.t.}\quad & -\pi p_j+\theta_j\geqslant -c_j,\ j\in \underline{J}\cup J, \\ & \pi p_j+\alpha_j\geqslant c_j,\ j\in J\cup \bar{J}, \\ & \theta_j\geqslant 0,\ j\in \underline{J}\cup J, \\ & \alpha_j\geqslant 0,\ j\in J\cup \bar{J}, \end{aligned}$$

where

$$\underline{J}=\{j\mid x_j^0=0\},\ J=\{j\mid 0<x_j^0<u_j\},\quad \text{and}\quad \bar{J}=\{j\mid x_j^0=u_j\}.$$

## 3. The solution of (ILP) under $l_1$ norm

Under the $l_1$ norm, problem (2.5) becomes

$$
\begin{aligned}
\text{(ILP1)} \quad & \text{Min} \quad \sum_{j=1}^{n} \theta_j + \sum_{j\in J} \alpha_j \\
& \text{s.t.} \quad -\pi p_j + \theta_j \geqslant -c_j, \ j \in \underline{J} \cup J, \\
& \qquad\quad \pi p_j + \alpha_j \geqslant c_j, \ j \in J, \\
& \qquad\quad \theta_j \geqslant 0, \ j \in \underline{J} \cup J, \\
& \qquad\quad \alpha_j \geqslant 0, \ j \in J,
\end{aligned}
$$

which is a LP problem with the dual

$$
\begin{aligned}
\text{Max} \quad & -c^{\mathrm{T}}x + c_J^{\mathrm{T}} y \\
\text{s.t.} \quad & Ax - A_J y = 0, \\
& 0 \leqslant x_j \leqslant 1, \ j \in \underline{J} \cup J, \\
& 0 \leqslant y_j \leqslant 1, \ j \in J,
\end{aligned}
$$

where $A_J$ is the submatrix of $A$ consisting of the columns $p_j$ for $j \in J$, and $c_J$ is the subvector of $c$ consisting of the components $c_j$ for $j \in J$. If we let

$$
z_j = \begin{cases} x_j, & j \in \underline{J}, \\ x_j - y_j, & j \in J, \end{cases}
$$

then the above dual problem can be rewritten as

$$
\begin{aligned}
\text{(DILP1)} \quad & \text{Max} \quad -c^{\mathrm{T}} z \\
& \text{s.t.} \quad Az = 0, \\
& \qquad\quad 0 \leqslant z_j \leqslant 1, \ j \in \underline{J}, \\
& \qquad\quad -1 \leqslant z_j \leqslant 1, \ j \in J.
\end{aligned}
$$

We now establish the main result of this section which shows that in some special cases, the optimal solution of the $l_1$-norm inverse problem of problem (LP) can be obtained from the dual optimal solution.

**Theorem 3.1.** *Suppose $x^0$ is a given 0–1 feasible solution of problem (LP) which has an optimal solution $x^*$ satisfying $0 \leqslant x^* \leqslant 1$. Let $\pi^*$ be the optimal solution of its dual problem (DLP). Define vectors $\theta^* = 0$ and $\alpha_j^* = \max\{0, c_j - \pi^* p_j\}$ for all $j \in J$. Then $\{\pi^*, \theta^*, \alpha^*\}$ is an optimal solution of the inverse problem (ILP1).*

**Proof.** The dual of problem (LP) is

$$
\begin{aligned}
\text{(DLP)} \quad & \max \quad \pi b \\
& \text{s.t.} \quad \pi p_j \leqslant c_j, \ j = 1, \ldots, n.
\end{aligned}
$$

We first show that $\{\pi^*, \theta^*, \alpha^*\}$ is a feasible solution to problem (ILP1). Obviously, $\theta^*, \alpha^* \geqslant 0$. As $\pi^*$ is a feasible solution of problem (DLP), for any $j \in \underline{J} \cup J$,

$$-\pi^* p_j + \theta_j^* = -\pi^* p_j \geqslant -c_j.$$

Also, by the definition of $\alpha_j^*$, for $j \in J$.

$$\begin{aligned}\pi^* p_j + \alpha_j^* &= \pi^* p_j + \max\{0, c_j - \pi^* p_j\} \\ &\geqslant c_j.\end{aligned}$$

So, $\{\pi^*, \theta^*, \alpha^*\}$ is feasible. Note that the objective value of problem (ILP1) is $\sum_{j \in J} \alpha_j^*$.

We now prove that $\{\pi^*, \theta^*, \alpha^*\}$ is an optimal solution. In order to do so it suffices if we can show that the dual problem (DILP1) has a feasible solution with the same objective value $\sum_{j \in J} \alpha_j^*$.

Since $x^*$ and $\pi^*$ are respectively the optimal solutions of (LP) and (DLP), by the complementary slackness condition,

$$\sum_{j \in \underline{J} \cup J} (c_j - \pi^* p_j) x_j^* = 0. \tag{3.1}$$

As $x^0$ is a 0–1 vector, for each $j \in J$, $x_j^0 = 1$. By the definition of $\alpha_j^*$ and the feasibility of $\pi^*$ to problem (DLP), for each $j \in J$, $\alpha_j^* = c_j - \pi^* p_j$. So,

$$\begin{aligned}\sum_{j \in J} \alpha_j^* &= \sum_{j \in J} \alpha_j^* x_j^0 \\ &= \sum_{j \in J} (c_j - \pi^* p_j) x_j^0 \\ &= \sum_{j \in \underline{J} \cup J} (c_j - \pi^* p_j) x_j^0.\end{aligned} \tag{3.2}$$

Combining Eqs. (3.1) and (3.2), we obtain

$$\begin{aligned}\sum_{j \in J} \alpha_j^* &= \sum_{j \in \underline{J} \cup J} (c_j - \pi^* p_j)(x_j^0 - x_j^*) \\ &= -c^{\mathrm{T}}(x^* - x^0) + \pi^* A (x^* - x^0) \\ &= -c^{\mathrm{T}}(x^* - x^0).\end{aligned}$$

Since

$$x_j^* - x_j^0 = \begin{cases} x_j^* \in [0,1], & \text{if } j \in \underline{J}, \\ x_j^* - 1 \in [-1, 0], & \text{if } j \in J, \end{cases}$$

$x^* - x^0$ is a feasible solution of problem (DILP1) with the objective value $-c^{\mathrm{T}}(x^* - x^0)$. Therefore, $(\pi^*, \theta^*, \alpha^*)$ and $x^* - x^0$ are, respectively, the optimal solutions of problems (ILP1) and (DILP1). $\square$

## 4. The solution of (IBLP) under $l_1$ norm

Under the $l_1$ norm, the inverse problem (ILBP) becomes

$$
\text{(IBLP1)} \quad \begin{aligned}
\text{Min} \quad & \sum_{j\in \underline{J}\cup J} \theta_j + \sum_{j\in J\cup\bar{J}} \alpha_j \\
\text{s.t.} \quad & -\pi p_j + \theta_j \geqslant -c_j, \ j\in \underline{J}\cup J, \\
& \pi p_j + \alpha_j \geqslant c_j, \ j\in J\cup\bar{J}, \\
& \theta_j \geqslant 0, \ j\in \underline{J}\cup J, \\
& \alpha_j \geqslant 0, \ j\in J\cup\bar{J}.
\end{aligned}
$$

Let $A_1$ and $A_2$ be the submatrices consisting of the columns $p_j$ of $A$, corresponding to $j\in \underline{J}\cup J$ and $j\in J\cup\bar{J}$, respectively. Then the dual of the above problem is

$$
\begin{aligned}
\text{Max} \quad & -\sum_{j\in \underline{J}\cup J} c_j x_j + \sum_{j\in J\cup\bar{J}} c_j y_j \\
\text{s.t.} \quad & A_1 x - A_2 y = 0, \\
& 0 \leqslant x_j \leqslant 1, \ j\in \underline{J}\cup J, \\
& 0 \leqslant y_j \leqslant 1, \ j\in J\cup\bar{J}.
\end{aligned}
$$

In what follows we consider a special case of the problem (BLP): $u=\mathbf{1}$, i.e. each variable $x_j$ has a unit upper-bound: $u_j=1$. Problem (BLP) becomes

$$
\text{(UBLP)} \quad \begin{aligned}
\text{Min} \quad & c^{\mathrm{T}} x \\
\text{s.t.} \quad & Ax = b, \\
& 0 \leqslant x \leqslant 1.
\end{aligned}
$$

Its dual is

$$
\text{(DUBLP)} \quad \begin{aligned}
\text{Max} \quad & \pi b - w\mathbf{1} \\
\text{s.t.} \quad & \pi A - w \leqslant c^{\mathrm{T}}, \\
& w \geqslant 0,
\end{aligned}
$$

in which $\pi\in R^m$ and $w\in R^n$ are two row vectors. Let $x^0$ be a 0–1 feasible solution of problem (UBLP), then the inverse problem with respect to $x^0$ is

$$
\text{(IUBLP1)} \quad \begin{aligned}
\text{Min} \quad & \sum_{j\in \underline{J}} \theta_j + \sum_{j\in \bar{J}} \alpha_j \\
\text{s.t.} \quad & -\pi p_j + \theta_j \geqslant -c_j, \ j\in \underline{J}, \\
& \pi p_j + \alpha_j \geqslant c_j, \ j\in \bar{J}, \\
& \theta_j \geqslant 0, \ j\in \underline{J}, \\
& \alpha_j \geqslant 0, \ j\in \bar{J},
\end{aligned}
$$

or equivalently,

$$\begin{array}{lll}
(\text{IUBLP1}') & \text{Min} & \sum_{j\in \underline{J}} \theta_j + \sum_{j\in \bar{J}} \alpha_j \\
& \text{s.t.} & -\pi p_j + \theta_j \geqslant -c_j, \ j \in \underline{J}, \\
& & -\pi p_j - \alpha_j + w_j = -c_j, \ j \in \bar{J}, \\
& & \theta_j \geqslant 0, \ j \in \underline{J}, \\
& & \alpha_j, w_j \geqslant 0, \ j \in \bar{J}.
\end{array}$$

Obviously the dual of (IUBLP1) is

$$\begin{array}{lll}
(\text{DIUBLP1}) & \text{Max} & -\sum_{j\in \underline{J}} c_j x_j + \sum_{j\in \bar{J}} c_j y_j \\
& \text{s.t.} & A_{\underline{J}} x - A_{\bar{J}} y = 0, \\
& & 0 \leqslant x_j \leqslant 1, \ j \in \underline{J}, \\
& & 0 \leqslant y_j \leqslant 1, \ j \in \bar{J}.
\end{array}$$

Or, if we define $x_j = -y_j$ for $j \in \bar{J}$, then the dual of the inverse problem can be expressed as

$$\begin{array}{lll}
(\text{DIUBLP1}') & \text{Max} & -c^{\mathrm{T}} x \\
& \text{s.t.} & Ax = 0, \\
& & 0 \leqslant x_j \leqslant 1, \ j \in \underline{J}, \\
& & -1 \leqslant x_j \leqslant 0, \ j \in \bar{J}.
\end{array}$$

We now give a very simple method to obtain an optimal solution of the inverse problem (IUBLP1).

**Theorem 4.1.** *Let $x^0$ be a 0–1 feasible solution of problem (UBLP), $x^*$ be an optimal solution of (UBLP), and $(\pi^*, w^*)$ be an optimal solution of the dual problem (DUBLP). Define*

$$\theta_j = \begin{cases} w_j^*, & j \in \underline{J} \cap \bar{I}, \\ 0, & j \in \underline{J} \cap \{\underline{I} \cup I\}, \end{cases}$$

$$\alpha_j = \begin{cases} c_j - \pi^* p_j, & j \in \bar{J} \cap \underline{I}, \\ 0, & j \in \bar{J} \cap \{I \cup \bar{I}\}, \end{cases}$$

$$w_j = \begin{cases} w_j^*, & j \in \bar{J} \cap \bar{I}, \\ 0, & j \in \bar{J} \cap \{\underline{I} \cup I\}, \end{cases}$$

$$\pi = \pi^*,$$

*where*

$$\underline{I} = \{j | x_j^* = 0\}, \qquad I = \{j | 0 < x_j^* < 1\}, \qquad \bar{I} = \{j | x_j^* = 1\},$$

*then $\{\pi, w, \theta, \alpha\}$ is an optimal solution of the problem (IUBLP1′).*

**Proof.** We first show that $\{\pi, w, \theta, \alpha\}$ is feasible to problem (IUBLP1′). Obviously $\theta$, $w \geqslant 0$. If $j \in \bar{J} \cap \underline{I}$, $x_j^* < 1$, and hence by the complementary slackness condition, $w_j^* = 0$ which implies that $c_j - \pi^* p_j \geqslant -w_j^* = 0$. So, $\alpha_j \geqslant 0$ for any $j \in \bar{J}$.

Since $(\pi^*, w^*)$ is feasible to (DUBLP),

$$\pi^* p_j - w_j^* \leqslant c_j \quad \forall j. \tag{4.1}$$

For each $j \in \underline{J} \cap \bar{I}$, $\theta_j = w_j^*$ and therefore Eq. (4.1) is equivalent to

$$\pi p_j - \theta_j \leqslant c_j. \tag{4.2}$$

On the other hand, if $j \in \underline{J} \cap \{\underline{I} \cap I\}$, as $\theta_j = 0$ and $x_j^* < 1$ which implies $w_j^* = 0$, (4.1) means $\pi p_j \leqslant c_j$ and hence Eq. (4.2) is still true. Therefore, for any $j \in \underline{J}$, Eq. (4.2) holds, i.e. the first set of constraints in (IUBLP1′) is true.

For $j \in \bar{J} \cap \underline{I}$, by definition, $w_j = 0$ and $\alpha_j = c_j - \pi p_j$. Hence,

$$\pi p_j + \alpha_j - w_j = c_j \tag{4.3}$$

is true. Now for $j \in \bar{J} \cap \{I \cup \bar{I}\}$, by definition $\alpha_j = 0$ and $x_j^* > 0$. Due to the complementary slackness condition,

$$\pi p_j - w_j^* = c_j. \tag{4.4}$$

Note that when $j \in \bar{I}$, $w_j = w_j^*$, whereas when $j \in I$, as $x_j^* < 1$, we must have $w_j^* = 0$, and by definition $w_j = 0 = w_j^*$. In other words, when $j \in \bar{J} \cap \{I \cup \bar{I}\}$, we always have $w_j = w_j^*$ and therefore, Eq. (4.4) means Eq. (4.3) still holds. So, we have proved that for any $j \in \bar{J}$, the second group of constraints of (IUBLP1′) holds. Thus $\{\pi, w, \theta, \alpha\}$ is a feasible solution of (IUBLP1′). Its objective value is

$$\gamma^* = \sum_{j \in \underline{J} \cap \bar{I}} \theta_j + \sum_{j \in \bar{J} \cap \underline{I}} \alpha_j. \tag{4.5}$$

It is easy to see that

$$\begin{aligned} &A(x^* - x^0) = 0, \\ &0 \leqslant x_j^* - x_j^0 = x_j^* \leqslant 1, \quad j \in \underline{J}, \\ &-1 \leqslant x_j^* - x_j^0 = x_j^* - 1 \leqslant 0, \quad j \in \bar{J}, \end{aligned}$$

i.e. $x^* - x^0$ is a feasible solution of the dual problem (DIUBLP1′). So, in order to prove that $\{\pi, w, \theta, \alpha\}$ is an optimal solution of (IUBLP1′), it suffices to show that the objective value $-c^{\mathrm{T}}(x^* - x^0)$ of problem (DIUBLP1′) at $x^* - x^0$ equals $\gamma^*$.

By the complementary slackness condition,

$$\begin{aligned} 0 &= \sum_{j=1}^{n} (c_j + w_j^* - \pi^* p_j) x_j^* \\ &= \sum_{j \in \underline{I} \cup I} (c_j - \pi p_j) x_j^* + \sum_{j \in \bar{I}} (c_j + w_j^* - \pi p_j) x_j^* \\ &\quad \text{(since for } j \in \underline{I} \cup I,\ x_j^* < 1 \text{ and } w_j^* = 0) \end{aligned}$$

$$= \sum_{j\in\underline{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)x_j^* + \sum_{j\in\bar{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)x_j^*$$

$$+ \sum_{j\in\underline{J}\cap\bar{I}} (c_j + w_j^* - \pi p_j)x_j^* + \sum_{j\in\bar{J}\cap\bar{I}} (c_j + w_j^* - \pi p_j)x_j^*$$

$$= \sum_{j\in\underline{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)(x_j^* - x_j^0) + \sum_{j\in\bar{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)(x_j^* - x_j^0)$$

$$+ \sum_{j\in\bar{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)x_j^0 + \sum_{j\in\underline{J}\cap\bar{I}} (c_j - \pi p_j)(x_j^* - x_j^0)$$

$$+ \sum_{j\in\underline{J}\cap\bar{I}} \theta_j x_j^* + \sum_{j\in\bar{J}\cap\bar{I}} (c_j + w_j^* - \pi p_j)(x_j^* - x_j^0).$$

In the above derivation, we used the facts that when $j \in \underline{J}$, $x_j^0 = 0$; when $j \in \underline{J} \cap \bar{I}$, $\theta_j = w_j^*$ and when $j \in \bar{I}$, as $x_j^* > 0$, $c_j + w_j^* - \pi^* p_j = 0$. Furthermore, since when $j \in \bar{J} \cap I$. $0 < x_j^* < 1$, we have $\pi^* p_j - w_j^* = c_j$ and $w_j^* = 0$, i.e., $\pi p_j = c_j$, the third summation of the last expression need to be carried only for $j \in \bar{J} \cap \underline{I}$. Also, when $j \in \bar{J} \cap \bar{I}$, $w_j^* = w_j$. So, we can obtain

$$0 = \sum_{j\in\underline{J}} (c_j - \pi p_j)(x_j^* - x_j^0) + \sum_{j\in\bar{J}\cap\{\underline{I}\cup I\}} (c_j - \pi p_j)(x_j^* - x_j^0)$$

$$+ \sum_{j\in\bar{J}\cap\underline{I}} (c_j - \pi p_j)x_j^0 + \sum_{j\in\underline{J}\cap\bar{I}} \theta_j + \sum_{j\in\bar{J}\cap\bar{I}} (c_j + w_j - \pi p_j)(x_j^* - x_j^0).$$

Notice that for $j \in \bar{J} \cap \underline{I}$, we have $x_j^0 = 1$ and $\alpha_j = c_j - \pi p_j$, which imply that the third sum above equals $\sum_{j\in\bar{J}\cap\underline{I}} \alpha_j$. Therefore, we have

$$0 = \sum_{j=1}^{n} (c_j - \pi p_j)(x_j^* - x_j^0) + \sum_{j\in\bar{J}\cap\bar{I}} w_j(x_j^* - x_j^0) + \gamma^*$$

$$= c^{\mathrm{T}}(x^* - x^0) - \pi A(x^* - x^0) + \gamma^*$$

$$= c^{\mathrm{T}}(x^* - x^0) + \gamma^*$$

as $x_j^* = x_j^0 = 1$ when $j \in \bar{J} \cap \bar{I}$ and $Ax^* = Ax^0 = b$. In other words, we have proved the required result

$$-c^{\mathrm{T}}(x^* - x^0) = \gamma^*. \qquad \square$$

## 5. The solutions of (ILP) and (IBLP) under $l_\infty$ norm

If the $l_\infty$ norm is concerned, problem (2.5) becomes

$$\begin{array}{ll} \text{Min} & v \\ \text{s.t.} & -\pi p_j + \theta_j \geqslant -c_j, \ j \in \underline{J} \cup J, \\ & \pi p_j + \alpha_j \geqslant c_j, \ j \in J, \\ & v - \theta_j \geqslant 0, \ j \in \underline{J} \cup J, \\ & v - \alpha_j \geqslant 0, \ j \in J, \\ & \theta_j \geqslant 0, \ j \in \underline{J} \cup J \\ & \alpha_j \geqslant 0, \ j \in J. \end{array} \tag{5.1}$$

If we replace all $\theta_j$ and $\alpha_j$ in Eq. (5.1) by $v$, the optimal value will not change:

$$\begin{array}{ll} \text{Min} & v \\ \text{s.t.} & -\pi p_j + v \geqslant -c_j, \ j \in \underline{J} \cup J, \\ & \pi p_j + v \geqslant c_j, \ j \in J, \\ & v \geqslant 0. \end{array} \tag{5.2}$$

When $x^0$ is not the optimal solution of the problem (LP), there must be some positive adjustments $\theta_j$ or $\alpha_j$ for the cost vector, which means the minimum value $v^*$ of the above problem must be positive. So, the constraint $v \geqslant 0$ would be satisfied automatically and thus can be removed. In this way we formulate the inverse problem under $l_\infty$ norm as follows:

$$\begin{array}{lll} (\text{ILP}\infty) & \text{Min} & v \\ & \text{s.t.} & -\pi p_j + v \geqslant -c_j, \ j \in \underline{J} \cup J, \\ & & \pi p_j + v \geqslant c_j, \ j \in J. \end{array}$$

Suppose the optimal solution of (ILP$\infty$) is $(\pi^*, v^*)$. If $j \in J$ and $\pi^* p_j < c_j$, then from the first two sets of constraints in Eqs. (5.1) and (5.2), it is easy to see that we can choose $\theta_j = 0$ and $\alpha_j = v^*$. We can discuss other cases in a similar way and obtain an optimal solution $\bar{c}$ for the inverse LP problem (ILP):

$$\bar{c}_j = \begin{cases} c_j + v^*, & j \in \underline{J} \cup J \quad \text{and} \quad \pi^* p_j > c_j, \\ c_j - v^*, & j \in J \quad \text{and} \quad \pi^* p_j < c_j, \\ c_j, & \text{otherwise}. \end{cases}$$

Obviously, when $c$ is replaced by $\bar{c}$, $\underline{x}^0$ would become an optimal solution of (LP). Of course such a solution $\bar{c}$ is not unique.

Similary, under $l_\infty$ norm, the problem (IBLP) can be expressed as

$$\begin{array}{lll} (\text{IBLP}_\infty) & \text{Min} & v \\ & \text{s.t.} & -\pi p_j + v \geqslant -c_j, \ j \in \underline{J} \cup J, \\ & & \pi p_j + v \geqslant c_j, \ j \in J \cup \bar{J}, \\ & & v \geqslant 0. \end{array}$$

In fact if $x^0$ is not the optimal solution of (BLP), then again the minimum value $v^*$ must be positive and thus the last constraint: $v \geqslant 0$ can be removed.

Once we solved problem (IBLP$\infty$) and obtained an optimal solution $(\pi^*, v^*)$, it is easy to obtain the optimal solution of the inverse problem under $l_\infty$ norm as shown by the following theorem.

**Theorem 5.1.** *Let*

$$\bar{c}_j = \begin{cases} c_j + v^*, & j \in \underline{J} \cup J \quad \textit{and} \quad \pi^* p_j > c_j, \\ c_j - v^*, & j \in J \cup \bar{J} \quad \textit{and} \quad \pi^* p_j < c_j, \\ c_j, & \textit{otherwise}. \end{cases}$$

*Then, $\bar{c}$ is the least-change cost vector from the given $c$ in the $l_\infty$ measure such that under the cost vector $\bar{c}$, $\underline{x}^0$ will become the optimal solution of problem (BLP).*

**Proof.** Obviously $\|\bar{c} - c\|_\infty = v^*$, which is the minimum change for the inverse problem (IBLP$\infty$). $(\pi^*, v^*)$ satisfies the constraints of (IBLP$\infty$):

$$-\pi^* p_j + v^* \geqslant -c_j, \quad j \in \underline{J} \cup J, \tag{5.3}$$

$$\pi^* p_j + v^* \geqslant c_j, \quad j \in J \cup \bar{J}. \tag{5.4}$$

In order to show that under the cost vector $\bar{c}$, $\underline{x}^0$ is an optimal solution of problem (BLP), it suffices to prove that the minimum value of the problem:

$$\begin{aligned} \text{Min} \quad & v \\ \text{s.t.} \quad & -\pi p_j + v \geqslant -\bar{c}_j, \ j \in \underline{J} \cup J, \\ & \pi p_j + v \geqslant \bar{c}_j, \ j \in J \cup \bar{J}, \\ & v \geqslant 0 \end{aligned} \tag{5.5}$$

is zero. To reach this purpose, we only need to show that $v = 0$ and $\pi = \pi^*$ is a feasible solution of the above problem. In other words, we need to show that the following two conditions hold:

$$-\pi^* p_j \geqslant -\bar{c}_j, \ j \in \underline{J} \cup J, \tag{5.6}$$

$$\pi^* p_j \geqslant \bar{c}_j, \ j \in J \cup \bar{J}. \tag{5.7}$$

In fact when $j \in \underline{J} \cup J$, if

$$\pi^* p_j \leqslant c_j,$$

then $\bar{c}_j = c_j$ and thus Eq. (5.6) is true; otherwise $\bar{c}_j = c_j + v^*$ and hence by Eq. (5.3), $\bar{c}_j \geqslant \pi^* p_j$, i.e. Eq. (5.6) also holds. In a similar way, we are able to show that when $j \in J \cup \bar{J}$, (5.7) always holds. □

## 6. Some applications to inverse network optimization

The above results for inverse LP problems can be used to analyse some inverse network optimization problems.

We know that the shortest path problem from node $s$ to $t$ in a network $N(V,A,C)$ can be formulated as a LP problem:

$$\begin{aligned}
\text{Min}\quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{s.t.}\quad & -\sum_{(i,j)\in A} x_{ij} + \sum_{(k,i)\in A} x_{ki} = \begin{cases} -1, & i=s, \\ 0, & i\in V\setminus\{s,t\}, \\ 1, & i=t, \end{cases} \\
& x_{ij}\geqslant 0, \quad (i,j)\in A.
\end{aligned} \tag{6.1}$$

Note that as the coefficient matrix is the node-edge incidence matrix of the network, which is totally unimodular, each basic feasible solution must be a 0–1 solution. So, a 0–1 optimal solution $x^*$ exists in which the components with $x^*_{ij}=1$ correspond to a shortest path. Now suppose a path $P$ from $s$ to $t$ is given. By defining

$$x^0_{ij} = \begin{cases} 1, & (i,j)\in P, \\ 0, & \text{otherwise.} \end{cases}$$

We have a 0–1 feasible solution $x^0$ to the above LP problem and thus the conditions of Theorem 3.1 are satisfied. Now the optimal solution of the inverse shortest path problem in the sense of this paper can be obtained very quickly by using Theorem 3.1. In fact the dual of problem (6.1) is

$$\begin{aligned}
\text{Max}\quad & \pi_t - \pi_s \\
\text{s.t.}\quad & \pi_j - \pi_i \leqslant c_{ij}, \quad (i,j)\in A,
\end{aligned}$$

and it is well known that $\pi_i$ represents the shortest length from $s$ to $i$ (possibly plus a constant). Therefore, the algorithm for such type of inverse shortest path problems consists of the following three steps:

*Step* 1. Find the shortest distance $\pi^*_i$ from $s$ to each node $i\in V$.

*Step* 2. For each $(i,j)\in P$, define

$$\alpha^*_{ij} = c_{ij} + \pi^*_i - \pi^*_j.$$

*Step* 3. Let

$$\bar{c}_{ij} = \begin{cases} c_{ij} - \alpha^*_{ij}, & (i,j)\in P, \\ c_{ij}, & \text{otherwise,} \end{cases}$$

then $\bar{c}$ is the least-change (under $l_1$ measure) cost vector to make $P$ become a shortest path from $s$ to $t$.

Note that when there is a negative cycle in the network, it may not have a shortest path from $s$ to $t$. But the inverse problem is still solvable, because we can insert the constraints

$$x_{ij}\leqslant 1, \quad (i,j)\in A,$$

to problem (6.1), and then use Theorem 4.1 to solve the inverse BLP problem.

Since the assignment problem can also be expressed as a LP problem with a totally unimodular coefficient matrix, its inverse problem can be solved by using Theorem 3.1. Note that such an algorithm has been given in [11], but now it can be regarded as a simple application of Theorem 3.1.

If the $l_\infty$ measure is concerned, for inverse shortest path problem we can first solve the LP problem:

$$\begin{aligned} \text{Min} \quad & v \\ \text{s.t.} \quad & \pi_i - \pi_j + v \geqslant -c_{ij}, \ (i,j) \in A, \\ & \pi_j - \pi_i + v \geqslant c_{ij}, \ (i,j) \in P, \end{aligned}$$

obtaining an optimal solution $(\pi^*, v^*)$, and then the least-change cost vector under $l_\infty$ norm is

$$\bar{c}_{ij} = \begin{cases} c_{ij} + v^*, & (i,j) \in A \text{ and } \pi_j^* - \pi_i^* > c_{ij}, \\ c_{ij} - v^*, & (i,j) \in P \text{ and } \pi_j^* - \pi_i^* < c_{ij}, \\ c_{ij}, & \text{otherwise.} \end{cases}$$

The inverse minimum spanning tree problem under $l_\infty$ norm is especially simple to solve. Let $N=(V,E,c)$ be a undirected network and $T$ be a given spanning tree in $N$. For each $e \in E \backslash T$, $T \cup \{e\}$ contains a unique cycle. It is easy to know that in order to let $T$ become a minimum spanning tree, we only need to reduce the weights on $T$ and increase the weights not on $T$. In particular, the inverse problem under $l_\infty$ measure can be formulated to:

$$\begin{aligned} (\text{IST}\infty) \quad \text{Min} \quad & \max\{\theta_e, \alpha_f\} \\ \text{s.t.} \quad & c_e + \theta_e \geqslant c_f - \alpha_f, \ e \notin T, \ f \in C(T,e), \\ & \theta_e \geqslant 0, \ e \notin T, \\ & \alpha_f \geqslant 0, \ f \in T, \end{aligned}$$

where $C(T,e)$ consists of the subset of edges in $T$ which together with $e$ form the unique cycle in $T \cup \{e\}$.

Using the method of Section 5, we can change problem (IST$\infty$) to

$$\begin{aligned} \text{Min} \quad & v \\ \text{s.t.} \quad & 2v \geqslant c_f - c_e, \ e \notin T, \ f \in C(T,e), \\ & v \geqslant 0, \end{aligned}$$

without affecting the optimal value. Obviously, the optimal solution (value) of the above problem is

$$v^* = \frac{1}{2}\max\left\{0, \max_{e \notin T} \max_{f \in C(T,e)} \{c_f - c_e\}\right\},$$

and the least-change cost vector $\bar{c}$ under the $l_\infty$ norm which lets $T$ become the minimum spanning tree in $N$ is

$$\bar{c}_g = \begin{cases} c_g - v^*, & g \in T, \\ c_g + v^*, & g \notin T, \end{cases}$$

for each edge $g$.

In fact more applications of the theorems established in this paper have been found and some of them shall be illustrated in a forthcoming paper.

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