# Logic - Homework I

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## 1 Problems

1. Question 1 **Solution:** $x_i \rightarrow x_j$ is a boolean function that represents logical implication. While $F_i \Rightarrow F_j$ is the provable statement that $I \vdash F_j$ for all interpretations $I$ such that $I \vdash F_j$ □

2. Question 2 **Solution:**

| | | |
|---|---|---|
| 1 | $P$ | Hypothesis |
| 2 | $Q \vee R$ | Hypothesis |
| 3 | $\neg Q \rightarrow R$ | Implication Equivalence $(2)$ |
| 4 | $\neg Q \vee \neg P$ | Assumption |
| 5 | $P \rightarrow \neg Q$ | Implication Equivalence $(4)$ |
| 6 | $\neg Q$ | Modus Ponens $(1, 5)$ |
| 7 | $R$ | Modus Ponens $(3, 6)$ |
| 8 | $P \wedge R$ | Conjunction $(1, 7)$ |
| 9 | $(\neg Q \vee \neg P) \rightarrow (P \wedge R)$ | Conditional $(4, 8)$ |
| 10 | $\neg(P \wedge Q) \rightarrow (P \wedge R)$ | deMorgan $(9)$ |
| 11 | $(P \wedge Q) \vee (P \wedge R)$ | Implication Equivalence $(10)$ |

□

3. Question 3 **Solution:** To get a contradiction we assume that there is an interpretation $I$ such that:

| | | |
|---|---|---|
| 1 | $I \not\vdash [(P \wedge Q) \rightarrow R] \rightarrow [P \rightarrow (Q \rightarrow R)]$ | Hypothesis |
| 2 | $I \vdash [(P \wedge Q) \rightarrow R] \wedge \neg[P \rightarrow (Q \rightarrow R)]$ | Implication equiv $(1)$ |
| 3 | $I \vdash [(P \wedge Q) \rightarrow R] \wedge P \wedge \neg(Q \rightarrow R)$ | Implication equiv $(2)$ |
| 4 | $I \vdash [(P \wedge Q) \rightarrow R] \wedge P \wedge Q \wedge \neg R$ | Implication equiv $(3)$ |
| 5 | $I \vdash [(P \wedge Q) \rightarrow R]$ | Simplification $(4)$ |
| 6 | $I \vdash P \wedge Q$ | Simplification $(4)$ |
| 7 | $I \vdash \neg R$ | Simplification $(4)$ |
| 8 | $I \vdash R$ | Modus Ponens $(5, 6)$ |
| 9 | $I \vdash \perp$ | Contradiction $(7, 8)$ |

□

4. Question 4 **Solution:** We will express the CNF formula as a 3CNF formula by transforming each CNF clause as a series of 3CNF clauses. Let the $j$th clause of the orininal formula be $(x_1, x_2, \ldots, x_m)$ we create the $m - 3$ variables $y_{j_1}, \ldots, y_{j_{m-3}}$ and replace the clause with the following:

$$(x_1, x_2, y_{j_1}) \wedge (\neg y_{j_1}, x_3, y_{j_2}) \wedge \ldots \wedge (\neg y_{j_{m-4}}, x_{m-2}, y_{j_{m-3}}) \wedge (\neg y_{j_{m-3}}, x_{m-1}, x_m)$$

if all the $x_i$ literals are false then this series of clauses becomes:

$$(y_{j_1}) \wedge (\neg y_{j_1}, y_{j_2}) \wedge \ldots \wedge (\neg y_{j_{m-4}}, y_{j_{m-3}}) \wedge (\neg y_{j_{m-3}})$$

This is equivalent to the implication chain:

$$(y_{j_1}) \wedge (y_{j_1} \rightarrow y_{j_2}) \wedge \ldots \wedge (y_{j_{m-4}} \rightarrow y_{j_{m-3}}) \wedge (\neg y_{j_{m-3}})$$

Which cannot be satisfied by any assignment to the $y_{j_k}$ literals.

However if any literal is true then the implication chain is broken and the clauses are satisfiable. $\square$

5. Question 5 **Solution:** For each pair of vertex, $v_i$, and color, $c_j$ create the variable $x_{(i,j)}$ which is true iff vertex $v_i$ is color $c_j$. We construct the clauses as follows.

   (a) For each vertex $v_i$ add clause $(x_{(i,1)}, \ldots, x_{(i,k)})$ so that vertex $v_i$ has at least one color.

   (b) For each vertex $v_i$ and pair of colors $c_j$ and $c_l$ add clause $(\neg x_{(i,j)}, \neg x_{(i,l)})$ so that vertex $v_i$ cannot be both colors $c_j$ and $c_l$.

   (c) For each edge $(v_i, v_h) \in E$ and color $c_j$ create clause $(\neg x_{(i,j)}, \neg x_{(h,j)})$ so that both endpoints are not color $c_j$.

   $\square$