Logic - Homework III

Piotr Wojciechowski
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

July 8, 2013

1 Problems

1. Question 1 **Solution:** The formula is valid. If we assume otherwise then we have that for some interpretation I of T_E,

$$I \not\models [f^{(3)}(a) = f^{(2)}(a) \wedge f^{(4)}(a) = a] \rightarrow [f(a) = a]$$

Which is equivalent to

$$I \models [f^{(3)}(a) = f^{(2)}(a) \wedge f^{(4)}(a) = a \wedge f(a) \neq a]$$

However, this leads to a contradiction as shown below.

1	$f^{(3)}(a) = f^{(2)}(a)$	Hypothesis
2	$f^{(4)}(a) = f^{(3)}(a)$	Axiom 4 (1)
3	$f^{(4)}(a) = f^{(3)}(a) \wedge f^{(3)}(a) = f^{(2)}(a)$	Conjunction $(1, 2)$
4	$f^{(4)}(a) = f^{(2)}(a)$	Axiom 3 (3)
5	$f^{(2)}(a) = f^{(4)}(a)$	Axiom 2 (4)
6	$f^{(4)}(a) = a$	Hypothesis
7	$f^{(2)}(a) = f^{(4)}(a) \wedge f^{(4)}(a) = a$	Conjunction $(5, 6)$
8	$f^{(2)}(a) = a$	Axiom 3 (7)
9	$f^{(3)}(a) = f(a)$	Axiom 4 (8)
10	$f(a) = f^{(3)}(a)$	Axiom 2 (9)
11	$f(a) = f^{(3)}(a) \wedge f^{(3)}(a) = f^{(2)}(a)$	Conjunction $(1, 10)$
12	$f(a) = f^{(2)}(a)$	Axiom 3 (11)
13	$f(a) = f^{(2)}(a) \wedge f^{(2)}(a) = a$	Conjunction $(8, 12)$
14	$f(a) = a$	Axiom 3 (13)
15	$f(a) \neq a$	Hypothesis
16	$\perp$	Contradiction $(14, 15)$

□

2. Question 2 **Solution:**

□

3. Question 3 **Solution:**

Base case.

When $n = 1$ we have that $(\cos\theta + i \cdot \sin\theta)^1 = \cos(1 \cdot \theta) + i \cdot \sin(1 \cdot \theta)$.

Inductive case.

Assume this is true for $n = k$. When $n = k + 1$ we have that

$$\begin{aligned}(\cos\theta + i \cdot \sin\theta)^{k+1} &= (\cos\theta + i \cdot \sin\theta) \cdot (\cos\theta + i \cdot \sin\theta)^k = (\cos\theta + i \cdot \sin\theta) \cdot (\cos(k \cdot \theta) + i \cdot \sin(k \cdot \theta)) \\ &= \cos\theta \cdot \cos(k \cdot \theta) - \sin\theta \sin(k \cdot \theta) + i \cdot (\sin\theta \cdot \cos(k \cdot \theta) + \cos\theta \cdot \sin(k \cdot \theta)) \\ &= \cos((k+1) \cdot \theta) + i \cdot \sin((k+1) \cdot \theta)\end{aligned}$$

Thus the formula holds for all n $\square$

4. Question 4 **Solution:**

(a) Base case: $atom(u)$

1	$flat(v)$	hypothesis
2	$atom(u)$	base case
3	$flat(concat(u, v))$	flat list $(1, 2)$

Inductive case: assume that for an arbitrary list x, $flat(x) \wedge flat(v) \to flat(concat(x, v))$.

We will now show that for all lists y, $flat(cons(y, x)) \wedge flat(v) \to flat(concat(cons(y, x), v))$

1	$flat(cons(y, x))$	hypothesis
2	$atom(y) \wedge flat(x)$	flat list (1)
3	$flat(x)$	simplification (2)
4	$atom(y)$	simplification (2)
5	$flat(v)$	hypothesis
6	$flat(concat(x, v))$	inductive hypothesis (5)
7	$flat(cons(y, concat(x, v))$	flat list (6)
8	$concat(cons(y, x), v) = cons(y, concat(x, v))$	concat. list
9	$flat(concat(cons(y, x), v))$	pred. cong. $(7, 8)$.

(b) Base case: $atom(u)$

1	$flat(u)$	hypothesis
2	$u = rvs(u)$	reverse atom
3	$flat(rvs(u))$	pred. cong. $(1, 2)$

Inductive case: assume that for an arbitrary list x, $flat(x) \to flat(rev(x))$.

We will now show that for all lists y, $flat(cons(y, x)) \to flat(rev(cons(y, x)))$

1	$flat(cons(y, x))$	hypothesis
2	$atom(y) \wedge flat(x)$	flat list (1)
3	$flat(x)$	simplification (2)
4	$atom(y)$	simplification (2)
5	$flat(y)$	flat atom (4)
6	$cons(y, x) = concat(y, x)$	concat. atom (4)
7	$rvs(cons(y, x)) = rvs(concat(y, x))$	func. cong. (6)
8	$rvs(concat(y, x)) = concat(rvs(x), rvs(y))$	reverse list
9	$flat(x) \to flat(rvs(x))$	inductive hyp.
10	$flat(rvs(x))$	m.p. $(3, 9)$
11	$rvs(y) = y$	reverse atom
12	$flat(rev(y))$	pred. cong. $(5, 11)$
13	$flat(concat(rvs(x), rvs(y)))$	part (a) $(10, 12)$
14	$flat(rvs(concat(y, x)))$	pred. cong. $(8, 13)$
15	$flat(rvs(cons(y, x)))$	pred. cong. $(7, 14)$

$\square$

5. Question 5 **Solution:**

We have that $p^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\cdot\sqrt{5}}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1 + p$

We also have that $q^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{6-2\cdot\sqrt{5}}{4} = 1 + \frac{1-\sqrt{5}}{2} = 1 + q$

Base cases.

When $n = 1$ then $F(1) = \frac{p-q}{p-q} = 1$ and $F(2) = \frac{p^2-q^2}{p-q} = p + q = 1$

Inductive case.

Assume that the formula holds for all $n < k$. So we have that

$$\begin{aligned} F(k) &= F(k-1) + F(k-2) = \frac{p^{k-1} - q^{k-1}}{p-q} + \frac{p^{k-2} - q^{k-2}}{p-q} = \frac{p^{k-2} \cdot (1+p) - q^{k-2} \cdot (1+q)}{p-q} \\ &= \frac{p^{k-2} \cdot p^2 - q^{k-2} \cdot q^2}{p-q} = \frac{p^k - q^k}{p-q} \end{aligned}$$

Thus the formula holds for all n. □