## Formal Logic - Revision

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# Outline





# Recap

### Notions

Propositions, connectives, truth-tables, tautologies, arguments and valid arguments, rules of derivation.

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## Definition

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## Definition

The argument

$$(P_1 \wedge P_2 \wedge \ldots P_n) \rightarrow Q$$

is said to be valid, if it is a tautology.

# Simplifications

## **Reducing Complexity**

- $(A \lor true) \Leftrightarrow true.$  $(A \lor false) \Leftrightarrow A.$
- $(A \land true) \Leftrightarrow A.$  $(A \land false) \Leftrightarrow false.$

- $(A \lor A') \Leftrightarrow \text{true.} \\ (A \land A') \Leftrightarrow \text{false.}$
- $(A \rightarrow \mathsf{false}) \Leftrightarrow A'. \\ (\mathsf{false} \rightarrow A) \Leftrightarrow \mathsf{true}.$

# Establishing validity of arguments

## Proof Techniques

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# Establishing validity of arguments

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Truth-tables

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The tautology algorithm

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- Q Rules of Derivation viz., Equivalence Rules and Inference Rules (works quite often but is less expensive than truth-tables).
- Intuitive argument (works all the time and is often times less expensive than using Rules of Derivation).
- The tautology algorithm (works in most cases).

# Applying the Methods

### Example

Consider the argument:

$$[(A \to B) \land (B \to C)] \to (A \to C) \tag{1}$$

In Argument 1,  $(A \rightarrow B)$  and  $(B \rightarrow C)$  are called the **hypotheses**, while  $(A \rightarrow C)$  is called the **conclusion**.

#### Recap

# The Truth-table method

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Let *G* denote the proposition  $[(A \rightarrow B) \land (B \rightarrow C)] \rightarrow (A \rightarrow C)$ .

### Recap

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$$A | B | C | A \rightarrow B | B \rightarrow C | (A \rightarrow B) \land (B \rightarrow C) | A \rightarrow C | G$$

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F	Т	Т	Т	Т	Т	Т	Т
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- (v) C (iii), (iv) Modus Ponens.

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- (iii) If *A* is **false**, the hypothesis of the argument is **false** and hence the argument itself is trivially **true**.
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- (v) Then the hypothesis of Argument 1 becomes (true  $\rightarrow B$ )  $\land$  ( $B \rightarrow C$ )  $\land$  true which can be simplified to  $B \land (B \rightarrow C)$ , since  $P \land$  true  $\Leftrightarrow P$  and (true  $\rightarrow B$ )  $\Leftrightarrow B$ .
- (vi) In other words, when A is **true**, the given argument reduces to:  $B \land (B \rightarrow C) \rightarrow C$ , which follows directly from Modus Ponens.
- (vii) Thus the argument holds, whether or not A is true.
- (viii) Since we have covered all the cases, it follows that the argument is a tautology, i.e., valid.

#### Recap

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- (iii) Since A is **true**,  $(A \rightarrow B)$  simplifies to B.
- (iv) Since C is false,  $(B \rightarrow C)$  simplifies to B'.
- (v) Thus, in order to make the antecedent **true**, we have to make  $B \wedge B'$  **true**, which is not possible.

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- (iv) Since C is false,  $(B \rightarrow C)$  simplifies to B'.
- (v) Thus, in order to make the antecedent true, we have to make B \land B' true, which is not possible.
- (vi) It follows that the given argument is valid.

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# One More Example

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## Proof.

Assume *A* is **true**. The argument becomes  $(B \rightarrow \text{true})$  which is **true**. Assume *A* is **false**; the entire argument is trivially **true**!

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Alternative Proof

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Using the Deduction Method, we can rewrite the above argument as:

$$(A \land B) \rightarrow A$$

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Using the Deduction Method, we can rewrite the above argument as:

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However, from  $A \land B$ , we can derive A, using rules of inference (Simplification)!



# Exercises

## Exercise

Prove the validity of the following arguments using as many techniques as you can:

- $\bigcirc [A \to (B \to C)] \to [B \to (A \to C)].$
- $(A \to C) \land (C \to B') \land A] \to A'.$
- $(A' \to B') \land (A \to C)] \to (B \to C).$
- If security is a problem, then regulation will be increased. If security is not a problem, then business on the Web will grow.

Therefore if regulation is not increased, then business on the Web will grow.

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Convert the following English statements to Predicate Logic.

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- (ii) Only John loves Mary.

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Convert the following English statements to Predicate Logic.

- (i) John loves only Mary.
- (ii) Only John loves Mary.

Use J(x) for "x is John", M(y) for "y is Mary" and L(x, y) for "x loves y."

# Solution

# Solution

## Problem 1

Subramani CS 220 - Discrete Mathematics

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John loves only Mary.
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O Formal rewriting:

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Or Formal rewriting: For any thing, if it is John, if it loves anything, then that thing is Mary.

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$$(\forall x) [J(x) \rightarrow (\forall y)(L(x, y) \rightarrow M(y))]$$

#### Recap

# Solution (contd.)

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## Problem 2

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# Solution (contd.)

## Problem 2

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Only John loves Mary.

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(i) Strip of quantifiers.

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### Technique

- (i) Strip of quantifiers.
- (ii) Use propositional logic.
- (iii) Re-insert quantifiers as needed.

#### Recap

# **Rules of Negation**

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## Example

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# $[(\exists x)A(x)]' \to (\forall x)[A(x)]'$

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## $[(\exists x)A(x)]' \to (\forall x)[A(x)]'$

#### Proof.

- (i)  $[(\exists x)A(x)]'$  hypothesis.
- (ii) A(x) temporary hypothesis.

Prove that

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#### Proof.

- (i)  $[(\exists x)A(x)]'$  hypothesis.
- (ii) A(x) temporary hypothesis.
- (iii)  $(\exists x)A(x)$  (ii), eg.

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- (ii) A(x) temporary hypothesis.
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Prove that

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- (v) [A(x)]' (i), (iv), Modus Tollens.

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- (v) [A(x)]' (i), (iv), Modus Tollens.
- (vi)  $(\forall x)[A(x)]'$  (v), ug.



# Exercises

#### Exercise

Prove that the following arguments are valid:

- $(\exists x)[P(x) \to Q(x)] \to [(\forall x)P(x) \to (\exists x)Q(x)].$
- $(\exists x) P(x) \land (\forall x) (P(x) \to Q(x))] \to (\exists x) Q(x).$
- $(\forall x) P(x) \land (\exists x) Q(x) ] \rightarrow [(\exists x) (P(x) \land Q(x))].$
- Output the second se

Someone has red hair and big feet.

Everybody who does not have green eyes does not have big feet.

Therefore, someone has green eyes and freckles.

(Hint: Use R(x), F(x), B(x), and G(x) for people with red hair, freckles, big feet and green eyes respectively.)