Functions - Permutation Functions and Counting

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Outline

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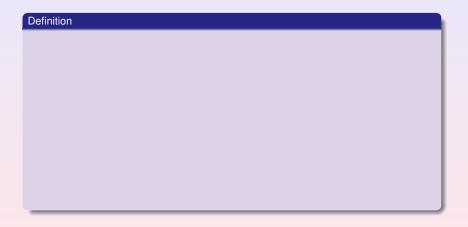
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It can be represented as: (1, 2, 3).





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 A transposition swaps two elements and leaves all other elements fixed.
- Two cycles are disjoint if they do not move the same element.



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② The composition of disjoint cycles is commutative, i.e., if f and g are two disjoint cycles, then $f \circ g = g \circ f$.

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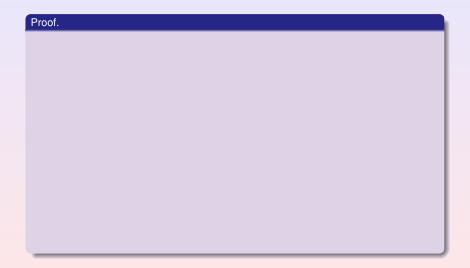
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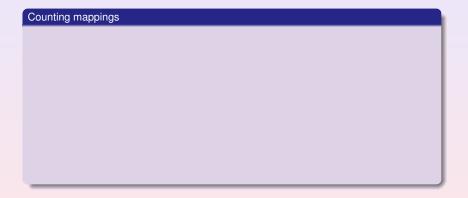
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Thus, the cycles of a and b must be either identical or disjoint.





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We count the number of non-onto functions and subtract this quantity from the total number of functions!





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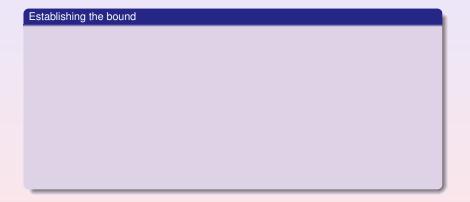
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Example

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(i) How many onto functions exist from S to T? Solution: 6.

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- (ii) How many derangements exist on S?

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Let $S = \{A, B, C\}$ and $T = \{a, b\}$.

- (i) How many onto functions exist from S to T? Solution: 6.
- (ii) How many derangements exist on S? Solution: 2.