Functions - Fundamentals and Order

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

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For $A \subseteq S$, $f(A) = \{f(a) : a \in A\}$.

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For instance, $f(x, y) = x^2 + y^2$ *is a function from* $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$.

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(i) *Pick an arbitrary element $t \in T$.*
(ii) *Show that there exists some $s \in S$, such that $f(s) = t$.*

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(i) *Show that for arbitrarily chosen $a, b \in S$, $a \neq b \rightarrow f(a) \neq f(b)$.*

(ii) *Alternatively, show that for arbitrarily chosen $f(a), f(b) \in T$, $f(a) = f(b) \rightarrow a = b$.*

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The domains and ranges have to be compatible.

Example

Let $f : \Re \rightarrow \Re$ be defined by $f(x) = x^2$ and $g : \Re \rightarrow \Re$ be defined by $g(x) = \lfloor x \rfloor$. Compute $(g \circ f)(2.3)$ and $(f \circ g)(2.3)$.

Solution: $(g \circ f)(2.3) = g(f(2.3)) = g(5.29) = \lfloor 5.29 \rfloor = 5.$
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Function composition

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Let $f : S \to T$ and $g : T \to U$ denote two functions.

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Function composition (contd.)

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□

Function composition (contd.)

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The composition of two surjective function is surjective.

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Proof.

Function composition (contd.)

Lemma

The composition of two surjective function is surjective.

Proof.

Let $f : S \to T$ and $g : T \to U$ denote two onto functions.

Function composition (contd.)

Lemma

The composition of two surjective function is surjective.

Proof.

Let $f : S \to T$ and $g : T \to U$ denote two onto functions.

Consider the composition $(g \circ f) : S \to U$.

Function composition (contd.)

Lemma

The composition of two surjective function is surjective.

Proof.

Let $f : S \to T$ and $g : T \to U$ denote two onto functions.

Consider the composition $(g \circ f) : S \to U$.

Pick an arbitrary element $u \in U$.

Function composition (contd.)

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Since g is surjective,

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Consider the composition $(g \circ f) : S \to U$.

Pick an arbitrary element $u \in U$.

Since g is surjective, there exists a $t \in T$, such that $g(t) = u$.

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Let $f : S \to T$ and $g : T \to U$ denote two onto functions.

Consider the composition $(g \circ f) : S \to U$.

Pick an arbitrary element $u \in U$.

Since g is surjective, there exists a $t \in T$, such that $g(t) = u$.

Since f is surjective, there exists an $s \in S$, such that $f(s) = t$.

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Let $f : S \to T$ and $g : T \to U$ denote two onto functions.

Consider the composition $(g \circ f) : S \to U$.

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Since g is surjective, there exists a $t \in T$, such that $g(t) = u$.

Since f is surjective, there exists an $s \in S$, such that $f(s) = t$.

In other words, $(g \circ f)(s) = u$.

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Since u was arbitrarily chosen, it follows that $(g \circ f)$ is surjective. □

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The composition of two surjective function is surjective.

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Theorem

Function composition (contd.)

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Theorem

The composition of two bijective functions is a bijective function.

Outline

Inverse functions

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Definition

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The function $i_S : S \to S$ which maps each element of S to itself,

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Example

Find the inverse of $f : \Re \rightarrow \Re$ defined as $f(x) = 3 \cdot x + 4$.

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Exercise

Argue that if a function $f : S \rightarrow T$ has an inverse function, then this inverse is unique.

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Show that if S is a finite set, S is not equivalent to $\mathcal{P}(S)$.

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Exercise

Are $\mathcal{Z}$ and $\mathcal{N}$ equivalent?

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Theorem (Cantor)

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Are $\mathcal{Z}$ and $\mathcal{N}$ equivalent?

Theorem (Cantor)

For any set S, S and $\mathcal{P}(S)$ are not equivalent.

Proof of Cantor's theorem

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Proof.

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Assume that S and $\mathcal{P}(S)$ are equivalent.

Proof of Cantor's theorem

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Assume that S and $\mathcal{P}(S)$ are equivalent.

Let f be a bijection between S and $\mathcal{P}(S)$.

Proof of Cantor's theorem

Proof.

Assume that S and $\mathcal{P}(S)$ are equivalent.

Let f be a bijection between S and $\mathcal{P}(S)$.

Pick $s \in S$.

Proof of Cantor's theorem

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In either case, there is a contradiction, which proves that S and $\mathcal{P}(S)$ are not equivalent. □

Consequences of Cantor's theorem

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How is ∞ related to $\infty + \infty$, $\infty \times \infty$, 2^{∞}?

Order of magnitude of functions

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(i) Which function grows faster: $100 \cdot x^2$ or $\frac{1}{10^6} \cdot x^3$?

(ii) Which function grows faster: $x^2 - 10$ or $x + 10$?

Order of Magnitude (contd.)

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Then $f = O(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) \leq c \cdot g(x)$.

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Then $f = o(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) < c \cdot g(x)$.

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Then $f = \Theta(g)$, if $f = O(g)$ and $g = O(f)$.

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(i) Let $f(x) = 2 \cdot x^2 - 2$ and $g(x) = \frac{1}{100} \cdot x^2 - 100$.

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Test to determine order

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The limit test

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The above rule is called L'Hospital's rule.

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(i) Show that $x = o(x^2)$.

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(i) Show that $x = o(x^2)$.

(ii) Show that $x = o(x \cdot \log x)$.

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(i) Show that $x = o(x^2)$.

(ii) Show that $x = o(x \cdot \log x)$.

(iii) Show that $\log x = o(x)$.