Permutations, Combinations and The Binomial Theorem

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Outline

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Motivating Examples

Motivating Examples

Example

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Motivating Examples

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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Motivating Examples

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Motivating Examples

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Example

In how many ways can 6 people be seated in a row?

Motivating Examples

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Motivating Examples

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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In how many ways can 6 people be seated in a row?

Example

In how many ways can 6 people be seated around a circular table with 6 chairs?

Motivating Examples

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Example

In how many ways can 6 people be seated in a row?

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In how many ways can 6 people be seated around a circular table with 6 chairs? (Only relative positions can be distinguished.)

Permutations

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A permutation is an ordered arrangement of objects.

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$$n! =$$

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Definition

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ \end{cases}$$

Permutations

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A permutation is an ordered arrangement of objects.

The number of distinct permutations of r distinct objects chosen from n distinct objects is denoted by $P(n, r)$.

Definition

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

Computing the number of permutations

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Computing $P(n, r)$

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Computing $P(n, r)$

Using the multiplication principle,

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$$P(n, r) \quad =$$

Computing the number of permutations

Computing $P(n, r)$

Using the multiplication principle,

$$P(n, r) \quad = \quad n$$

Computing the number of permutations

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Using the multiplication principle,

$$P(n,r) = n \cdot (n-1)$$

Computing the number of permutations

Computing $P(n, r)$

Using the multiplication principle,

$$P(n,r) \quad = \quad n \cdot (n-1) \cdot \ldots (n-r+1)$$

Computing the number of permutations

Computing $P(n, r)$

Using the multiplication principle,

$$
\begin{aligned}
P(n, r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\
&=
\end{aligned}
$$

Computing the number of permutations

Computing $P(n, r)$

Using the multiplication principle,

$$
\begin{aligned}
P(n,r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\
&= n \cdot (n-1) \cdot \ldots (n-r+1) \cdot \frac{(n-r) \cdot (n-r-1) \cdot \ldots 1}{(n-r) \cdot (n-r-1) \cdot \ldots 1}
\end{aligned}
$$

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Computing $P(n, r)$

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Computing the number of permutations

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Using the multiplication principle,

$$\begin{aligned} P(n,r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\ &= n \cdot (n-1) \cdot \ldots (n-r+1) \cdot \frac{(n-r) \cdot (n-r-1) \cdot \ldots 1}{(n-r) \cdot (n-r-1) \cdot \ldots 1} \\ &= \frac{n!}{(n-r)!},\ 0 \leq r \leq n \end{aligned}$$

Permutations (contd.)

Permutations (contd.)

Example

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution:

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210,

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1,

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution: 210, 1, n,

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Solution:

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Solution: $P(8,3)$.

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Solution: $P(8,3)$.

Example

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8, 3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Solution: $P(8,3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Solution:

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word “compiler”?

Solution: $P(8, 3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Solution: $P(20, 2)$.

One more example

One more example

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One more example

Example

A library has 4 books on programming, 7 on algorithms and 3 on complexity.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

Provided that the books of a subject are required to be together?

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If there is no restriction, the number of arrangements is

One more example

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Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

Provided that the books of a subject are required to be together?

Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

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If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

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Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

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Corresponding to each such arrangement,

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the programming books can be permuted in

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If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Corresponding to each such arrangement,

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One more example

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the programming books can be permuted in $P(4, 4) = 4!$ ways,

the algorithms books can be permuted in $P(7, 7) = 7!$ ways,

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Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

Corresponding to each such arrangement,

the programming books can be permuted in $P(4, 4) = 4!$ ways,

the algorithms books can be permuted in $P(7, 7) = 7!$ ways,

and the complexity books can be permuted in $P(3, 3) = 3!$ ways.

Using the multiplication principle, the total number of arrangements is $3! \cdot 4! \cdot 7! \cdot 3!$.

More Examples

More Examples

Example

More Examples

Example

Solve the motivating examples.

Motivating Examples

Motivating Examples

Example

Motivating Examples

Example

How many 5-card hands are possible with a 52 card deck?

Motivating Examples

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How many 5-card hands are possible with a 52 card deck?

Example

Ten athletes compete in an Olympic event.

Motivating Examples

Example

How many 5-card hands are possible with a 52 card deck?

Example

Ten athletes compete in an Olympic event. Three will be declared winners.

Motivating Examples

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Ten athletes compete in an Olympic event. Three will be declared winners.

In how many ways can the winners be selected?

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A committee of 3 is to be formed from 5 men and 2 women.

Motivating Examples

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A committee of 3 is to be formed from 5 men and 2 women.

In how many ways can the committee be formed, if

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In how many ways can the winners be selected?

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A committee of 3 is to be formed from 5 men and 2 women.

In how many ways can the committee be formed, if

1. The committee must include at least one woman.

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In how many ways can the winners be selected?

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A committee of 3 is to be formed from 5 men and 2 women.

In how many ways can the committee be formed, if

1. The committee must include at least one woman.
2. There cannot be more than two men on the committee.

Combinations

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The objects in this combination can be permuted in $r!$ different ways to get $r!$ distinct permutations.

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It follows that $C(n, r) \cdot r! = P(n, r)$,

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It follows that $C(n, r) \cdot r! = P(n, r)$, i.e., $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$, $0 \leq r \leq n$.

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Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

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Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution:

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Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution: 35,

Combinations

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It follows that $C(n, r) \cdot r! = P(n, r)$, i.e., $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$, $0 \leq r \leq n$.

Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution: 35, 1,

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Solution: 35, 1, n,

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Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution: 35, 1, n, 1.

Combinations (examples)

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Example

A committee of 8 students is to be selected from 19 freshmen and 34 sophomores.

Combinations (examples)

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A committee of 8 students is to be selected from 19 freshmen and 34 sophomores.

In how many ways, can this committee be formed, if

Combinations (examples)

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A committee of 8 students is to be selected from 19 freshmen and 34 sophomores.

In how many ways, can this committee be formed, if

1. it must contain 3 freshmen and 5 sophomores.

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Combinations (examples)

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A committee of 8 students is to be selected from 19 freshmen and 34 sophomores.

In how many ways, can this committee be formed, if

1. it must contain 3 freshmen and 5 sophomores. **Solution:** $C(19, 3) \cdot C(34, 5)$.

Combinations (examples)

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1. it must contain 3 freshmen and 5 sophomores. **Solution:** $C(19, 3) \cdot C(34, 5)$.
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More examples

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Solve the motivating examples.

Handling Duplicates

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$$\frac{11!}{4! \cdot 4! \cdot 2!}.$$

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In how many ways can six children choose one lollipop from a selection of red, green and yellow lollipops,

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So we have to place select two positions for the markers from a total of 6 (children) $+2$ placeholders.

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Exercises

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1. In how many ways, can you seat 11 men and 8 women in a row, so that no two women sit together?

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1. In how many ways, can you seat 11 men and 8 women in a row, so that no two women sit together?
2. A committee of three has to be chosen from five Democrats, three Republicans and four independents.

   In how many ways can the committee be chosen, if it cannot include both Democrats and Republicans?

Motivation

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(i) $(a+b)^1 =$

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(iv) $(a+b)^4 =$???

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(i) $(a+b)^1 = a+b$.
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(iii) $(a+b)^3 = a^3 + 3\cdot a^2\cdot b + 3\cdot a\cdot b^2 + b^3$.
(iv) $(a+b)^4 =???$

We want a general formula that permits us to write down the terms of $(a+b)^n$ without actual multiplication.

Pascal's Triangle

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The coefficient table

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Row 0: $C(0, 0)$

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Row 0:			$C(0,0)$			
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Pascal's Triangle

The coefficient table

Consider the following table:

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Pascal’s triangle (contd.)

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The Value Table

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Pascal's triangle (contd.)

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Pascal's formula

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&= \frac{(n-1)!}{k! \cdot (n-k)!} [k + (n-k)]
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&= \frac{n!}{k! \cdot (n-k)!} \\
&= C(n, k).
\end{aligned}$$

□

Alternative Proof

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(viii) Using the addition principle, $C(n, k) = T_1 + T_2 = C(n-1, k-1) + C(n-1, k)$.

Note on Proof Techniques

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The above proof is called a combinatorial proof and is always preferred on account of its elegance.

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Recall the combinatorial proof for proving that $C(n, r) = C(n, n - r)$.

The Theorem

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$$(a+b)^n =$$

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$$(a+b)^n = \sum_{i=0}^{n} C(n,i) \cdot a^{n-i} \cdot b^i, \ \forall n \geq 0.$$

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BASIS: At $n = 0$, the LHS is $(a+b)^0 = 1$ and the RHS is $\sum_{i=0}^{0} C(0,i) \cdot a^{0-i} \cdot b^i$.

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Since the only value for i is also 0, the RHS is $C(0,0)\ a^0 \cdot b^0 = 1$.

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BASIS: At $n = 0$, the LHS is $(a+b)^0 = 1$ and the RHS is $\sum_{i=0}^{0} C(0,i) \cdot a^{0-i} \cdot b^i$.

Since the only value for i is also 0, the RHS is $C(0,0)\ a^0 \cdot b^0 = 1$.

Thus, LHS = RHS and the basis is proven. □

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&= a \cdot (\sum_{i=0}^{k} C(k,i) \cdot a^{k-i} \cdot b^i) + b \cdot (\sum_{i=0}^{k} C(k,i) \cdot a^{k-i} \cdot b^i)
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□

Proof (contd.)

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We focus on the quantity

$$(\mathbf{F})\ \sum_{i=1}^{k} C(k,i)\cdot a^{k+1-i}\cdot b^{i} + (\mathbf{S})\ \sum_{i=0}^{k-1} C(k,i)\cdot a^{k-i}\cdot b^{i+1} \quad (1)$$

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Observe that the first k terms in **F** are $a^{k}\cdot b^{1},\ a^{k-1}\cdot b^{2},\ldots,\ a^{1}\cdot b^{k}$, while the first k terms in **S** are also $a^{k}\cdot b^{1},\ a^{k-1}\cdot b^{2},\ldots,\ a^{1}\cdot b^{k}$.

□

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Observe that the coefficient of $a^{k+1-p} \cdot b^p$ is $C(k, p)$ in **F** and $C(k, p-1)$ in **S**. (This requires some thought!)

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= \quad & C(k+1, 0) \cdot a^{k+1} \cdot b^0 + \sum_{i=1}^{k} C(k+1, i) \cdot a^{k+1-i} \cdot b^i + C(k+1, k+1) \cdot a^0 \cdot b^{k+1} \\
& \text{since } \mathrm{C(k, 0) = C(k, k) = C(k+1, 0) = C(k+1, k+1) = 1}
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$$

□

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We have thus shown that $P(k) \rightarrow P(k+1)$ and hence by applying the first principle of mathematical induction, we can conclude that $P(n)$ is true, for all $n \geq 0$. □

Application

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Expand $(x - 3)^4$.

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Solution:

$$\begin{aligned}(x-3)^4 &= C(4,0)\cdot x^4\cdot(-3)^0 + C(4,1)\cdot x^3\cdot(-3)^1 + C(4,2)\cdot x^2\cdot(-3)^2 \\ &\quad + C(4,3)\cdot x^1\cdot(-3)^3 + C(4,4)\cdot x^0\cdot(-3)^4\end{aligned}$$

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One more example

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Show that

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Show that

$$\sum_{i=0}^{n} C(n,i) = 2^n$$

Proof

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Proof using the binomial theorem

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As per the binomial theorem,

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Completing the induction

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Inductive proof (contd.)

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&= 1 + \sum_{i=1}^{k} [C(k, i) + C(k, i-1)] + 1, \ \text{Pascal}'\text{s formula}
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&= (1 + \sum_{i=1}^{k} C(k, i)) + (\sum_{i=1}^{k} C(k, i-1) + 1)) \\
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Inductive proof (contd.)

The last steps

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$$= \quad \sum_{i=0}^{k} C(k, i) + \sum_{j=0}^{k} C(k, j)$$

Inductive proof (contd.)

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$$
\begin{aligned}
&= \sum_{i=0}^{k} C(k,i) + \sum_{j=0}^{k} C(k,j) \\
&= 2 \cdot \sum_{i=0}^{k} C(k,i)
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Inductive proof (contd.)

The last steps

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\begin{aligned}
&= \sum_{i=0}^{k} C(k,i) + \sum_{j=0}^{k} C(k,j) \\
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&= 2 \cdot 2^k, \text{ using the inductive hypothesis}
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Thus LHS=RHS and the inductive step is proven.

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Thus LHS=RHS and the inductive step is proven.

Applying the first principle of mathematical induction, we conclude that the conjecture is true.

Example

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Example

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Prove the following identity:

Example

Example

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Example

Example

Prove the following identity:

1. $$\sum_{i=1}^{n} i \cdot C(n, i) = n \cdot 2^{n-1}$$

Solution

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1. Expand the expression on the LHS for the identity.

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   $$n \cdot (1+x)^{n-1} = \sum_{i=0}^{n} C(n,i) \cdot i \cdot x^{i-1}.$$
4. Put $x = 1$ to get the identity.