# Predicate Logic (First Order Logic)

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21, 26 January 2016

Quantifiers and Predicates

- Quantifiers and Predicates
- 2 Translation

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- Validity

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- Rules of Inference

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### **Basics**

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- (iii) The universal quantifier  $(\forall x)P(x)$  indicates that property P holds for all x in the domain.
- (iv) The existential quantifier  $(\exists x)P(x)$  indicates that property P holds for some x in the domain.



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For instance,  $(\exists x)(\forall y)Q(x,y)$  does not mean the same thing as  $(\forall y)(\exists x)Q(x,y)$ .

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- (iii) An assignment of a particular object in the domain to each constant symbol in the expression.

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What happens when the domain is the set of all integers?



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The scope of a quantifier is the portion of the predicate formula to which it applies.

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First order theories

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The axioms of the theory will be added to the axioms of Predicate Logic and we will reason using the tools of Predicate Logic.

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(ii) Propositional rules are not sufficient. For instance, you cannot use propositional rules to conclude validity in the Socrates example.

## Rule Enumeration

## Rules of Predicate Logic

Universal Instantiation.

- Universal Instantiation.
- 2 Existential Instantiation.

- Universal Instantiation.
- 2 Existential Instantiation.
- Universal Generalization.

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- Universal Generalization.
- Existential Generalization.
- Temporary Hypothesis.

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Let us prove that the following argument is valid, using ui.

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- (iv) M(s) (ii), (iii), Modus Ponens.

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Prove that the following argument is valid.

$$[(\forall x)[P(x) \to R(x)] \land (R(y))'] \to (P(y))'$$

#### **Details**

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Show that  $[(\forall x)[P(x) \to Q(x)] \land (\exists y)P(y)] \to Q(a)$  is valid.

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Consider the following proof sequence.

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#### Proof

- (i)  $(\exists y)P(y)$  hypothesis.
- (ii) P(a) (i), ei.

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- (i)  $(\exists y)P(y)$  hypothesis.
- (ii) P(a) (i), ei.
- (iii)  $(\forall x)[P(x) \rightarrow Q(x)]$  hypothesis.

## Example

Show that  $[(\forall x)[P(x) \to Q(x)] \land (\exists y)P(y)] \to Q(a)$  is valid.

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- (i)  $(\exists y)P(y)$  hypothesis.
- (ii) P(a) (i), ei.
- (iii)  $(\forall x)[P(x) \rightarrow Q(x)]$  hypothesis.
- (iv)  $P(a) \rightarrow Q(a)$  (iii), ui.

## Example

Show that  $[(\forall x)[P(x) \to Q(x)] \land (\exists y)P(y)] \to Q(a)$  is valid.

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- (i)  $(\exists y)P(y)$  hypothesis.
- (ii) P(a) (i), ei.
- (iii)  $(\forall x)[P(x) \rightarrow Q(x)]$  hypothesis.
- (iv)  $P(a) \rightarrow Q(a)$  (iii), ui.
- (v) Q(a) (ii), (iv), Modus Ponens.

### Example

Show that  $[(\forall x)[P(x) \to Q(x)] \land (\exists y)P(y)] \to Q(a)$  is valid.

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- (i)  $(\exists y)P(y)$  hypothesis.
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- (iii)  $(\forall x)[P(x) \rightarrow Q(x)]$  hypothesis.
- (iv)  $P(a) \rightarrow Q(a)$  (iii), ui.
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#### Note

## Example

Show that  $[(\forall x)[P(x) \to Q(x)] \land (\exists y)P(y)] \to Q(a)$  is valid.

#### Proof

Consider the following proof sequence.

- (i)  $(\exists y)P(y)$  hypothesis.
- (ii) P(a) (i), ei.
- (iii)  $(\forall x)[P(x) \rightarrow Q(x)]$  hypothesis.
- (iv)  $P(a) \rightarrow Q(a)$  (iii), ui.
- (v) Q(a) (ii), (iv), Modus Ponens.

#### Note

Steps (i)-(ii) and (iii)-(iv) cannot be interchanged.

#### Details

(i) From P(x), you can conclude  $(\forall x)P(x)$ .

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- (i) From P(x), you can conclude  $(\forall x)P(x)$ .
- (ii) Rule is abbreviated as u.g. (or ug).
- (iii) P(x) has not been deduced from a hypothesis in which x is a free variable. Also, P(x) has not been deduced using e.i. from any wff, in which x is a free variable.

Example

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Show that the following argument is valid.

$$[(\forall x)[P(x) \to Q(x)] \land (\forall x)P(x)] \to (\forall x)Q(x).$$

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- (ii)  $P(x) \rightarrow Q(x)$  (i), ui.

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- (ii)  $P(x) \rightarrow Q(x)$  (i), ui.
- (iii)  $(\forall x)P(x)$  hypothesis.
- (iv) P(x) (iii), ui.
- (v) Q(x) (ii), (iv) Modus Ponens.
- (vi)  $(\forall x)Q(x)$  (v), ug. (Neither restriction has been violated.)

Free Variable rule violation

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Subramani

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- (iii) Q(x, a) (ii), ei.
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#### Details

- (i) From P(x) or P(a), you can conclude  $(\exists x)P(x)$ .
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The argument  $Q(a, y) \rightarrow (\exists y)Q(y, y)$  is simply not valid. (Why?)

Example

### Example

Show that the following argument is valid.

$$(\forall x)P(x) \rightarrow (\exists x)P(x)$$

### Example

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$$(\forall x)P(x)\to (\exists x)P(x)$$

### Proof

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Consider the following proof sequence:

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Show that the following argument is valid.

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### Proof

- (i)  $(\forall x)P(x)$  hypothesis.
- (ii) P(x) (i), ui.

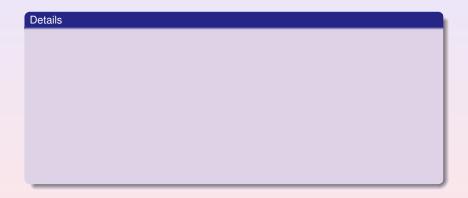
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- (i)  $(\forall x)P(x)$  hypothesis.
- (ii) P(x) (i), ui.
- (iii)  $(\exists x)P(x)$  (ii), eg.





### Details

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- In the midst of a proof, you can assume a suitable wff formula T.
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- Observe that this method is a generalization of the Deduction Method to Predicate Logic.

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- **3** If you can deduce the wff W, as a consequence of this assumption, then this means that the wff  $T \to W$  has been deduced from the initial hypotheses.
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- T is called the "temporary hypothesis".
- **3** If you can deduce the wff W, as a consequence of this assumption, then this means that the wff  $T \to W$  has been deduced from the initial hypotheses.
- Observe that this method is a generalization of the Deduction Method to Predicate Logic.
- ullet In the Deduction Method, if the desired conclusion is of the form  $P \to Q$ , then we simply assume P and deduce Q.
- **3** However, this cannot be done, if the conclusion is of the form:  $(\forall x)[P(x) \to Q(x)]$  or  $(\exists x)[P(x) \to Q(x)]$ .

# Temporary Hypothesis (example)

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Example

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#### Proof

(i) 
$$P(x) \rightarrow (\forall y)Q(x,y)$$
 hypothesis.

### Example

Show that the following formula is valid:

$$[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)].$$

#### Proof

- (i)  $P(x) \rightarrow (\forall y)Q(x,y)$  hypothesis.
- (ii) P(x) temporary hypothesis.

#### Example

Show that the following formula is valid:

$$[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)].$$

#### Proof

- (i)  $P(x) \rightarrow (\forall y)Q(x,y)$  hypothesis.
- (ii) P(x) temporary hypothesis.
- (iii)  $(\forall y)Q(x,y)$  (i), (ii), Modus Ponens.

#### Example

Show that the following formula is valid:

$$[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)].$$

#### Proof

- (i)  $P(x) \rightarrow (\forall y)Q(x,y)$  hypothesis.
- (ii) P(x) temporary hypothesis.
- (iii)  $(\forall y)Q(x,y)$  (i), (ii), Modus Ponens.
- (iv) Q(x, y) (iii), ui.

### Example

Show that the following formula is valid:

$$[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)].$$

#### Proof

- (i)  $P(x) \rightarrow (\forall y)Q(x,y)$  hypothesis.
- (ii) P(x) temporary hypothesis.
- (iii)  $(\forall y)Q(x,y)$  (i), (ii), Modus Ponens.
- (iv) Q(x, y) (iii), ui.
- (v)  $P(x) \rightarrow Q(x,y)$  temporary hypothesis discharged.

### Example

Show that the following formula is valid:

$$[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)].$$

#### Proof

- (i)  $P(x) \rightarrow (\forall y)Q(x,y)$  hypothesis.
- (ii) P(x) temporary hypothesis.
- (iii)  $(\forall y)Q(x,y)$  (i), (ii), Modus Ponens.
- (iv) Q(x, y) (iii), ui.
- (v)  $P(x) \rightarrow Q(x, y)$  temporary hypothesis discharged.
- (vi)  $(\forall y)[P(x) \rightarrow Q(x,y)]$  (v), ug.

Main points of predicate rules

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- (ii) Work with separate wffs.
- (iii) Insert quantifiers as necessary.

Example

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Show that the following argument is valid:

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#### Proof

(i) 
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 hypothesis.

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Show that the following argument is valid:

$$(\forall x)[P(x) \land Q(x)] \rightarrow (\forall x)P(x) \land (\forall x)Q(x).$$

#### Proof

- (i)  $(\forall x)[P(x) \land Q(x)]$  hypothesis.
- (ii)  $P(x) \wedge Q(x)$  (i), ui.

## Example

Show that the following argument is valid:

$$(\forall x)[P(x) \land Q(x)] \rightarrow (\forall x)P(x) \land (\forall x)Q(x).$$

#### Proof

- (i)  $(\forall x)[P(x) \land Q(x)]$  hypothesis.
- (ii)  $P(x) \wedge Q(x)$  (i), ui.
- (iii) P(x) (ii), Simplification.

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- (ii)  $P(x) \wedge Q(x)$  (i), ui.
- (iii) P(x) (ii), Simplification.
- (iv)  $(\forall x)P(x)$  (iii), ug.

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- (vii)  $(\forall x)P(x) \land (\forall x)Q(x)$  (iv), (vi), Conjunction.

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- (vii)  $(\forall x)P(x) \land (\forall x)Q(x)$  (iv), (vi), Conjunction.

Note that neither restriction has been violated in the ug steps.

Example

## Example

Show that the following expression is valid:

$$[(\forall y)[P(x) \to Q(x,y)]] \to [P(x) \to (\forall y)Q(x,y)].$$

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#### Proof

Using the Deduction Method, rewrite the argument as:

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Show that the following expression is valid:

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Example

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Every microcomputer has a serial interface port.

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  - (ix)  $M(a) \wedge S(a) \wedge P(a)$  (viii), commutativity.

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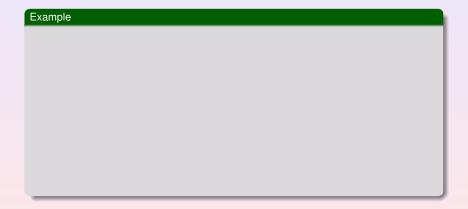
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The proofs of the negation rules are tedious and I do not expect you to know them. Feel free to use the above two rules as and when you need them, without proof.



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In English, the negation is: There is somebody who does not love everybody, all the time.  $\hfill\Box$ 

## One more example

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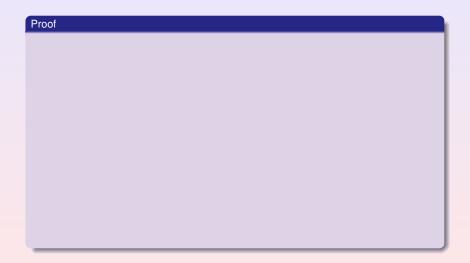
Example

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Establish the validity of the following statement:

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We first rewrite the argument as:  $(\forall x)[P(x) \lor Q(x)] \to [[(\exists x)P(x)]' \to (\forall x)Q(x)].$ 

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