Propositional Logic

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Outline

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Building blocks

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Basics

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1. Constants -

Building blocks

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1. Constants - **true** and **false**.

Building blocks

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1. Constants - **true** and **false**.
2. Atoms - Propositions.

Building blocks

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2. Atoms - Propositions.
3. Connectives.

Building blocks

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1. Constants - **true** and **false**.
2. Atoms - Propositions.
3. Connectives.
4. Semantics.

Propositions

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Definition

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Statement (or Atomic Proposition) -

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Statement (or Atomic Proposition) - A sentence that is either **true** or **false**.

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(i) The board is black.

Propositions

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(i) The board is black.
(ii) Are you John?

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Example

(i) The board is black.
(ii) Are you John?
(iii) The moon is made of green cheese.

Propositions

Definition

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Example

(i) The board is black.
(ii) Are you John?
(iii) The moon is made of green cheese.
(iv) I am a liar.

Propositions

Definition

Statement (or Atomic Proposition) - A sentence that is either **true** or **false**.

Example

(i) The board is black.
(ii) Are you John?
(iii) The moon is made of green cheese.
(iv) I am a liar. (Paradox).

Boolean Connectives

Motivation

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To make compound statements from simple ones.

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Basic Connectives are:

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1. **conjunction** (AND) ($\wedge$),

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To make compound statements from simple ones.

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To make compound statements from simple ones.

Basic Connectives are:

1. **conjunction** (AND) ($\wedge$),
2. **disjunction** (OR) ($\vee$),
3. **Negation** (NOT) ($'$),
4. **implication** (IF) ($\rightarrow$), and
5. **Equivalence** (IF AND ONLY IF) ($\leftrightarrow$).

Conjunction

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Semantics of Conjunction

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Semantics of Conjunction

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Conjunction

Semantics of Conjunction

A	B	$A \wedge B$
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Note

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Note

The above table is called a truth-table.

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The above table is called a truth-table.

Disjunction

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Semantics of Disjunction

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A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
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Negation

Negation

Semantics of Negation

Negation

Semantics of Negation

Negation

Semantics of Negation

A	A'
T	**F**
F	**T**

Implication

Implication

Semantics of Implication

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Semantics of Implication

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Implication

Semantics of Implication

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
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F	**F**	**T**

Note

Implication

Semantics of Implication

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Note

Note that $A \rightarrow B$ *is the same as* $A' \vee B$.

Implication

Semantics of Implication

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Note

Note that $A \rightarrow B$ is the same as $A' \vee B$.

A is called the antecedent and B is the consequent of the implication.

Equivalence

Equivalence

Semantics of Equivalence

Equivalence

Semantics of Equivalence

A	B	$A \leftrightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Equivalence

Semantics of Equivalence

A	B	$A \leftrightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Note

Equivalence

Semantics of Equivalence

A	B	$A \leftrightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Note

Note that $A \leftrightarrow B$ *is the same as* $(A \rightarrow B) \wedge (B \rightarrow A)$.

Well-formed Formulas

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(i) A simple proposition is a well-formed formula (wff).

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Example

$A \vee) B$ is not a wff.

Resolving Ambiguity

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Precedence

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Ambiguity is resolved using the following order of precedence.

(i) parentheses.
(ii) negation.
(iii) conjunction, disjunction.
(iv) implication.
(v) equivalence.

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Precedence

Ambiguity is resolved using the following order of precedence.

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Use brackets and forget about precedence!

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If A and B are two wffs, and $A \leftrightarrow B$ is a tautology, then A and B are said to be **equivalent wffs** (denoted by $A \Leftrightarrow B$) and can be substituted for one another.

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How do you check if a wff is a tautology?

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How do you check if a wff is a tautology? Truth-tables!

Example

$(A \rightarrow B) \Leftrightarrow (B' \rightarrow A')$.

Common Tautological Equivalences

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De Morgan's Laws

$$(A \vee B)' \Leftrightarrow (A' \wedge B')$$

Common Tautological Equivalences

De Morgan's Laws

$(A \vee B)' \Leftrightarrow (A' \wedge B')$

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Common Tautological Equivalences

De Morgan's Laws

$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\quad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$

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De Morgan's Laws

$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\quad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

Commutativity

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$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\quad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

Commutativity

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Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$

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Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$ $\qquad$ $(A \wedge B) \Leftrightarrow (B \wedge A)$

Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $\qquad$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$

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$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\quad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$ $\quad$ $(A \wedge B) \Leftrightarrow (B \wedge A)$

Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $\quad$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$

Distributivity

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$

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$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\qquad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

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Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $\qquad$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$

Distributivity

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$ $\qquad$ $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

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Exercise

Common Tautological Equivalences

De Morgan's Laws

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Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$ $\qquad$ $(A \wedge B) \Leftrightarrow (B \wedge A)$

Associativity

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Distributivity

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$ $\qquad$ $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

Exercise

Prove the above assertions.

Arguments

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An argument is a statement of the form:

$$(P_1 \wedge P_2 \wedge \ldots P_n) \rightarrow Q$$

Arguments

Definition

An argument is a statement of the form:

$$(P_1 \land P_2 \land \ldots P_n) \rightarrow Q$$

where each of the $P_i s$ and Q are propositions.

Arguments

Definition

An argument is a statement of the form:

$$(P_1 \land P_2 \land \ldots P_n) \to Q$$

where each of the $P_i s$ and Q are propositions.

The $P_i s$ are called the hypotheses and Q is called the conclusion.

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Semantics

If all the P_i are **true**, is Q necessarily **true**?

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If all the P_i are **true**, is Q necessarily **true**?

When can Q be logically deduced from $P_1, P_2, \ldots, P_n$?

Valid Arguments

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Definition

The argument

$$(P_1 \wedge P_2 \wedge \ldots P_n) \rightarrow Q$$

is said to be *valid*, if it is a tautology.

Valid Arguments

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Example

$2 + 2 = 4$ and $7 + 3 = 10$. Therefore, a minute has 60 seconds. Is this valid?

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Note

The validity of an argument is based purely on its intrinsic structure and not on the specific meanings attached to its constituent propositions.

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Valid Arguments

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$2+2=4$ and $7+3=10$. Therefore, a minute has 60 seconds. Is this valid?

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Example

If John is hungry, he will eat. John is hungry. Therefore, he will eat.

Valid Arguments

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Example

If John is hungry, he will eat. John is hungry. Therefore, he will eat.

Symbolically,

$$[(H \rightarrow E) \wedge H] \rightarrow E.$$

Checking Validity

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Truth-table Method

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Simply check if all rows of the truth-table are **true**.

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Derivation Rules

Checking Validity

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Derivation Rules

We will use a set of derivation rules and manipulate the hypotheses to arrive at the desired conclusion.

Checking Validity

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Proof Sequence

Checking Validity

Truth-table Method

Simply check if all rows of the truth-table are **true**. Horribly expensive!

Derivation Rules

We will use a set of derivation rules and manipulate the hypotheses to arrive at the desired conclusion.

Proof Sequence

A proof sequence is a sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system's derivation rules to earlier wffs in the sequence.

Derivation Rules

Rule Types

Derivation Rules

Rule Types

(i) Equivalence Rules.

Derivation Rules

Rule Types

(i) Equivalence Rules.

(ii) Inference Rules.

Equivalence Rules

Equivalence Rules

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Equivalence Rules

Expression	Equivalent to	Name of Rule
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan
$P \rightarrow Q$	$P' \vee Q$	Implication - imp
P	$(P')'$	Double negation - dn
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	Definition of equivalence

Inference Rules

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Inference Rules

From	Can Derive	Name of Rule
P, $P \to Q$	Q	*Modus Ponens* (mp)
$P \to Q$, Q'	P'	*Modus Tollens* (mt)
P, Q	$P \wedge Q$	Conjunction
$P \wedge Q$	P, Q	Simplification
P	$P \vee Q$	Addition

A proof derivation

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Example

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

A proof derivation

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Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i) A hypothesis.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i) A hypothesis.

(ii) B hypothesis.

A proof derivation

Example

Argue that

$$[A \wedge (B \to C) \wedge [(A \wedge B) \to (D \vee C')] \wedge B] \to D$$

is a valid argument.

Proof

(i) A hypothesis.

(ii) B hypothesis.

(iii) $B \to C$ hypothesis.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \rightarrow C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \rightarrow C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.
(v)	$A \wedge B$	(i), (ii), Conjunction.

A proof derivation

Example

Argue that

$$[A \land (B \to C) \land [(A \land B) \to (D \lor C')] \land B] \to D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \to C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.
(v)	$A \land B$	(i), (ii), Conjunction.
(vi)	$(A \land B) \to (D \lor C')$	hypothesis.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \rightarrow C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.
(v)	$A \wedge B$	(i), (ii), Conjunction.
(vi)	$(A \wedge B) \rightarrow (D \vee C')$	hypothesis.
(vii)	$(D \vee C')$	(v), (vi), Modus Ponens.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
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(vii)	$(D \vee C')$	(v), (vi), Modus Ponens.
(viii)	$(C \rightarrow D)$	(vii), Implication.
(ix)	D	(iv), (viii), Modus Ponens.

Two more rules

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Deduction Method

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Show that $[A \rightarrow (B \rightarrow C)] \Leftrightarrow [(A \wedge B) \rightarrow C]$.

Two more rules

Deduction Method

Show that $[A \to (B \to C)] \Leftrightarrow [(A \wedge B) \to C]$.

This leads us to conclude that the argument $[P_1 \wedge P_2 \ldots P_n] \to (R \to S)$ is tautologically equivalent to the argument $[P_1 \wedge P_2 \ldots P_n \wedge R] \to S$.

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Example

Prove that $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$.

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Example

Prove that $[(A \to B) \wedge (B \to C)] \to (A \to C)$.

Technique

Using the Deduction Method, the above argument can be rewritten as:
$[(A \to B) \wedge (B \to C) \wedge A] \to C$.

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Deduction Method

Show that $[A \to (B \to C)] \Leftrightarrow [(A \wedge B) \to C]$.

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Using the Deduction Method, the above argument can be rewritten as: $[(A \to B) \wedge (B \to C) \wedge A] \to C$.

Is it easy now?

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Show that $[A \rightarrow (B \rightarrow C)] \Leftrightarrow [(A \wedge B) \rightarrow C]$.

This leads us to conclude that the argument $[P_1 \wedge P_2 \dots P_n] \rightarrow (R \rightarrow S)$ is tautologically equivalent to the argument $[P_1 \wedge P_2 \dots P_n \wedge R] \rightarrow S$.

Example

Prove that $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$.

Technique

Using the Deduction Method, the above argument can be rewritten as: $[(A \rightarrow B) \wedge (B \rightarrow C) \wedge A] \rightarrow C$.

Is it easy now?

Note

The rule $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$ *is called hypothesis syllogism and can be used directly.*

Proving validity of Verbal Arguments

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Methodology

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(i) Symbolize the argument.

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If interest rates drop, the housing market will improve.

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If interest rates drop, the housing market will improve.

Either the federal discount rate will drop or the housing market will not improve.

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Interest rates will drop. Therefore, the federal discount rate will drop.

Proving validity of Verbal Arguments

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Example

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Either the federal discount rate will drop or the housing market will not improve.

Interest rates will drop. Therefore, the federal discount rate will drop.

Is this argument valid?

Verbal argument validity

Verbal argument validity

Proof.

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Let I denote the event that interest rates will drop.

Verbal argument validity

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Let F denote the event that the federal discount rate will drop.

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Verbal argument validity

Proof.

Let I denote the event that interest rates will drop.

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The symbolic argument is $[(I \rightarrow H)$

Verbal argument validity

Proof.

Let I denote the event that interest rates will drop.

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Let F denote the event that the federal discount rate will drop.

The symbolic argument is $[(I \rightarrow H) \land (F \lor H')$

Verbal argument validity

Proof.

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The symbolic argument is $[(I \rightarrow H) \wedge (F \vee H') \wedge I]$

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Proof.

Let I denote the event that interest rates will drop.

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The symbolic argument is $[(I \rightarrow H) \wedge (F \vee H') \wedge I] \rightarrow F$.

Consider the following proof sequence:

1. I.
2. $I \rightarrow H$.
3. H.

Verbal argument validity

Proof.

Let I denote the event that interest rates will drop.

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The symbolic argument is $[(I \rightarrow H) \wedge (F \vee H') \wedge I] \rightarrow F$.

Consider the following proof sequence:

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3. H.
4. $F \vee H'$.

Verbal argument validity

Proof.

Let I denote the event that interest rates will drop.

Let H denote the event that the housing market will improve.

Let F denote the event that the federal discount rate will drop.

The symbolic argument is $[(I \to H) \wedge (F \vee H') \wedge I] \to F$.

Consider the following proof sequence:

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2. $I \to H$.
3. H.
4. $F \vee H'$.
5. $H \to F$.

Verbal argument validity

Proof.

Let I denote the event that interest rates will drop.

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The symbolic argument is $[(I \rightarrow H) \wedge (F \vee H') \wedge I] \rightarrow F$.

Consider the following proof sequence:

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2. $I \rightarrow H$.
3. H.
4. $F \vee H'$.
5. $H \rightarrow F$.
6. F.

$\square$

One More Example

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Show that the argument $[A' \wedge (B \to A)] \to B'$ is valid.

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One More Example

Example

Show that the argument $[A' \wedge (B \rightarrow A)] \rightarrow B'$ is valid.

Proof

Consider the following proof sequence:

(i) $(B \rightarrow A)$ hypothesis.

One More Example

Example

Show that the argument $[A' \wedge (B \rightarrow A)] \rightarrow B'$ is valid.

Proof

Consider the following proof sequence:

(i) $(B \rightarrow A)$ hypothesis.
(ii) $(B' \vee A)$ (i), Implication.

One More Example

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Proof

Consider the following proof sequence:

(i) $(B \rightarrow A)$ hypothesis.

(ii) $(B' \vee A)$ (i), Implication.

(iii) $(A \vee B')$ (ii), Commutativity.

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Consider the following proof sequence:

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(iv) $(A' \to B')$ (iii), Implication.

(v) A' hypothesis.

One More Example

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(iv) $(A' \to B')$ (iii), Implication.
(v) A' hypothesis.
(vi) B' (iv), (v) Modus Ponens.

Caveat on using Rules

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Lots of additional rules are provided on Page 37.

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2. the rules enumerated in the inference rules table,
3. the deduction method, and
4. hypothesis syllogism.

Additional Exercises

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Exercise

Show that the following arguments are valid:

1. $[A \land (B \to C)] \to (B \to (A \land C))$.
2. $[(A \lor B')' \land (B \to C)] \to (A' \land C)$.
3. $[(C \to D) \to C] \to [(C \to D) \to D)]$.
4. *If Jose took the jewelry or Mrs. Krasov lied, then a crime was committed.*
   Mr. Krasov was not in town.
   If a crime was committed, then Mr. Krasov was in town.
   Therefore, Jose did not take the jewelry.