Recursion and Recurrence Relations

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Outline

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1. Recursive Definitions
2. Solving Recurrences

Recursive Definitions

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Note

Recursive Definitions

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Parts of a recursive definition

(i) A basis, where some simple cases of the object being defined are explicitly provided,

(ii) An inductive or recursive step, where new cases of the item being defined are given in terms of previous cases.

Note

Strong connection between induction and recursion.

Types of objects defined recursively

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Recursive Objects

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Recursive Objects

(i) Sequences.

Types of objects defined recursively

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(ii) Sets.

Types of objects defined recursively

Recursive Objects

(i) Sequences.
(ii) Sets.
(iii) Operations.

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Recursive Objects

(i) Sequences.
(ii) Sets.
(iii) Operations.
(iv) Algorithms.

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Write down the first 5 elements of the following recursively defined sequence:

$$S(1) = 2$$

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Write down the first 5 elements of the following recursively defined sequence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

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Write down the first 5 elements of the following recursively defined sequence:

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The second part of the definition is called a recurrence relation.

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Example

Write down the first 5 elements of the following recursively defined sequence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3, \; n \geq 2. \end{aligned}$$

Sequences (contd.)

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Fibonacci Sequence

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Fibonacci Sequence

$$F(1) = 1$$

Sequences (contd.)

Fibonacci Sequence

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\begin{aligned}
F(1) &= 1 \\
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\end{aligned}
$$

Sequences (contd.)

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$$\begin{aligned} F(1) &= 1 \\ F(2) &= 1 \\ F(n) &= F(n-1) + F(n-2), \ n \geq 3 \end{aligned}$$

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Enumerate the first 5 elements of the Fibonacci sequence.

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Enumerate the first 5 elements of the Fibonacci sequence.
Show that

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F(1) &= 1 \\
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F(n) &= F(n-1) + F(n-2), \ n \geq 3
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Example

Enumerate the first 5 elements of the Fibonacci sequence.
Show that

$$F(n+4) = 3 \cdot F(n+2) - F(n), \ \text{for all } n \geq 1$$

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Observe that,

$$F(k+1+4) = F(k+5)$$

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We now need to show that

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Observe that,

$$\begin{aligned} F(k + 1 + 4) &= F(k + 5) \\ &= F(k + 4) + F(k + 3), \quad \text{by definition} \end{aligned}$$

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&= F(k+4) + F((k-1)+4)) \\
&= 3 \cdot F(k+2) - F(k) + 3 \cdot F((k-1)+2) - F(k-1), \quad \text{using the i.h.}
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&= 3 \cdot (F(k+2) + F(k+1)) - (F(k) + F(k-1)) \\
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\end{aligned}$$

Applying the second principle of mathematical induction, we conclude that the conjecture is true for all $n \geq 1$. $\square$

Direct proof

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$$F(n+4) = F(n+3) + F(n+2)$$

Direct proof

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&= F(n+2) + F(n+1) + F(n+2)
\end{aligned}
$$

Direct proof

Proof.

$$
\begin{aligned}
F(n+4) &= F(n+3)+F(n+2) \\
&= F(n+2)+F(n+1)+F(n+2) \\
&= 2\cdot F(n+2)+F(n+1)
\end{aligned}
$$

Direct proof

Proof.

$$
\begin{aligned}
F(n+4) &= F(n+3)+F(n+2) \\
&= F(n+2)+F(n+1)+F(n+2) \\
&= 2\cdot F(n+2)+F(n+1) \\
&= 2\cdot F(n+2)+(F(n+2)-F(n))
\end{aligned}
$$

Direct proof

Proof.

$$\begin{aligned} F(n+4) &= F(n+3) + F(n+2) \\ &= F(n+2) + F(n+1) + F(n+2) \\ &= 2 \cdot F(n+2) + F(n+1) \\ &= 2 \cdot F(n+2) + (F(n+2) - F(n)) \\ &= 3 \cdot F(n+2) - F(n) \end{aligned}$$

$\square$

Recursively Defined Sets

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Example

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Define the set of ancestors of John.

Recursively Defined Sets

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Define the set of ancestors of John.

(i) John's parents are his ancestors.

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(i) The empty string λ is a word.

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Define the set of all possible word combinations using small-case letters from the English alphabet.

(i) The empty string λ is a word.
(ii) $\{\mathbf{a, b, c, \ldots, z}\}$ are words.
(iii) If x and y are words, then so is $x \cdot y$.

Recursively Defined Sets (contd.)

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Example

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Define the set of binary palindromes.

Recursively Defined Sets (contd.)

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Define the set of binary palindromes.

(i) The empty string λ is a palindrome.

Recursively Defined Sets (contd.)

Example

Define the set of binary palindromes.

(i) The empty string λ is a palindrome.

(ii) 0 and 1 are palindromes.

Recursively Defined Sets (contd.)

Example

Define the set of binary palindromes.

(i) The empty string λ is a palindrome.
(ii) 0 and 1 are palindromes.
(iii) If x is a palindrome, then so are $0 \cdot x \cdot 0$ and $1 \cdot x \cdot 1$.

Recursively Defined Operations

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Certain binary operations can be defined recursively in terms of “less powerful” operations.

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$$a^0 = 1$$

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Define exponentiation in terms of multiplication.

$$\begin{aligned} a^0 &= 1 \\ a^m &= a \cdot (a^{m-1}),\ m \geq 1. \end{aligned}$$

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Define multiplication in terms of addition.

$$x \cdot 0 = 0$$

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Certain binary operations can be defined recursively in terms of "less powerful" operations.

Example

Define exponentiation in terms of multiplication.

$$\begin{aligned} a^0 &= 1 \\ a^m &= a \cdot (a^{m-1}),\ m \geq 1. \end{aligned}$$

Example

Define multiplication in terms of addition.

$$\begin{aligned} x \cdot 0 &= 0 \\ x \cdot y &= x + x \cdot (y-1),\ y \geq 1. \end{aligned}$$

Recursively Defined Algorithms

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Function MAX(a, b)

```
if (a ≥ b) then
    return(a).
else
    return(b).
end if
```

Recursively Defined Algorithms

Function MAX(a, b)

if ($a \geq b$) **then**
    return(a).
else
    return(b).
end if

The Find-Max Algorithm

Recursively Defined Algorithms

Function MAX(a, b)
if $(a \geq b)$ **then**
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The Find-Max Algorithm

Function FIND-MAX$(\mathbf{A}, n)$
if $(n = 1)$ **then**
  return$(A[1])$.
else
  return(MAX$(A[n]$, FIND-MAX$(\mathbf{A}, n - 1)))$.
end if

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  return(MAX$(A[n]$, FIND-MAX$(\mathbf{A}, n-1)))$.
end if

Note

Recursively Defined Algorithms

Function MAX(a, b)
1: **if** $(a \geq b)$ **then**
2: **return**(a).
3: **else**
4: **return**(b).
5: **end if**

The Find-Max Algorithm

Function FIND-MAX$(\mathbf{A}, n)$
1: **if** $(n = 1)$ **then**
2: **return**$(A[1])$.
3: **else**
4: **return**(MAX$(A[n]$, FIND-MAX$(\mathbf{A}, n-1)$)).
5: **end if**

Note

Can you prove the correctness of the above algorithm?

Solving recurrences

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Given a recurrence relation describing a function, say $T(n)$,

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Given a recurrence relation describing a function, say $T(n)$,
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Given a recurrence relation describing a function, say $T(n)$,
we want to find a closed form expression which exactly describes $T(n)$,
i.e., an expression not involving calls to the function $T()$.

Solving recurrences

Problem definition

Given a recurrence relation describing a function, say $T(n)$,
we want to find a closed form expression which exactly describes $T(n)$,
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Two methods

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(i) Expand-Guess-Verify (EGV).

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Two methods

(i) Expand-Guess-Verify (EGV).
(ii) Formula.

Expand-Guess-Verify

Expand-Guess-Verify

Example

Expand-Guess-Verify

Example

Consider the recurrence:

Expand-Guess-Verify

Example

Consider the recurrence:

$$S(1) = 1$$

Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1)+1, \;\; n \geq 2. \end{aligned}$$

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Since LHS=RHS, the basis is proven.

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$$\begin{aligned} S(k+1) &= S(k)+1, \quad \text{by definition} \\ &= k+1, \quad \text{by inductive hypothesis} \end{aligned}$$

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Applying the first principle of mathematical induction, we conclude that $S(n) = n$.

EGV (contd.)

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Example

EGV (contd.)

Example

Consider the recurrence:

EGV (contd.)

Example

Consider the recurrence:

$$S(1) = 2$$

EGV (contd.)

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 2 \\
S(n) &= 2 \cdot S(n-1), \quad n \geq 2.
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EGV (contd.)

Example

Consider the recurrence:

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EGV (contd.)

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EGV (contd.)

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EGV (contd.)

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EGV (contd.)

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EGV (contd.)

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Consider the recurrence:

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BASIS: $n = 1$

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$$LHS = 2$$

EGV (contd.)

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Since LHS=RHS, the basis is proven.

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$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \\ &= 2^{k+1}. \end{aligned}$$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

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Applying the first principle of mathematical induction, we conclude that $S(n) = 2^n$.

EGV (contd.)

EGV (contd.)

Example

EGV (contd.)

Example

Solve the recurrence:

EGV (contd.)

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Solve the recurrence:

$$T(1) = 1$$

EGV (contd.)

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3, \; n \geq 2. \end{aligned}$$

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Solve the recurrence:

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EGV (contd.)

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Solve the recurrence:

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EGV (contd.)

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Solve the recurrence:

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EGV (contd.)

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Solve the recurrence:

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(ii) Guess: $T(n) = 3 \cdot n - 2$.

EGV (contd.)

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Solve the recurrence:

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\begin{aligned}
T(1) &= 1 \\
T(n) &= T(n-1) + 3, \; n \geq 2.
\end{aligned}
$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$, $T(3) = T(2) + 3 = 7, \ldots$

(ii) Guess: $T(n) = 3 \cdot n - 2$.

(iii) Verify: Somebody from class!

Formula approach

Formula approach

Definition

Formula approach

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A general linear recurrence has the form:

Formula approach

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$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n).$$

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The above formula is called linear, *because the* $S()$ *terms occur only in the first power.*

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It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*.*

The recurrence is called homogeneous, if $g(n) = 0$*, for all* n*.*

Linear first-order recurrence with constant coefficients

Linear first-order recurrence with constant coefficients

Formula for Linear first-order recurrence

Linear first-order recurrence with constant coefficients

Formula for Linear first-order recurrence

$$S(1) = k_0$$

Linear first-order recurrence with constant coefficients

Formula for Linear first-order recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

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$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

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Linear first-order recurrence with constant coefficients

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$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

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Another Example

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Solve the recurrence:

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$$S(1) = 4$$

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$$S(n) = 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3$$

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$$\begin{aligned} S(n) &= 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3 \\ &= 2^{n+1} + 3 \cdot \sum_{i=2}^{n} 2^{n-i} \end{aligned}$$

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Second Order homogeneous Linear Recurrence with constant coefficients

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Let r_1 and r_2 denote the roots.

(a) If $r_1 \neq r_2$, solve

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(b) If $r_1 = r_2 = r$, solve

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Examples of second order recurrences

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Solve the recurrence relation

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Solve the recurrence relation

$$T(1) = 5$$

Examples of second order recurrences

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Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3. \end{aligned}$$

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Solution:

(i) $c_1 = 6, \; c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.

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$$p + q = T(1) = 5$$

Examples of second order recurrences

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p + q &= T(1) = 5 \\
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\end{aligned}
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Examples of second order recurrences

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Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3. \end{aligned}$$

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(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.
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We get $p = 3$ and $q = 2$.

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3. \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.
Solution is: $r_1 = 1$, $r_2 = 5$.

(ii) Solve the equations:

$$\begin{aligned} p + q &= T(1) = 5 \\ p \cdot 1 + q \cdot 5 &= T(2) = 13 \end{aligned}$$

We get $p = 3$ and $q = 2$.

(iii) Accordingly, the solution is $T(n) =$

Examples of second order recurrences

Example

Solve the recurrence relation

$$
\begin{aligned}
T(1) &= 5 \\
T(2) &= 13 \\
T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3.
\end{aligned}
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We get $p = 3$ and $q = 2$.

(iii) Accordingly, the solution is $T(n) = 3 \cdot 1^{n-1} +$

Examples of second order recurrences

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We get $p = 3$ and $q = 2$.

(iii) Accordingly, the solution is $T(n) = 3 \cdot 1^{n-1} + 2 \cdot 5^{n-1} =$

Examples of second order recurrences

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Solve the recurrence relation

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\begin{aligned}
T(1) &= 5 \\
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T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \;\; n \geq 3.
\end{aligned}
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(iii) Accordingly, the solution is $T(n) = 3 \cdot 1^{n-1} + 2 \cdot 5^{n-1} = 3 + 2 \cdot 5^{n-1}$.

□

One More Example

One More Example

Example

One More Example

Example

Solve the recurrence relation:

One More Example

Example

Solve the recurrence relation:

$$S(1) = 1$$

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \end{aligned}$$

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \;\; n \geq 3 \end{aligned}$$

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \ n \geq 3 \end{aligned}$$

Solution:

One More Example

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Solve the recurrence relation:

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S(1) &= 1\\
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$$

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One More Example

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(i) $c_1 = 8$, $c_2 = -16$.

One More Example

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Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$.

One More Example

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Solve the recurrence relation:

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Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

One More Example

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$$p = 1$$

One More Example

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Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

(ii) Solve the equations:

$$\begin{aligned} p &= 1 \\ p \cdot 4 + q \cdot 4 &= 12 \end{aligned}$$

One More Example

Example

Solve the recurrence relation:

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\begin{aligned}
S(1) &= 1 \\
S(2) &= 12 \\
S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \ n \geq 3
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Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

(ii) Solve the equations:

$$
\begin{aligned}
p &= 1 \\
p \cdot 4 + q \cdot 4 &= 12
\end{aligned}
$$

We get $p = 1$ and $q = 2$.

One More Example

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(iii) Accordingly, the solution is $S(n) = 4^{n-1} + 2 \cdot (n-1) \cdot 4^{n-1} =$

One More Example

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Solve the recurrence relation:

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\begin{aligned}
S(1) &= 1\\
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\begin{aligned}
p &= 1\\
p \cdot 4 + q \cdot 4 &= 12
\end{aligned}
$$

We get $p = 1$ and $q = 2$.

(iii) Accordingly, the solution is $S(n) = 4^{n-1} + 2 \cdot (n-1) \cdot 4^{n-1} = (2 \cdot n - 1) \cdot 4^{n-1}$.

□

Divide and Conquer Recurrences

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

$$S(1) = k_0$$

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m. \end{aligned}$$

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m. \end{aligned}$$

$\Rightarrow S(n) =$

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$\Rightarrow S(n) = c^{\log n} \cdot k_0 +$

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$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i).$

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m. \end{aligned}$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i)$. (All logarithms are to base 2).

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

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\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m.
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i)$. (All logarithms are to base 2).

Note that $c^{\log n - i}$ in the expression above stands for $\frac{c^{\log n}}{c^i}$.

Example

Example

Example

Example

Example

Solve the recurrence:

Example

Example

Solve the recurrence:

$$C(1) = 1$$

Example

Example

Solve the recurrence:

$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m. \end{aligned}$$

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Note that $k_0 = 1$, $c = 1$ and $g(i) = 1,\ \forall i$.

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Solve the recurrence:

$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}), \; n \geq 2, \; n = 2^m. \end{aligned}$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1, \; \forall i$. Hence, $g(2^i) = 1, \; \forall i$.
As per the formula,

$$C(n) = 1^{\log n} \cdot 1 +$$

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As per the formula,

$$C(n) = 1^{\log n} \cdot 1 + \sum_{i=1}^{\log n} 1^{\log n - i} \cdot (1)$$

Example

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$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}), \ n \geq 2, \ n = 2^m. \end{aligned}$$

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As per the formula,

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Example

Example

Solve the recurrence:

$$
\begin{aligned}
T(1) &= 3 \\
T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m.
\end{aligned}
$$

Example

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Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \ \ \forall i$. It follows that $g(2^i) = 2 \cdot 2^i$, $\forall i$.

As per the formula,

$$
T(n) = 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} .
$$

Example

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$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m. \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. It follows that $g(2^i) = 2 \cdot 2^i$, $\forall i$.

As per the formula,

$$T(n) \quad = \quad 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i)$$

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\begin{aligned}
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Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. It follows that $g(2^i) = 2 \cdot 2^i$, $\forall i$.

As per the formula,

$$
\begin{aligned}
T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\
&= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1}
\end{aligned}
$$

Example

Solve the recurrence:

$$T(1) = 3$$
$$T(n) = 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m.$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. It follows that $g(2^i) = 2 \cdot 2^i, \forall i$.

As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \end{aligned}$$

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T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\
&= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\
&= 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \quad \text{since } (a^{\log_a n} = n, \;\; a \neq 0)
\end{aligned}
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Solve the recurrence:

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Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m. \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. It follows that $g(2^i) = 2 \cdot 2^i, \forall i$.

As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \quad \text{since } (a^{\log_a n} = n, \;\; a \neq 0) \\ &= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\ &= 3 \cdot n + n \cdot 2 \cdot \log n, \end{aligned}$$

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if $(a \geq b)$ **then**
    return(a).
else
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Function MAX(a, b)

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if (a ≥ b) then
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Function FIND-MAX($\mathbf{A}$, n)

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if (n = 1) then
    return(A[1]).
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    return(MAX(A[n], FIND-MAX(A, n − 1))).
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Note

How many element-to-element comparisons are performed by the FIND-MAX() *algorithm on an array of size n?*