Relations

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Outline

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Given any two sets S and T, a binary relation on $S \times T$, (or a binary relation from S to T) is any subset of $S \times T$.

Given n sets $S_1, S_2, \ldots, S_n$, an n-ary relation on $S_1 \times S_2 \ldots S_n$ is any subset of $S_1 \times S_2 \ldots S_n$.

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Note

A binary relation on $A \times B$ is a **pairing** *of elements in A, with the elements in B.*

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   - Classification of binary relations
   - Properties of relations
   - Closures of relations
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Relations on the power set $\mathcal{P}(S)$ of a set S

(i) The relation $\subseteq$ is reflexive, transitive and antisymmetric.

Examples

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Outline

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We use $[x]$ *to denote the set* $\{y \mid y \in S \wedge x \,\rho\, y\}$.

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We use $[x]$ to denote the set $\{y \mid y \in S \wedge x\,\rho\,y\}$.

$[x]$ is said to be the equivalence class of x.

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Note

The proof is somewhat tedious but the main idea is that if there is an element common to two distinct equivalence classes, then these classes coincide.

Partitions as equivalences

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Define ρ as follows: $x \; \rho \; y$, if x and y are in the same partition.

Clearly, ρ is reflexive, symmetric and transitive, i.e., an equivalence relation. □

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It follows that $S \subseteq U$.

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We have thus shown the union of the equivalence classes is S. $\square$

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It follows that $z \, \rho \, q$. From the previous discussion, we have, $x \, \rho \, z$.

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In similar fashion, we can show that $[x] \subseteq [z]$.

We have thus shown that every equivalence relation on S, induces a collection of disjoint sets, whose union is S, i.e., a partition on S. □

Partition Examples

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How does the equivalence relation $x \, \rho \, y \leftrightarrow x + y$ is even partition $\mathcal{N}$?

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How does the equivalence relation $x \, \rho \, y \leftrightarrow x + y$ is even partition $\mathcal{N}$?

Solution: All odd numbers are in one partition and all even numbers are in the other partition!

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Enumerate the equivalence classes of congruence modulo 4 *on* $\mathcal{Z}$.

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Enumerate the equivalence classes of congruence modulo 4 on $\mathcal{Z}$.

Solution

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

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$$\begin{aligned} [0] &= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\ [1] &= \{\ldots, -7, -3, 1, 5, \ldots\} \end{aligned}$$

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Enumerate the equivalence classes of congruence modulo 4 on $\mathcal{Z}$.

Solution

$$\begin{aligned} [0] &= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\ [1] &= \{\ldots, -7, -3, 1, 5, \ldots\} \\ [2] &= \{\ldots, -6, -2, 2, 6, \ldots\} \\ [3] &= \{\ldots, -5, -1, 3, 7, \ldots\} \end{aligned}$$