Set Theory and Countability

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Outline

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$$A = B \Leftrightarrow (\forall x)[x \in A \leftrightarrow x \in B].$$

Fundamentals (contd.)

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$A = B \Leftrightarrow (A \subseteq B)$ and $(B \subseteq A)$.

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Hence, $x \in B$.

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Exercise

Show that if a set has n elements, then its power set will have 2^n elements.

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The property that $x \circ y \in S$ is called the closure property, i.e., S is closed under operation $\circ$.

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(iv) Is $\circ$ an operation on $\mathcal{N}$, where

$$x \circ y = \begin{cases} 1 & \text{if } x \geq 5 \\ 0 & \text{if } x \leq 5 \end{cases}$$

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But it is a unary operation on $\Re_{+}$ (the set of non-negative reals).

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$\circ$	**1**	**2**	**3**
1	*2*	*3*	*3*
2	*1*	*2*	*1*
3	*2*	*3*	*1*

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Two sets are said to be **disjoint** if $A \cap B = \emptyset$.

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Operations on Sets

Fundamentals

For discussing operations on sets, we assume the existence of a ground set S and its power set $\mathcal{P}(S)$.

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Note

$A \times A$ is referred to as A^2, $A \times A \times A$ as A^3 and so on.

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Set Identities (contd.)

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$$\textit{DeMorgan}: \begin{cases} (A \cup B)' &= A' \cap B'. \\ (A \cap B)' &= A' \cup B'. \end{cases}$$

Proving set identities

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(v) Convert the deduction into a statement in set theory.

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Simply reverse the above argument! □

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\begin{aligned}
x \in A \cup (B \cap C) \quad & \rightarrow \quad x \in A \textbf{ or } x \in (B \cap C), \text{ definition of union} \\
& \rightarrow \quad (x \in A) \textbf{ or } (x \in B \textbf{ and } x \in C), \text{ definition of intersection} \\
& \rightarrow \quad P \vee (Q \wedge R) \\
& \rightarrow \quad (P \vee Q) \wedge (P \vee R), \text{ since } [P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \\
& \rightarrow \quad (x \in A \textbf{ or } x \in B) \textbf{ and } (x \in A \textbf{ or } x \in C) \\
& \rightarrow \quad (x \in A \cup B) \textbf{ and } (x \in A \cup C), \text{ definition of union} \\
& \rightarrow \quad x \in (A \cup B) \cap (A \cup C), \text{ definition of intersection}
\end{aligned}
$$

Another example

Example

Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

Let P denote $x \in A$, Q denote $x \in B$ and R denote $x \in C$.

Observe that,

$$
\begin{aligned}
x \in A \cup (B \cap C) \quad & \rightarrow \quad x \in A \textbf{ or } x \in (B \cap C), \text{ definition of union} \\
& \rightarrow \quad (x \in A) \textbf{ or } (x \in B \textbf{ and } x \in C), \text{ definition of intersection} \\
& \rightarrow \quad P \vee (Q \wedge R) \\
& \rightarrow \quad (P \vee Q) \wedge (P \vee R), \text{ since } [P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \\
& \rightarrow \quad (x \in A \textbf{ or } x \in B) \textbf{ and } (x \in A \textbf{ or } x \in C) \\
& \rightarrow \quad (x \in A \cup B) \textbf{ and } (x \in A \cup C), \text{ definition of union} \\
& \rightarrow \quad x \in (A \cup B) \cap (A \cup C), \text{ definition of intersection}
\end{aligned}
$$

Simply reverse the argument to show that every element in the set represented by the RHS is also an element of the set represented by the LHS. □

Two more examples

Two more examples

Examples

Two more examples

Examples

Prove De Morgan's Laws

Two more examples

Examples

Prove De Morgan's Laws

(i) $(A \cup B)' = A' \cap B'$.

Two more examples

Examples

Prove De Morgan's Laws

(i) $(A \cup B)' = A' \cap B'$.
(ii) $(A \cap B)' = A' \cup B'$.

Proof of the Union Law

Proof of the Union Law

Proof

Proof of the Union Law

Proof

Let $P \equiv x \in A$ and $Q \equiv x \in B$.

Proof of the Union Law

Proof

Let $P \equiv x \in A$ and $Q \equiv x \in B$. It follows that $P' \equiv x \notin A$ and $Q' \equiv x \notin B$.

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$$x \in (A \cup B)' \quad \rightarrow$$

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Let $P \equiv x \in A$ and $Q \equiv x \in B$. It follows that $P' \equiv x \notin A$ and $Q' \equiv x \notin B$. Observe that,

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Observe that,

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&\rightarrow (P \vee Q)'
\end{aligned}
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&\rightarrow (P \vee Q)' \\
&\rightarrow P' \wedge Q', \text{ De Morgan's law for propositional logic} \\
&\rightarrow (x \notin A) \textbf{ and } (x \notin B) \\
&\rightarrow (x \in A') \textbf{ and } (x \in B') \\
&\rightarrow x \in (A' \cap B'),
\end{aligned}
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Let $P \equiv x \in A$ and $Q \equiv x \in B$. It follows that $P' \equiv x \notin A$ and $Q' \equiv x \notin B$. Observe that,

$$\begin{aligned}
x \in (A \cup B)' &\to x \notin (A \cup B) \\
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&\to [(x \in A) \textbf{ or } (x \in B)]', \text{ definition of union} \\
&\to (P \vee Q)' \\
&\to P' \wedge Q', \text{ De Morgan's law for propositional logic} \\
&\to (x \notin A) \textbf{ and } (x \notin B) \\
&\to (x \in A') \textbf{ and } (x \in B') \\
&\to x \in (A' \cap B'), \text{ definition of intersection} \\
\Rightarrow (A \cup B)' &\subseteq (A' \cap B').
\end{aligned}$$

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&\rightarrow (x \notin A) \textbf{ and } (x \notin B) \\
&\rightarrow (x \in A') \textbf{ and } (x \in B') \\
&\rightarrow x \in (A' \cap B'), \text{ definition of intersection} \\
\Rightarrow (A \cup B)' &\subseteq (A' \cap B').
\end{aligned}
$$

You can reverse the argument to show that $(A' \cap B') \subseteq (A \cup B)'$.

Proof of the Intersection Law

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&\rightarrow [(x \in A) \wedge (x \in B)]' \\
&\rightarrow (P \wedge Q)' \\
&\rightarrow P' \vee Q', \text{ De Morgan's law for propositional logic}
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&\rightarrow [(x \in A) \wedge (x \in B)]' \\
&\rightarrow (P \wedge Q)' \\
&\rightarrow P' \vee Q', \text{ De Morgan's law for propositional logic} \\
&\rightarrow (x \notin A) \textbf{ or } (x \notin B)
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 & \rightarrow & [(x \in A) \wedge (x \in B)]' \\
 & \rightarrow & (P \wedge Q)' \\
 & \rightarrow & P' \vee Q', \text{ De Morgan's law for propositional logic} \\
 & \rightarrow & (x \notin A) \textbf{ or } (x \notin B) \\
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Proof of the Intersection Law

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Let $P \equiv x \in A$ and $Q \equiv x \in B$. It follows that $P' \equiv x \notin A$ and $Q' \equiv x \notin B$. Observe that,

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x \in (A \cap B)' &\rightarrow x \notin (A \cap B) \\
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Hence, L_2 can be rewritten as:

$L_3 = [(B \cap C) \cup (A \cap A')] \cap (B \cap C)'$.

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But $P \cup \emptyset = P$, for any set P.

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Countable and Uncountable sets

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A set S is said to be countable, if it is either finite or denumerable.
If it is not countable, it is said to be uncountable.

Countability examples

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Cantor's Theorem

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The set of all real numbers in the interval $[0, 1]$ *is uncountable.*

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Does p belong to the enumeration?

It follows that the set of reals in $[0, 1]$ cannot be enumerated, i.e., the set is uncountable. □