Algebra of the Simplex Method

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Outline

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1. Representation Issues
2. Checking Optimality

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Overview

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Principal Ideas

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$$z = \mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b} - \sum_{j \in J} (\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j) \cdot x_j$$

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Main idea

The *key idea* of the simplex method is to move from an extreme point to an improving adjacent extreme point by interchanging a column in **B** and a column in **N**.

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Based on the derived expression for z, the *rate of change* of z with respect to the nonbasic variable x_j is:

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Based on the derived expression for z, the *rate of change* of z with respect to the nonbasic variable x_j is:

$$\frac{\partial z}{\partial x_j} = -(\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j)$$

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Based on the derived expression for z, the *rate of change* of z with respect to the nonbasic variable x_j is:

$$\frac{\partial z}{\partial x_j} = -(\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j)$$

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What is $(z_j - c_j)$ for a basic variable?

Determining the entering variable and departing variable

Linear Programming

Determining the entering variable and departing variable

Entering Variable

Pick the non-basic variable for which $\frac{\partial z}{\partial x_j}$ is the largest.

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x_k is determined by a blocking constraint.

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$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \cdot \mathbf{b}_j, \text{where } \lambda_j \in E^1,\ \text{for all } j = 1, \ldots, m \qquad \text{(I)}$$

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□

Proof (contd.)

Linear Programming

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Subtracting (II) from (I) yields $\mathbf{0} = \sum_{j=1}^{m} (\lambda_j - \mu_j) \cdot \mathbf{b_j}$.

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The theorem is proven!

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Can $\delta = 0$?

□

Exchanging columns (contd.)

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Proof.

Linear Programming

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Thus $\mathbf{b}_1,\ \mathbf{b}_2, \ldots,\ \mathbf{b}_{m-1},\ \mathbf{a}$ are linearly independent and form a basis for E^m.

□

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Basic and Non-basic variables

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Basic and Non-basic variables

We recall the equation relating the basic and non-basic variables of our standard linear program:

$$\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b} - \sum_{j \in J} (\mathbf{B}^{-1} \cdot \mathbf{a_j}) \cdot x_j$$

Clearly, the vector of coefficients associated with non-basic variable x_k is:

$$\alpha_k = \mathbf{B}^{-1} \cdot \mathbf{a}_k$$

$$\Rightarrow \mathbf{a}_k = \mathbf{B} \cdot \alpha_k = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m) \cdot \begin{pmatrix} \boldsymbol{\alpha}_{1,j} \\ \boldsymbol{\alpha}_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\alpha}_{m,j} \end{pmatrix} = \sum_{i=1}^{m} \boldsymbol{\alpha}_{i,j} \cdot \mathbf{b}_j$$

Departing Variable

Basic and Non-basic variables

We recall the equation relating the basic and non-basic variables of our standard linear program:

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Departing Variable (contd.)

Departing Variable (contd.)

Identifying the departing variable

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix $\mathbf{B}$.

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix $\mathbf{B}$.

From a previous theorem, we know that $\mathbf{a}_k$ can be exchanged with any column $\mathbf{b}_j$ of $\mathbf{B}$, such that $\alpha_{j,k} \neq 0$.

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix $\mathbf{B}$.

From a previous theorem, we know that $\mathbf{a}_k$ can be exchanged with any column $\mathbf{b}_j$ of $\mathbf{B}$, such that $\alpha_{j,k} \neq 0$.

Next observe that,

$$\frac{\partial \mathbf{x}_B}{\partial x_k} = -\mathbf{B}^{-1} \cdot \mathbf{a}_k = -\boldsymbol{\alpha}_k.$$

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix **B**.

From a previous theorem, we know that $\mathbf{a}_k$ can be exchanged with any column $\mathbf{b}_j$ of **B**, such that $\alpha_{j,k} \neq 0$.

Next observe that,

$$\frac{\partial \mathbf{x}_B}{\partial x_k} = -\mathbf{B}^{-1} \cdot \mathbf{a}_k = -\boldsymbol{\alpha}_k.$$

This means that if we raise x_k from its current value of 0 and keep all other non-basic variables at 0,

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix $\mathbf{B}$.

From a previous theorem, we know that $\mathbf{a}_k$ can be exchanged with any column $\mathbf{b}_j$ of $\mathbf{B}$, such that $\alpha_{j,k} \neq 0$.

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Departing Variable (contd.)

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This means that if we raise x_k from its current value of 0 and keep all other non-basic variables at 0, then the basic variables will change as per the relationship:

$$\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b} + x_j \cdot (\mathbf{B}^{-1} \cdot \mathbf{a}_j) = \mathbf{B}^{-1} \cdot \mathbf{b} - x_j \cdot \boldsymbol{\alpha}_j$$

Departing Variable (contd.)

Identifying the departing variable

The above equation represents $\mathbf{a}_k$ as a unique linear combination of the columns of the basis matrix $\mathbf{B}$.

From a previous theorem, we know that $\mathbf{a}_k$ can be exchanged with any column $\mathbf{b}_j$ of $\mathbf{B}$, such that $\alpha_{j,k} \neq 0$.

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Departing Variable (contd.)

Departing Variable (contd.)

Identifying the departing variable

Departing Variable (contd.)

Identifying the departing variable

All variables must be non-negative;

Departing Variable (contd.)

Identifying the departing variable

All variables must be non-negative; hence,

Departing Variable (contd.)

Identifying the departing variable

All variables must be non-negative; hence,

$$\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b} - x_j \cdot \alpha_j \geq \mathbf{0}$$

Departing Variable (contd.)

Identifying the departing variable

All variables must be non-negative; hence,

$$\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b} - x_j \cdot \boldsymbol{\alpha}_j \geq \mathbf{0}$$

Let

$$\mathbf{B}^{-1} \cdot \mathbf{b} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{pmatrix}$$

Departing Variable (contd.)

Identifying the departing variable

All variables must be non-negative; hence,

$$\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b} - x_j \cdot \boldsymbol{\alpha}_j \geq \mathbf{0}$$

Let

$$\mathbf{B}^{-1} \cdot \mathbf{b} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_m \end{pmatrix}$$

We thus have,

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_m \end{pmatrix} - x_j \cdot \begin{pmatrix} \boldsymbol{\alpha}_{1,j} \\ \boldsymbol{\alpha}_{2,j} \\ . \\ . \\ . \\ \boldsymbol{\alpha}_{m,j} \end{pmatrix} \geq 0$$

Departing Variable (contd.)

Departing Variable (contd.)

Final Step

Departing Variable (contd.)

Final Step

We get an upper bound on x_k as:

$$x_k \leq \text{minimum} \left\{ \frac{\beta_i}{\boldsymbol{\alpha}_{i,k}} : \boldsymbol{\alpha}_{i,k} > 0 \right\}$$

Departing Variable (contd.)

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We get an upper bound on x_k as:

$$x_k \leq \text{minimum} \left\{ \frac{\beta_i}{\boldsymbol{\alpha}_{i,k}} : \boldsymbol{\alpha}_{i,k} > 0 \right\}$$

The above test is called the minimum ratio test.

Departing Variable (contd.)

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Unboundedness

Departing Variable (contd.)

Final Step

We get an upper bound on x_k as:

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The above test is called the minimum ratio test.

Unboundedness

If we attempt to bring non-basic variable x_k into the basis and $\alpha_k \leq \mathbf{0}$, then the objective function can be increased indefinitely and no finite optimal solution exists.

Example

Example

Example

Example

Example

Example

Example

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

Example

Example

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

subject to

Example

Example

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
$$\text{subject to}$$
$$x_1 - 2 \cdot x_2 \quad \leq \quad 4$$

Example

Example

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_1 - 2 \cdot x_2 &\leq 4 \\ 2 \cdot x_1 + x_2 &\leq 18 \end{aligned}$$

Example

Example

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

subject to

$$\begin{aligned} x_1 - 2 \cdot x_2 &\leq 4 \\ 2 \cdot x_1 + x_2 &\leq 18 \\ x_2 &\leq 10 \end{aligned}$$

Example

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$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

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Note

Solve the above problem graphically.

Standardization

Linear Programming

Standardization

Standardizing the constraints

Standardization

Standardizing the constraints

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

subject to

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
$$\text{subject to}$$
$$x_1 - 2 \cdot x_2 + x_3 \quad = \quad 4$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_1 - 2 \cdot x_2 + x_3 &= 4 \\ 2 \cdot x_1 + x_2 + x_4 &= 18 \end{aligned}$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

subject to

$$\begin{aligned} x_1 - 2 \cdot x_2 + x_3 &= 4 \\ 2 \cdot x_1 + x_2 + x_4 &= 18 \\ x_2 + x_5 &= 10 \end{aligned}$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
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$$\begin{array}{rcl} x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

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Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

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$$\begin{array}{rcl} x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

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Summary

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
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$$\begin{array}{rcl} x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Summary

This problem can be summarized as follows:

Standardization

Standardizing the constraints

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Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
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$$\begin{aligned} x_1 - 2 \cdot x_2 + x_3 &= 4 \\ 2 \cdot x_1 + x_2 + x_4 &= 18 \\ x_2 + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Summary

This problem can be summarized as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
$$\text{subject to}$$
$$\begin{array}{rcl} x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Summary

This problem can be summarized as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix}$$

Standardization

Standardizing the constraints

$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
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$$\begin{array}{rcl} x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Summary

This problem can be summarized as follows:

$$\begin{array}{rcl} \mathbf{A} & = & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \\ \mathbf{b} & = & \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \\ \mathbf{c} & = & (2\ \ 3\ \ 0\ \ 0\ \ 0) \end{array}$$

Standardization

Standardizing the constraints

$$\begin{array}{rcl} & \text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2 & \\ & \text{subject to} & \\ x_1 - 2 \cdot x_2 + x_3 & = & 4 \\ 2 \cdot x_1 + x_2 + x_4 & = & 18 \\ x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Summary

This problem can be summarized as follows:

$$\begin{array}{rcl} \mathbf{A} & = & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \\ \mathbf{b} & = & \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \\ \mathbf{c} & = & (2 \;\; 3 \;\; 0 \;\; 0 \;\; 0) \end{array}$$

Ploughing through

Linear Programming

Ploughing through

Locate the initial basis

Ploughing through

Locate the initial basis

An obvious choice is **I**.

Ploughing through

Locate the initial basis

An obvious choice is **I**.

$$\mathbf{B} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Linear Programming

Ploughing through

Locate the initial basis

An obvious choice is **I**.

$$\mathbf{B} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{x_B} = \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Ploughing through

Locate the initial basis

An obvious choice is **I**.

$$\mathbf{B} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{x_B} = \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Is this basis feasible?

Moving from one basis to the next

Linear Programming

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$z = 2 \cdot x_1 + 3 \cdot x_2$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \end{aligned}$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \end{aligned}$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$
\begin{aligned}
z &= 2 \cdot x_1 + 3 \cdot x_2 \\
x_3 &= 4 - x_1 + x_2 \\
x_4 &= 18 - 2 \cdot x_1 - x_2 \\
x_5 &= 10 - x_2
\end{aligned}
$$

Starting solution is obtained by setting the nonbasic variables equal to zero

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

Starting solution is obtained by setting the nonbasic variables equal to zero

$$z = 0$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

Starting solution is obtained by setting the nonbasic variables equal to zero

$$\begin{aligned} z &= 0 \\ \mathbf{x}_B &= \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \end{aligned}$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

Starting solution is obtained by setting the nonbasic variables equal to zero

$$\begin{aligned} z &= 0 \\ \mathbf{x}_B &= \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \\ \mathbf{x}_N &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Moving from one basis to the next

Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$
\begin{aligned}
z &= 2 \cdot x_1 + 3 \cdot x_2 \\
x_3 &= 4 - x_1 + x_2 \\
x_4 &= 18 - 2 \cdot x_1 - x_2 \\
x_5 &= 10 - x_2
\end{aligned}
$$

Starting solution is obtained by setting the nonbasic variables equal to zero

$$
\begin{aligned}
z &= 0 \\
\mathbf{x}_B &= \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \\
\mathbf{x}_N &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Is the current basic solution optimal?

Choosing the departing variables

Choosing the departing variables

Choosing the entering variable

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2.$

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$

Linear Programming

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal).

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

How to pick the departing variable

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

How to pick the departing variable

As x_2 is increased, we must ensure that x_3 and x_4 and x_5 remain nonnegative.

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

How to pick the departing variable

As x_2 is increased, we must ensure that x_3 and x_4 and x_5 remain nonnegative.

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

How to pick the departing variable

As x_2 is increased, we must ensure that x_3 and x_4 and x_5 remain nonnegative.

x_2 needs to satisfy the most restrictive upper bound $x_2 \leq 10$ due to x_5.

Choosing the departing variables

Choosing the entering variable

$\partial z/\partial x_1 = 2$. $\partial z/\partial x_2 = 3$ (maximal). We choose x_2 as the entering variable.

How to pick the departing variable

As x_2 is increased, we must ensure that x_3 and x_4 and x_5 remain nonnegative.

x_2 needs to satisfy the most restrictive upper bound $x_2 \leq 10$ due to x_5.

x_5 is the *departing variable* and the corresponding constant is called the blocking constraint.

Pivoting

Pivoting

Pivot

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2;

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

Linear Programming

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

$$z = \quad 2 \cdot x_1 + 3 \cdot (10 - x_5) = 30 + 2 \cdot x_1 - 3 \cdot x_5$$

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

$$\begin{aligned} z = &\quad 2 \cdot x_1 + 3 \cdot (10 - x_5) = 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 = &\quad 4 - x_1 + 2 \cdot (10 - x_5) = 24 - x_1 - 2 \cdot x_5 \end{aligned}$$

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

$$\begin{aligned} z = &\quad 2 \cdot x_1 + 3 \cdot (10 - x_5) = 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 = &\quad 4 - x_1 + 2 \cdot (10 - x_5) = 24 - x_1 - 2 \cdot x_5 \\ x_4 = &\quad 18 - 2 \cdot x_1 - (10 - x_5) = 8 - 2 \cdot x_1 + x_5 \end{aligned}$$

Pivoting

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

$$\begin{aligned} z = &\quad 2 \cdot x_1 + 3 \cdot (10 - x_5) = 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 = &\quad 4 - x_1 + 2 \cdot (10 - x_5) = 24 - x_1 - 2 \cdot x_5 \\ x_4 = &\quad 18 - 2 \cdot x_1 - (10 - x_5) = 8 - 2 \cdot x_1 + x_5 \\ x_2 = &\quad 10 - x_5 \end{aligned}$$

New basis

Linear Programming

New basis

Summary

New basis

Summary

The current solution and basis matrix can be summarized as follows:

New basis

Summary

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$$z = 30$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

$$
\begin{aligned}
z &= 30 \\
\mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix}
\end{aligned}
$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

$$
\begin{aligned}
z &= 30 \\
\mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

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z &= 30 \\
\mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &=
\end{aligned}
$$

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\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) =
\end{aligned}
$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

$$
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z &= 30 \\
\mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

$$\begin{aligned} z &= 30 \\ \mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\ \mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Is the current solution optimal?

New basis

Summary

The current solution and basis matrix can be summarized as follows:

$$
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z &= 30 \\
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\end{aligned}
$$

Is the current solution optimal? Clearly not, since $\partial z / \partial x_1 = 2 \geq 0$.

New basis

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The current solution and basis matrix can be summarized as follows:

$$
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\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
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\end{aligned}
$$

Is the current solution optimal? Clearly not, since $\partial z / \partial x_1 = 2 \geq 0$.

This also means that x_1 is the entering variable.

Final move

Linear Programming

Final move

Departing variable

Final move

Departing variable

$$z = \quad 30 + 2 \cdot x_1 - 3 \cdot x_5$$

Final move

Departing variable

$$
\begin{aligned}
z = &\quad 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 = &\quad 24 - x_1 - 2 \cdot x_5
\end{aligned}
$$

Final move

Departing variable

$$
\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
x_4 &= 8 - 2 \cdot x_1 + x_5
\end{aligned}
$$

Final move

Departing variable

$$
\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
x_4 &= 8 - 2 \cdot x_1 + x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

Linear Programming

Final move

Departing variable

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\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
x_4 &= 8 - 2 \cdot x_1 + x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

Final move

Departing variable

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\begin{aligned}
z = &\quad 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 = &\quad 24 - x_1 - 2 \cdot x_5 \\
x_4 = &\quad 8 - 2 \cdot x_1 + x_5 \\
x_2 = &\quad 10 - x_5
\end{aligned}
$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

Thus x_1 can be raised up to 4.

Final move

Departing variable

$$
\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
x_4 &= 8 - 2 \cdot x_1 + x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

Thus x_1 can be raised up to 4. x_4 is now the departing variable.

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Departing variable

$$\begin{aligned} z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 &= 24 - x_1 - 2 \cdot x_5 \\ x_4 &= 8 - 2 \cdot x_1 + x_5 \\ x_2 &= 10 - x_5 \end{aligned}$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

Thus x_1 can be raised up to 4. x_4 is now the departing variable.

Replacing x_1 with $4 - \frac{1}{2} \cdot x_4 + \frac{1}{2} \cdot x_5$, we get,

Final move

Departing variable

$$\begin{aligned} z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 &= 24 - x_1 - 2 \cdot x_5 \\ x_4 &= 8 - 2 \cdot x_1 + x_5 \\ x_2 &= 10 - x_5 \end{aligned}$$

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Replacing x_1 with $4 - \frac{1}{2} \cdot x_4 + \frac{1}{2} \cdot x_5$, we get,

$$z = 30 + 2 \cdot (4 - \frac{1}{2} \cdot x_4 - \frac{1}{2} \cdot x_5) - 3 \cdot x_5 = 38 - x_4 - 2 \cdot x_5$$

Final move

Departing variable

$$
\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
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Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

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\begin{aligned}
z &= 30 + 2 \cdot (4 - \frac{1}{2} \cdot x_4 - \frac{1}{2} \cdot x_5) - 3 \cdot x_5 = 38 - x_4 - 2 \cdot x_5 \\
x_3 &= 24 - (4 + \frac{1}{2} \cdot x_5 - \frac{1}{2} \cdot x_4) - 2 \cdot x_5 = 20 + \frac{1}{2} \cdot x_4 - \frac{5}{2} \cdot x_5
\end{aligned}
$$

Final move

Departing variable

$$
\begin{aligned}
z = &\quad 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 = &\quad 24 - x_1 - 2 \cdot x_5 \\
x_4 = &\quad 8 - 2 \cdot x_1 + x_5 \\
x_2 = &\quad 10 - x_5
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x_3 = &\quad 24 - (4 + \frac{1}{2} \cdot x_5 - \frac{1}{2} \cdot x_4) - 2 \cdot x_5 = 20 + \frac{1}{2} \cdot x_4 - \frac{5}{2} \cdot x_5 \\
x_1 = &\quad 4 - \frac{1}{2} x_4 + \frac{1}{2} x_5
\end{aligned}
$$

Final move

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$$\begin{aligned} z = &\quad 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 = &\quad 24 - x_1 - 2 \cdot x_5 \\ x_4 = &\quad 8 - 2 \cdot x_1 + x_5 \\ x_2 = &\quad 10 - x_5 \end{aligned}$$

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Final move

Departing variable

$$\begin{aligned} z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\ x_3 &= 24 - x_1 - 2 \cdot x_5 \\ x_4 &= 8 - 2 \cdot x_1 + x_5 \\ x_2 &= 10 - x_5 \end{aligned}$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.

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Replacing x_1 with $4 - \frac{1}{2} \cdot x_4 + \frac{1}{2} \cdot x_5$, we get,

$$\begin{aligned} z &= 30 + 2 \cdot (4 - \frac{1}{2} \cdot x_4 - \frac{1}{2} \cdot x_5) - 3 \cdot x_5 = 38 - x_4 - 2 \cdot x_5 \\ x_3 &= 24 - (4 + \frac{1}{2} \cdot x_5 - \frac{1}{2} \cdot x_4) - 2 \cdot x_5 = 20 + \frac{1}{2} \cdot x_4 - \frac{5}{2} \cdot x_5 \\ x_1 &= 4 - \frac{1}{2}x_4 + \frac{1}{2}x_5 \\ x_2 &= 10 - x_5 \end{aligned}$$

Is the new solution optimal?

Important observations

Important observations

Note

Linear Programming

Important observations

Note

1. *There is finite progress being made at each pivot.*

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Important observations

Note

1. *There is finite progress being made at each pivot. If the optimum is finite, the algorithm will converge (barring cycling).*
2. *How do we get the initial bfs?*

Finding an initial basis

Finding an initial basis

Initial Basis

Finding an initial basis

Initial Basis

Consider the system:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Finding an initial basis

Initial Basis

Consider the system:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Change the system to:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} -2 & -3 & -1 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Finding initial basis (contd.)

Finding initial basis (contd.)

Finding a bfs

Finding initial basis (contd.)

Finding a bfs

Insert an artificial basis as follows:

Finding initial basis (contd.)

Finding a bfs

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$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} -2 & -3 & -1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Finding initial basis (contd.)

Finding a bfs

Insert an artificial basis as follows:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} -2 & -3 & -1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Finally drive x_5 and x_6 out of the system, by changing the system to:

Finding initial basis (contd.)

Finding a bfs

Insert an artificial basis as follows:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} -2 & -3 & -1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Finally drive x_5 and x_6 out of the system, by changing the system to:

$$\max -x_5 - x_6 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} -2 & -3 & -1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$