Basics

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

November 1, 2016

Outline

Outline

The Linear Decision Model

The Linear Decision Model

Introduction

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

We normally use **models** such as mathematical formulas, decision trees, and flow charts as aids for these decision making problems.

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

We normally use **models** such as mathematical formulas, decision trees, and flow charts as aids for these decision making problems.

These models do not make the decisions, but are used to complement and improve the decision process.

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

We normally use **models** such as mathematical formulas, decision trees, and flow charts as aids for these decision making problems.

These models do not make the decisions, but are used to complement and improve the decision process.

Definition

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

We normally use **models** such as mathematical formulas, decision trees, and flow charts as aids for these decision making problems.

These models do not make the decisions, but are used to complement and improve the decision process.

Definition

A **linear model** is a mathematical model consisting solely of linear functions.

The Linear Decision Model

Introduction

When dealing with real-world problems, performing a thorough, objective, and systematic analysis of the majority of decisions is more complex than we expect.

We normally use **models** such as mathematical formulas, decision trees, and flow charts as aids for these decision making problems.

These models do not make the decisions, but are used to complement and improve the decision process.

Definition

A **linear model** is a mathematical model consisting solely of linear functions.

A **linear function** is one in which all terms consist of a single continuous-valued variable and in which each variable is raised to the power of 1.

Example

Example

Examples of linear functions:

Example

Examples of linear functions:

$$f(x_1, x_2) = x_1 + x_2$$

Example

Examples of linear functions:

$$f(x_1, x_2) = x_1 + x_2$$

$$f(x_1, x_2, x_3) = 3 \cdot x_1 + 7 \cdot x_2 - 8 \cdot x_3$$

Example

Examples of linear functions:

$$f(x_1, x_2) = x_1 + x_2$$

$$f(x_1, x_2, x_3) = 3 \cdot x_1 + 7 \cdot x_2 - 8 \cdot x_3$$

Examples of nonlinear functions:

Example

Examples of linear functions:

$$f(x_1, x_2) = x_1 + x_2$$

$$f(x_1, x_2, x_3) = 3 \cdot x_1 + 7 \cdot x_2 - 8 \cdot x_3$$

Examples of nonlinear functions:

$$f(x_1, x_2) = 2 \cdot x_1^2 + 3 \cdot x_2^{3.7}$$

Example

Examples of linear functions:

$$f(x_1, x_2) = x_1 + x_2$$

$$f(x_1, x_2, x_3) = 3 \cdot x_1 + 7 \cdot x_2 - 8 \cdot x_3$$

Examples of nonlinear functions:

$$f(x_1, x_2) = 2 \cdot x_1^2 + 3 \cdot x_2^{3.7}$$

$$f(x_1, x_2) = 4 \cdot x_1 - 3 \cdot x_1 \cdot x_2 + 2 \cdot x_2^2$$

What is Linear Programming?

What is Linear Programming?

Definition

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

Example

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

Example

$$\text{maximize} \quad 7 \cdot x_1 + 10 \cdot x_2$$

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

Example

maximize	$7 \cdot x_1 + 10 \cdot x_2$	Objective Function
subject to	$x_1 + x_2 \leq 10$	

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

Example

maximize	$7 \cdot x_1 + 10 \cdot x_2$	Objective Function
subject to	$x_1 + x_2 \leq 10$	Constraints
	$3 \cdot x_1 - x_2 \geq 4$	
	$x_1, x_2 \geq 0$	

What is Linear Programming?

Definition

Linear programming, or linear optimization, is an approach to the modeling and solution of linear mathematical models.

Specifically, we are concerned with those models that seek to optimize a linear measure of performance.

Example

maximize	$7 \cdot x_1 + 10 \cdot x_2$	Objective Function
subject to	$x_1 + x_2 \leq 10$	Constraints
	$3 \cdot x_1 - x_2 \geq 4$	
	$x_1, x_2 \geq 0$	Variables

Notes

Linear Programming

Notes

If we can place a problem in the form of a linear programming model, then we have some powerful tools for obtaining a solution and analyzing how the solution changes when changing the variables.

Notes

If we can place a problem in the form of a linear programming model, then we have some powerful tools for obtaining a solution and analyzing how the solution changes when changing the variables.

The important thing to note is that the key ingredient in linear programming is the **model**.

Applications of Linear Programming

Applications of Linear Programming

Examples

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)
2. Mixes (weapons to use against a given threat, investment mixes, capital budgeting)

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)
2. Mixes (weapons to use against a given threat, investment mixes, capital budgeting)
3. Diet Problem (determining foods that satisfy calories, vitamins, and minerals while limiting fat and cholesterol)

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)
2. Mixes (weapons to use against a given threat, investment mixes, capital budgeting)
3. Diet Problem (determining foods that satisfy calories, vitamins, and minerals while limiting fat and cholesterol)
4. Production Scheduling (how many items to produce to satisfy customer demand, storage limitations)

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)
2. Mixes (weapons to use against a given threat, investment mixes, capital budgeting)
3. Diet Problem (determining foods that satisfy calories, vitamins, and minerals while limiting fat and cholesterol)
4. Production Scheduling (how many items to produce to satisfy customer demand, storage limitations)
5. Assignment (workers to tasks to minimize cost, missiles to targets)

Applications of Linear Programming

Examples

1. Blending (petroleum products to produce gasoline, ingredients to form a chemical product)
2. Mixes (weapons to use against a given threat, investment mixes, capital budgeting)
3. Diet Problem (determining foods that satisfy calories, vitamins, and minerals while limiting fat and cholesterol)
4. Production Scheduling (how many items to produce to satisfy customer demand, storage limitations)
5. Assignment (workers to tasks to minimize cost, missiles to targets)
6. Transportation/Dispatching (shipping scheme to satisfy customer demand while minimizing transport cost)

Outline

Vectors

Vectors

Definition

Vectors

Definition

A **vector** is an ordered array of numbers.

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

We use lowercase boldface type to denote vectors.

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

We use lowercase boldface type to denote vectors.

Example

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

We use lowercase boldface type to denote vectors.

Example

$$\mathbf{a} = \begin{pmatrix} 15 \\ 24 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & -2 & 3 & 1 \end{pmatrix} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

We use lowercase boldface type to denote vectors.

Example

$$\mathbf{a} = \begin{pmatrix} 15 \\ 24 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & -2 & 3 & 1 \end{pmatrix} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We denote an *m*-dimensional column vector **b** as

Vectors

Definition

A **vector** is an ordered array of numbers.

It can be either a row or a column of elements.

We use lowercase boldface type to denote vectors.

Example

$$\mathbf{a} = \begin{pmatrix} 15 \\ 24 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & -2 & 3 & 1 \end{pmatrix} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We denote an m-dimensional column vector $\mathbf{b}$ as $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Vectors

Vectors

Geometric Representation

Vectors

Geometric Representation

The collection of all m-dimensional vectors is called **Euclidean m-space** and is denoted by E^m.

Vectors

Geometric Representation

The collection of all *m*-dimensional vectors is called **Euclidean *m*-space** and is denoted by E^m.

Vectors can be represented geometrically, where a vector can be thought of as either a point or as an arrow directed from the origin to the point.

Vectors

Geometric Representation

The collection of all m-dimensional vectors is called **Euclidean m-space** and is denoted by E^m.

Vectors can be represented geometrically, where a vector can be thought of as either a point or as an arrow directed from the origin to the point.

Example

Vectors

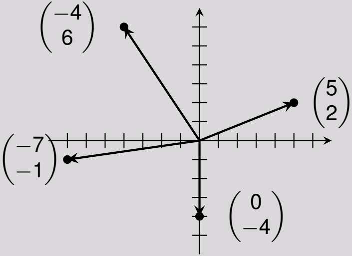
Geometric Representation

The collection of all m-dimensional vectors is called **Euclidean m-space** and is denoted by E^m.

Vectors can be represented geometrically, where a vector can be thought of as either a point or as an arrow directed from the origin to the point.

Example

Euclidean 2-space, E^2.



Vectors

Vectors

Vector Addition

Vectors

Vector Addition

Vectors of the same type (row or column) can be added if they have the same number of entries

Vectors

Vector Addition

Vectors of the same type (row or column) can be added if they have the same number of entries (dimension).

Vectors

Vector Addition

Vectors of the same type (row or column) can be added if they have the same number of entries (dimension).

Given two vectors **a** and **b**, we simply add one element in **a** with the corresponding element in **b** that is in the same position.

Vectors

Vector Addition

Vectors of the same type (row or column) can be added if they have the same number of entries (dimension).

Given two vectors **a** and **b**, we simply add one element in **a** with the corresponding element in **b** that is in the same position.

In other words, given $\mathbf{c} = \mathbf{a} + \mathbf{b}$ where c_i is the element in the ith position, we have $c_i = a_i + b_i$.

Vectors

Vector Addition

Vectors of the same type (row or column) can be added if they have the same number of entries (dimension).

Given two vectors **a** and **b**, we simply add one element in **a** with the corresponding element in **b** that is in the same position.

In other words, given $\mathbf{c} = \mathbf{a} + \mathbf{b}$ where c_i is the element in the ith position, we have $c_i = a_i + b_i$.

Vector addition satisfies both the commutative ($\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$) and associative ($\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c}$) laws.

Vector Addition Example

Vector Addition Example

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 6 & 8 & 0 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

Vector Addition Example

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 6 & 8 & 0 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \end{pmatrix}$$

Vector Addition Example

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 6 & 8 & 0 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \end{pmatrix}$$

$\mathbf{a} + \mathbf{c}$ is undefined (not the same type)

Vector Addition Example

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 6 & 8 & 0 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \end{pmatrix}$$

$\mathbf{a} + \mathbf{c}$ is undefined (not the same type)
$\mathbf{a} + \mathbf{d}$ is undefined (different number of elements)

Vectors

Vectors

Multiplication of a Vector by a Scalar

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Some examples include $3, 19, 37.5$, and $\frac{2}{3}$.

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Some examples include $3, 19, 37.5$, and $\frac{2}{3}$.

To multiply a vector by a scalar, we simply multiply each element in the vector by the scalar.

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Some examples include $3, 19, 37.5$, and $\frac{2}{3}$.

To multiply a vector by a scalar, we simply multiply each element in the vector by the scalar.

For example, if we are given a scalar α, a row vector **a**, and a column vector **b**, we have

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Some examples include $3, 19, 37.5$, and $\frac{2}{3}$.

To multiply a vector by a scalar, we simply multiply each element in the vector by the scalar.

For example, if we are given a scalar α, a row vector **a**, and a column vector **b**, we have

$$\alpha \cdot \mathbf{a} = \alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha \cdot a_1, \alpha \cdot a_2, \dots, \alpha \cdot a_n)$$

Vectors

Multiplication of a Vector by a Scalar

We define a **scalar** as an element of E^1, Euclidean 1-space.

Some examples include $3, 19, 37.5$, and $\frac{2}{3}$.

To multiply a vector by a scalar, we simply multiply each element in the vector by the scalar.

For example, if we are given a scalar α, a row vector **a**, and a column vector **b**, we have

$$\alpha \cdot \mathbf{a} = \alpha \cdot (a_1, a_2, \ldots, a_n) = (\alpha \cdot a_1, \alpha \cdot a_2, \ldots, \alpha \cdot a_n)$$

$$\alpha \cdot \mathbf{b} = \alpha \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \alpha \cdot b_1 \\ \alpha \cdot b_2 \\ \vdots \\ \alpha \cdot b_m \end{pmatrix}$$

Vectors

Vector Multiplication

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

The result, often called the **dot product**, is a scalar.

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

The result, often called the **dot product**, is a scalar.

By convention, having $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$ means $\mathbf{a}$ is the row vector and $\mathbf{b}$ is the column vector.

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

The result, often called the **dot product**, is a scalar.

By convention, having $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$ means $\mathbf{a}$ is the row vector and $\mathbf{b}$ is the column vector.

To multiply the vectors, we multiply the corresponding entries and add the results.

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

The result, often called the **dot product**, is a scalar.

By convention, having $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$ means $\mathbf{a}$ is the row vector and $\mathbf{b}$ is the column vector.

To multiply the vectors, we multiply the corresponding entries and add the results.

What this means that if we assume the vectors have m entries, we have

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{ab} = \sum_{i=1}^{m} a_i \cdot b_i = \alpha.$$

Vectors

Vector Multiplication

We can multiply two vectors, if both have the same number of entries, one of them is a row vector, and the other is a column vector.

The result, often called the **dot product**, is a scalar.

By convention, having $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$ means $\mathbf{a}$ is the row vector and $\mathbf{b}$ is the column vector.

To multiply the vectors, we multiply the corresponding entries and add the results.

What this means that if we assume the vectors have m entries, we have

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{ab} = \sum_{i=1}^{m} a_i \cdot b_i = \alpha.$$

We should also note that vector multiplication satisfies the distributive law $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

Vectors

Vector Multiplication Example

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

$$\mathbf{c} \cdot \mathbf{a} =$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

$$\mathbf{c} \cdot \mathbf{a} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} = 12 + 0 + 14 = 26$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

$$\mathbf{c} \cdot \mathbf{a} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} = 12 + 0 + 14 = 26$$

$$\mathbf{c} \cdot \mathbf{b} =$$

Vectors

Vector Multiplication Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} 5 & 1 & 4 & 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 3 & -2 \end{pmatrix}$$

$$\mathbf{c} \cdot \mathbf{a} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} = 12 + 0 + 14 = 26$$

$$\mathbf{c} \cdot \mathbf{b} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} = -8 + 90 + 2 = 84$$

Vectors

Vectors

Norm of a Vector

Vectors

Norm of a Vector

The **Euclidean norm** of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|$,

Vectors

Norm of a Vector

The **Euclidean norm** of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|$, is a measure of the size of $\mathbf{a}$ and is given by $\|\mathbf{a}\| = \left(\sum_{i=1}^{n}(a_i)^2\right)^{1/2}$.

Vectors

Norm of a Vector

The **Euclidean norm** of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|$, is a measure of the size of $\mathbf{a}$ and is given by $\|\mathbf{a}\| = \left(\sum_{i=1}^{n}(a_i)^2\right)^{1/2}$. The Euclidean norm is also denoted by $\|\mathbf{a}\|_2$

Vectors

Norm of a Vector

The **Euclidean norm** of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|$, is a measure of the size of $\mathbf{a}$ and is given by $\|\mathbf{a}\| = \left(\sum_{i=1}^{n} (a_i)^2 \right)^{1/2}$. The Euclidean norm is also denoted by $\|\mathbf{a}\|_2$

Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \qquad \|\mathbf{a}\| = [3^2 + 2^2 + (-1)^2]^{1/2} = (14)^{1/2}$$

Vectors

Norm of a Vector

The **Euclidean norm** of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|$, is a measure of the size of $\mathbf{a}$ and is given by $\|\mathbf{a}\| = \left(\sum_{i=1}^{n}(a_i)^2\right)^{1/2}$. The Euclidean norm is also denoted by $\|\mathbf{a}\|_2$

Example

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \qquad \|\mathbf{a}\| = [3^2 + 2^2 + (-1)^2]^{1/2} = (14)^{1/2}$$

Note

The dot product of two vectors can also be defined by using the Euclidean norm, which is given by $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$, where θ is the angle between the two vectors.

Special Vectors

Special Vectors

Special Vector Types

Special Vectors

Special Vector Types

Unit Vector -

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the *j*th position and 0's elsewhere.

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the jth position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the jth position.

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the *j*th position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the *j*th position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the jth position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the jth position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Null or Zero Vector -

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the *j*th position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the *j*th position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Null or Zero Vector - Denoted by **0**, is a vector having only 0's.

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the jth position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the jth position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Null or Zero Vector - Denoted by $\mathbf{0}$, is a vector having only 0's.

Sum Vector -

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the jth position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the jth position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Null or Zero Vector - Denoted by **0**, is a vector having only 0's.

Sum Vector - Denoted by **1**, is a vector having only 1's.

Special Vectors

Special Vector Types

Unit Vector - Has a 1 in the jth position and 0's elsewhere. We normally denote this by $\mathbf{e}_j$, where 1 appears in the jth position.
For example, if $\mathbf{e}_j \in E^3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Null or Zero Vector - Denoted by $\mathbf{0}$, is a vector having only 0's.

Sum Vector - Denoted by $\mathbf{1}$, is a vector having only 1's.

We call this the sum vector because the dot product of $\mathbf{1}$ and some vector $\mathbf{a}$ is a scalar that is equal to the sum of the elements in $\mathbf{a}$.

Special Vectors (contd.)

Special Vectors (contd.)

Sum Vector

Special Vectors (contd.)

Sum Vector

$$\mathbf{1} \cdot \mathbf{a} =$$

Special Vectors (contd.)

Sum Vector

$$\mathbf{1} \cdot \mathbf{a} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Special Vectors (contd.)

Sum Vector

$$\mathbf{1} \cdot \mathbf{a} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 + a_2 + \cdots + a_n$$

Special Vectors (contd.)

Sum Vector

$$\mathbf{1} \cdot \mathbf{a} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Vector Combinations

Vector Combinations

Combinations

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2,

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*,

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*,

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*, if $\alpha_1, \alpha_2 \geq 0$.

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*, if $\alpha_1, \alpha_2 \geq 0$.

The combination is said to be *affine*,

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*, if $\alpha_1, \alpha_2 \geq 0$.

The combination is said to be *affine*, if $\alpha_1 + \alpha_2 = 1$.

Vector Combinations

Combinations

Given two vectors **x** and **y** and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors **x** and **y**.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*, if $\alpha_1, \alpha_2 \geq 0$.

The combination is said to be *affine*, if $\alpha_1 + \alpha_2 = 1$.

Observation

Vector Combinations

Combinations

Given two vectors $\mathbf{x}$ and $\mathbf{y}$ and two scalars α_1 and α_2, $\alpha_1 \cdot \mathbf{x} + \alpha_2 \cdot \mathbf{y}$ is called a linear combination of the vectors $\mathbf{x}$ and $\mathbf{y}$.

The combination is said to be *convex*, if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

The combination is said to be *conical*, if $\alpha_1, \alpha_2 \geq 0$.

The combination is said to be *affine*, if $\alpha_1 + \alpha_2 = 1$.

Observation

The above notions can be generalized to vector sets of 3 *or more vectors.*

Vectors

Vectors

Linear Dependence and Independence

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**,

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Affine Dependence

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Affine Dependence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **affinely dependent**,

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Affine Dependence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **affinely dependent**, if it is linearly dependent **and** the scalars establishing the linear dependence sum to 0.

Vectors

Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent**, if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0}$$

If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Affine Dependence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **affinely dependent**, if it is linearly dependent **and** the scalars establishing the linear dependence sum to 0.

A set of vectors which is not affinely dependent, is said to be affinely independent.

Linear Dependence

Linear Dependence

Example

Linear Dependence

Example

Prove that the following vectors are linearly dependent:

Linear Dependence

Example

Prove that the following vectors are linearly dependent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

Linear Dependence

Example

Prove that the following vectors are linearly dependent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$2\mathbf{a}_1 + 3\mathbf{a}_2 - 1\mathbf{a}_3 = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Vectors

Vectors

Example

Vectors

Example

Prove that the following vectors are linearly independent:

Vectors

Example

Prove that the following vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Vectors

Example

Prove that the following vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Consider the equation

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Vectors

Example

Prove that the following vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Consider the equation

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned}$$

Vectors

Example

Prove that the following vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Consider the equation

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned}$$

We can see that the only solution is $\alpha_1 = \alpha_2 = 0$. This means $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent.

Exercise

Exercise

Exercise

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Observe that $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$ and hence the vectors are linearly dependent.

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Observe that $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$ and hence the vectors are linearly dependent.

Assume that they are affinely dependent.

Exercise

Exercise

Give an example of a vector set which is linearly dependent, but not affinely dependent.

Example

Consider the following set of vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Observe that $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$ and hence the vectors are linearly dependent.

Assume that they are affinely dependent. Then, as per the definition of affine independence, there must exist scalars, $\alpha_1, \alpha_2, \alpha_3$ not all 0, such that $\alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} = \mathbf{0}$ and $\sum_{i=1}^{3} \alpha_i = 0$.

Example (contd.)

Example (contd.)

Example

Example (contd.)

Example

Observe that,

Example (contd.)

Example

Observe that,

$$\alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} = \mathbf{0}$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix}1\\0\end{pmatrix} + \alpha_2 \cdot \begin{pmatrix}0\\1\end{pmatrix} + \alpha_3 \cdot \begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$

$$\alpha_2 + \alpha_3 = 0$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$
$$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$$

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \binom{1}{0} + \alpha_2 \cdot \binom{0}{1} + \alpha_3 \cdot \binom{1}{1} &= \binom{0}{0} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$
$$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$$

Finally, since $\sum_{i=1}^{3} \alpha_i = 0$,

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$
$$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$$

Finally, since $\sum_{i=1}^{3} \alpha_i = 0$, we have, $-\alpha_3 - \alpha_3 + \alpha_3 = 0$.

Example (contd.)

Example

Observe that,

$$\begin{aligned} \alpha_1 \cdot \mathbf{a} + \alpha_2 \cdot \mathbf{b} + \alpha_3 \cdot \mathbf{c} &= \mathbf{0} \\ \Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This means that

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$$
$$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$$

Finally, since $\sum_{i=1}^{3} \alpha_i = 0$, we have, $-\alpha_3 - \alpha_3 + \alpha_3 = 0$.

It follows that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and the vectors are affinely independent.

Vectors

Vectors

Spanning Sets and Bases

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**,

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**, if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**, if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

In other words, if $\mathbf{v} \in E^n$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\mathbf{v} = \alpha_1 \cdot \mathbf{b}_1 + \alpha_2 \cdot \mathbf{b}_2 + \cdots + \alpha_p \cdot \mathbf{b}_p$.

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**, if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

In other words, if $\mathbf{v} \in E^n$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\mathbf{v} = \alpha_1 \cdot \mathbf{b}_1 + \alpha_2 \cdot \mathbf{b}_2 + \cdots + \alpha_p \cdot \mathbf{b}_p$.

We say that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in E^n$ are called a **basis** for E^n if they are linearly independent, and form a spanning set for E^n.

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**, if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

In other words, if $\mathbf{v} \in E^n$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\mathbf{v} = \alpha_1 \cdot \mathbf{b}_1 + \alpha_2 \cdot \mathbf{b}_2 + \cdots + \alpha_p \cdot \mathbf{b}_p$.

We say that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in E^n$ are called a **basis** for E^n if they are linearly independent, and form a spanning set for E^n.

Note that a basis is a minimal spanning set.

Vectors

Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are called a **spanning set**, if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

In other words, if $\mathbf{v} \in E^n$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\mathbf{v} = \alpha_1 \cdot \mathbf{b}_1 + \alpha_2 \cdot \mathbf{b}_2 + \cdots + \alpha_p \cdot \mathbf{b}_p$.

We say that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in E^n$ are called a **basis** for E^n if they are linearly independent, and form a spanning set for E^n.

Note that a basis is a minimal spanning set.

This is because adding a new vector would make the set linearly dependent and removing one of the vectors would mean the remaining ones no longer span E^n.

Outline

Matrices

Matrices

Definition

Matrices

Definition

A **matrix** is a rectangular array of numbers.

Matrices

Definition

A **matrix** is a rectangular array of numbers.

We represent them by uppercase boldface type with m rows and n columns.

Matrices

Definition

A **matrix** is a rectangular array of numbers.

We represent them by uppercase boldface type with *m* rows and *n* columns.

The **order** of a matrix is the number of rows and columns of the matrix, so the example below would be an *m* x *n* matrix.

Matrices

Definition

A **matrix** is a rectangular array of numbers.

We represent them by uppercase boldface type with *m* rows and *n* columns.

The **order** of a matrix is the number of rows and columns of the matrix, so the example below would be an *m* x *n* matrix.

Example

Matrices

Definition

A **matrix** is a rectangular array of numbers.

We represent them by uppercase boldface type with *m* rows and *n* columns.

The **order** of a matrix is the number of rows and columns of the matrix, so the example below would be an *m* x *n* matrix.

Example

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Matrices

Matrices

Matrix Addition

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j.

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Example

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \\ 9 & 2 \end{pmatrix}$$

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \\ 9 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 2 \\ 4 & 8 & 9 \end{pmatrix}$$

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \\ 9 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 2 \\ 4 & 8 & 9 \end{pmatrix}$$

$\mathbf{A} + \mathbf{C}$ is undefined

Matrices

Matrix Addition

If two matrices are of the same order, then we can add them together.

To add two matrices, we add the elements in each corresponding position.

For example, if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $c_{i,j} = a_{i,j} + b_{i,j}$ for every i and j. Matrix addition satisfies both the commutative and associative laws.

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \\ 9 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 2 \\ 4 & 8 & 9 \end{pmatrix}$$

$\mathbf{A} + \mathbf{C}$ is undefined
$\mathbf{B} + \mathbf{C}$ is undefined

Matrices

Matrices

Multiplication by a Scalar

Matrices

Multiplication by a Scalar

As with vectors, if we have a scalar α and a matrix **A**, the product $\alpha \cdot \mathbf{A}$ is obtained by multiplying each element $a_{i,j}$ by α.

Matrices

Multiplication by a Scalar

As with vectors, if we have a scalar α and a matrix $\mathbf{A}$, the product $\alpha \cdot \mathbf{A}$ is obtained by multiplying each element $a_{i,j}$ by α.

$$\alpha \cdot \mathbf{A} = \begin{pmatrix} \alpha \cdot a_{1,1} & \alpha \cdot a_{1,2} & \cdots & \alpha \cdot a_{1,n} \\ \alpha \cdot a_{2,1} & \alpha \cdot a_{2,2} & \cdots & \alpha \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \cdot a_{m,1} & \alpha \cdot a_{m,2} & \cdots & \alpha \cdot a_{m,n} \end{pmatrix}$$

Matrices

Multiplication by a Scalar

As with vectors, if we have a scalar α and a matrix $\mathbf{A}$, the product $\alpha \cdot \mathbf{A}$ is obtained by multiplying each element $a_{i,j}$ by α.

$$\alpha \cdot \mathbf{A} = \begin{pmatrix} \alpha \cdot a_{1,1} & \alpha \cdot a_{1,2} & \cdots & \alpha \cdot a_{1,n} \\ \alpha \cdot a_{2,1} & \alpha \cdot a_{2,2} & \cdots & \alpha \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \cdot a_{m,1} & \alpha \cdot a_{m,2} & \cdots & \alpha \cdot a_{m,n} \end{pmatrix}$$

Example

$$\beta = 3, \quad \mathbf{A} = \begin{pmatrix} 8 & 3 \\ -1 & 2 \\ 7 & 1 \end{pmatrix} \quad \beta \cdot \mathbf{A} = 3 \cdot \begin{pmatrix} 8 & 3 \\ -1 & 2 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 24 & 9 \\ -3 & 6 \\ 21 & 3 \end{pmatrix}$$

Matrices

Matrices

Matrix Multiplication

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

If **A** is an m x n matrix, and **B** is a p x q matrix, then $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is defined as an m x q matrix if and only if $n = p$.

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

If **A** is an m x n matrix, and **B** is a p x q matrix, then $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is defined as an m x q matrix if and only if $n = p$.

Each element in **C** is given by $c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$, where n is the number of columns of **A** (or rows of **B**).

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

If **A** is an m x n matrix, and **B** is a p x q matrix, then $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is defined as an m x q matrix if and only if $n = p$.

Each element in **C** is given by $c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$, where n is the number of columns of **A** (or rows of **B**).

Observe that i varies from 1 through m, where m is the number of rows of **A**.

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

If **A** is an m x n matrix, and **B** is a p x q matrix, then $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is defined as an m x q matrix if and only if $n = p$.

Each element in **C** is given by $c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$, where n is the number of columns of **A** (or rows of **B**).

Observe that i varies from 1 through m, where m is the number of rows of **A**. Likewise, j varies from 1 through q, where q is the number of columns of **B**.

Matrices

Matrix Multiplication

Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** is equal to the number of rows in **B**.

If **A** is an m x n matrix, and **B** is a p x q matrix, then $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is defined as an m x q matrix if and only if $n = p$.

Each element in **C** is given by $c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$, where n is the number of columns of **A** (or rows of **B**).

Observe that i varies from 1 through m, where m is the number of rows of **A**. Likewise, j varies from 1 through q, where q is the number of columns of **B**.

Matrix multiplication satisfies the associative and distributive laws, but it does **not** satisfy the commutative law in general.

Matrix Multiplication example

Matrix Multiplication example

Example

Matrix Multiplication example

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix}$$

Matrix Multiplication example

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} =$$

Matrix Multiplication example

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix} =$$

Matrix Multiplication example

Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 6 & 53 \\ 8 & 7 & 16 \\ 4 & 2 & 14 \end{pmatrix}$$

Matrices

Matrices

Special Matrices

Matrices

Special Matrices

Diagonal Matrix

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1.

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Null or Zero Matrix

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Null or Zero Matrix - All elements are equal to zero and is denoted as $\mathbf{0}$.

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Null or Zero Matrix - All elements are equal to zero and is denoted as $\mathbf{0}$. Note that this does not have to be a square matrix.

Matrices

Special Matrices

Diagonal Matrix - A square matrix ($m = n$) whose entries that are not on the diagonal are zero.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

Identity Matrix - A diagonal matrix where all diagonal elements are equal to 1. We denote this matrix as $\mathbf{I}_m$ or $\mathbf{I}$.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Null or Zero Matrix - All elements are equal to zero and is denoted as $\mathbf{0}$. Note that this does not have to be a square matrix.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrices

Special Matrices (Contd.)

Matrix Transpose

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of $\mathbf{A}$, denoted as $\mathbf{A}^t$, is a reordering of $\mathbf{A}$ by interchanging the rows and columns.

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of **A**, denoted as $\mathbf{A}^t$, is a reordering of **A** by interchanging the rows and columns. For example, row 1 of **A** would be column 1 of $\mathbf{A}^t$.

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of **A**, denoted as $\mathbf{A}^t$, is a reordering of **A** by interchanging the rows and columns. For example, row 1 of **A** would be column 1 of $\mathbf{A}^t$.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of **A**, denoted as $\mathbf{A}^t$, is a reordering of **A** by interchanging the rows and columns. For example, row 1 of **A** would be column 1 of $\mathbf{A}^t$.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \quad \mathbf{A}^t = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of **A**, denoted as $\mathbf{A}^t$, is a reordering of **A** by interchanging the rows and columns. For example, row 1 of **A** would be column 1 of $\mathbf{A}^t$.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \quad \mathbf{A}^t = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

Symmetric Matrix - A matrix **A**, where $\mathbf{A} = \mathbf{A}^t$.

Matrices

Special Matrices (Contd.)

Matrix Transpose - The transpose of $\mathbf{A}$, denoted as $\mathbf{A}^t$, is a reordering of $\mathbf{A}$ by interchanging the rows and columns. For example, row 1 of $\mathbf{A}$ would be column 1 of $\mathbf{A}^t$.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \quad \mathbf{A}^t = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

Symmetric Matrix - A matrix $\mathbf{A}$, where $\mathbf{A} = \mathbf{A}^t$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

Matrices

Matrices

Special Matrices (Contd.)

Matrices

Special Matrices (Contd.)

Augmented Matrix

Matrices

Special Matrices (Contd.)

Augmented Matrix - A matrix where the rows and columns of another matrix are appended to the original matrix.

Matrices

Special Matrices (Contd.)

Augmented Matrix - A matrix where the rows and columns of another matrix are appended to the original matrix. If $\mathbf{A}$ is augmented with $\mathbf{B}$, we get $(\mathbf{A}, \mathbf{B})$ or $(\mathbf{A}|\mathbf{B})$.

Matrices

Special Matrices (Contd.)

Augmented Matrix - A matrix where the rows and columns of another matrix are appended to the original matrix. If **A** is augmented with **B**, we get $(\mathbf{A}, \mathbf{B})$ or $(\mathbf{A}|\mathbf{B})$.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 5 & 6 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 9 \end{pmatrix} \quad (\mathbf{A}\,|\,\mathbf{B}) = \left(\begin{array}{cc|cc} 1 & 4 & 3 & 2 \\ 5 & 6 & 1 & 9 \end{array}\right)$$

Matrices

Matrices

Determinants

Matrices

Determinants

Given a square matrix **A**, the **determinant** denoted by $|\mathbf{A}|$ or $det(\mathbf{A})$ is a number associated with **A**.

Matrices

Determinants

Given a square matrix **A**, the **determinant** denoted by $|\mathbf{A}|$ or $det(\mathbf{A})$ is a number associated with **A**.

The determinant of a 1 x 1 matrix: $|a_{1,1}| = a_{1,1}$

Matrices

Determinants

Given a square matrix **A**, the **determinant** denoted by $|\mathbf{A}|$ or $det(\mathbf{A})$ is a number associated with **A**.

The determinant of a 1 x 1 matrix: $|a_{1,1}| = a_{1,1}$

The determinant of a 2 x 2 matrix: $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1} \cdot a_{2,2} - a_{1,2} \cdot a_{2,1}$

Matrices

Determinants

Given a square matrix **A**, the **determinant** denoted by $|\mathbf{A}|$ or $det(\mathbf{A})$ is a number associated with **A**.

The determinant of a 1 x 1 matrix: $|a_{1,1}| = a_{1,1}$

The determinant of a 2 x 2 matrix: $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1} \cdot a_{2,2} - a_{1,2} \cdot a_{2,1}$

Cofactor of a matrix

Cofactor of a matrix

Cofactor

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Example

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Example

Let

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & 0 \\ 3 & 2 & 1 \\ 8 & 1 & -4 \end{bmatrix}$$

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Example

Let

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & 0 \\ 3 & 2 & 1 \\ 8 & 1 & -4 \end{bmatrix}$$

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Example

Let

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & 0 \\ 3 & 2 & 1 \\ 8 & 1 & -4 \end{bmatrix}$$

The cofactor for $a_{2,1} = 3$ is

Cofactor of a matrix

Cofactor

Every element of a determinant, except for a 1 x 1 matrix, has an associated **minor**.

To get the minor, we remove the row and column corresponding to the element and find the determinant of the new matrix.

We denote the minor of and element $a_{i,j}$ in matrix **A** as $|\mathbf{A}_{i,j}|$.

The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ associated to it.

Example

Let

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & 0 \\ 3 & 2 & 1 \\ 8 & 1 & -4 \end{bmatrix}$$

The cofactor for $a_{2,1} = 3$ is

$$(-1)^{2+1} \cdot |\mathbf{A}_{2,1}| = (-1) \begin{vmatrix} -1 & 0 \\ 1 & -4 \end{vmatrix} = -4$$

Matrices

Matrices

Computing Determinants

Matrices

Computing Determinants

The determinant of a matrix **A** of order n is found by expanding along a row or a column.

Matrices

Computing Determinants

The determinant of a matrix **A** of order n is found by expanding along a row or a column.

For any row i, this would be

Matrices

Computing Determinants

The determinant of a matrix **A** of order n is found by expanding along a row or a column.

For any row i, this would be

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{i,j} \cdot (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$$

Matrices

Computing Determinants

The determinant of a matrix **A** of order n is found by expanding along a row or a column.

For any row i, this would be

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{i,j} \cdot (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$$

and for any column j, this would be

Matrices

Computing Determinants

The determinant of a matrix **A** of order n is found by expanding along a row or a column.

For any row i, this would be

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{i,j} \cdot (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$$

and for any column j, this would be

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{i,j} \cdot (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$$

Matrices

Matrices

Example

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$|\mathbf{A}| \quad =$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$|\mathbf{A}| = 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$|\mathbf{A}| \quad = 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} +$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$|\mathbf{A}| = 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 5 \cdot (-1)^{3}$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$\begin{aligned} |\mathbf{A}| &= 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 5 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} \\ &= 3 \cdot (6) - 2 \cdot (-1) + 5 \cdot (-8) = \end{aligned}$$

Matrices

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 3 & 5 \end{bmatrix}.$$

Compute $|\mathbf{A}|$.
Expanding along column 3, we get

$$\begin{aligned} |\mathbf{A}| &= 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 5 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} \\ &= 3 \cdot (6) - 2 \cdot (-1) + 5 \cdot (-8) = -20 \end{aligned}$$

Matrices

Matrices

Computing Determinants (contd.)

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

1. If every element of a row of **A** is zero, then $det(\mathbf{A}) = 0$.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

1. If every element of a row of **A** is zero, then $det(\mathbf{A}) = 0$.
2. If two rows of **A** have elements that are proportional to one another, then $det(\mathbf{A}) = 0$.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

1. If every element of a row of **A** is zero, then $det(\mathbf{A}) = 0$.
2. If two rows of **A** have elements that are proportional to one another, then $det(\mathbf{A}) = 0$.
3. If two rows of a determinant are interchanged, the $det(\mathbf{A})$ changes sign.
4. Elements of any row may be multiplied by a nonzero constant if the entire determinant is multiplied by the reciprocal of the constant.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

1. If every element of a row of **A** is zero, then $det(\mathbf{A}) = 0$.
2. If two rows of **A** have elements that are proportional to one another, then $det(\mathbf{A}) = 0$.
3. If two rows of a determinant are interchanged, the $det(\mathbf{A})$ changes sign.
4. Elements of any row may be multiplied by a nonzero constant if the entire determinant is multiplied by the reciprocal of the constant.
5. To the elements of any row, you may add a constant times the corresponding element of any other row without changing the determinant.

Matrices

Computing Determinants (contd.)

The expansion of determinants can become complex for larger orders.

Let **A** denote a matrix and let $det(\mathbf{A})$ denote its determinant.

We can simplify the process by utilizing five properties.

1. If every element of a row of **A** is zero, then $det(\mathbf{A}) = 0$.
2. If two rows of **A** have elements that are proportional to one another, then $det(\mathbf{A}) = 0$.
3. If two rows of a determinant are interchanged, the $det(\mathbf{A})$ changes sign.
4. Elements of any row may be multiplied by a nonzero constant if the entire determinant is multiplied by the reciprocal of the constant.
5. To the elements of any row, you may add a constant times the corresponding element of any other row without changing the determinant.

Note that we can interchange the words “row” and “column”.

Matrices

Matrices

Adjoint

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Example

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Example

Let $\gamma_{i,j} = (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$ be the cofactor for $a_{i,j}$.

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Example

Let $\gamma_{i,j} = (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$ be the cofactor for $a_{i,j}$. Then,

Matrices

Adjoint

If **A** is a square matrix, the **adjoint** of **A**, denoted as $\mathbf{A}^{\alpha}$, can be found using the following process:

1. Replace each element $a_{i,j}$ of **A** by its cofactor.
2. Take the transpose of the matrix of cofactors found in step 1.

Example

Let $\gamma_{i,j} = (-1)^{i+j} \cdot |\mathbf{A}_{i,j}|$ be the cofactor for $a_{i,j}$. Then,

$$\mathbf{A}^{\alpha} = \begin{pmatrix} \gamma_{1,1} & \gamma_{2,1} & \cdots & \gamma_{n,1} \\ \gamma_{1,2} & \gamma_{2,2} & \cdots & \gamma_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1,n} & \gamma_{2,n} & \cdots & \gamma_{n,n} \end{pmatrix}$$

Matrices

Matrices

Inverse

Matrices

Inverse

The **inverse** of a square matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$.

Matrices

Inverse

The **inverse** of a square matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$.

For a matrix to have an inverse, it must be nonsingular, i.e., its determinant cannot be zero.

Matrices

Inverse

The **inverse** of a square matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$.

For a matrix to have an inverse, it must be nonsingular, i.e., its determinant cannot be zero.

Given a nonsingular matrix $\mathbf{A}$, we find the inverse by

Matrices

Inverse

The **inverse** of a square matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$.

For a matrix to have an inverse, it must be nonsingular, i.e., its determinant cannot be zero.

Given a nonsingular matrix $\mathbf{A}$, we find the inverse by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{\alpha}$$

Inverse (example)

Inverse (example)

Example

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \qquad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{\alpha} =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{\alpha} = \frac{1}{4} \cdot \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix} =$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{\alpha} = \frac{1}{4} \cdot \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Inverse (example)

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad |\mathbf{A}| = 2 \cdot (5) - 1 \cdot (6) = 10 - 6 = 4$$

$$cof(\mathbf{A}) = \begin{pmatrix} |5| & -|6| \\ -|1| & |2| \end{pmatrix}$$

$$\mathbf{A}^{\alpha} = \begin{pmatrix} |5| & -|1| \\ -|6| & |2| \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{\alpha} = \frac{1}{4} \cdot \begin{pmatrix} 5 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

We can then check $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$ to make sure we are correct.

Matrices

Matrices

Gauss-Jordan Elimination

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

The idea is to augment the matrix with the identity matrix and then perform elementary row operations.

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

The idea is to augment the matrix with the identity matrix and then perform elementary row operations.

Elementary Row Operations

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

The idea is to augment the matrix with the identity matrix and then perform elementary row operations.

Elementary Row Operations

1. Interchange a row i with a row j.

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

The idea is to augment the matrix with the identity matrix and then perform elementary row operations.

Elementary Row Operations

1. Interchange a row *i* with a row *j*.
2. Multiply a row *i* by a nonzero scalar α.

Matrices

Gauss-Jordan Elimination

This is another method for computing the inverse of a matrix.

The idea is to augment the matrix with the identity matrix and then perform elementary row operations.

Elementary Row Operations

1. Interchange a row *i* with a row *j*.
2. Multiply a row *i* by a nonzero scalar α.
3. Replace a row *i* by a row *i* plus a multiple of some row *j*.

Matrices

Matrices

Finding the Inverse of a matrix using Gauss-Jordan Elimination

Matrices

Finding the Inverse of a matrix using Gauss-Jordan Elimination

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad (\mathbf{A} \,|\, \mathbf{I}) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array} \right)$$

Matrices

Finding the Inverse of a matrix using Gauss-Jordan Elimination

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad (\mathbf{A} \,|\, \mathbf{I}) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array} \right)$$

Multiply the first row by $\frac{1}{2}$:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 5 & 0 & 1 \end{array} \right)$$

Matrices

Finding the Inverse of a matrix using Gauss-Jordan Elimination

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad (\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array} \right)$$

Multiply the first row by $\frac{1}{2}$:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 5 & 0 & 1 \end{array} \right)$$

Multiply the first row by -6 and add the result to the second row:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & -3 & 1 \end{array} \right)$$

Matrices

Finding the Inverse of a matrix using Gauss-Jordan Elimination

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 5 \end{pmatrix} \quad (\mathbf{A}\,|\,\mathbf{I}) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right)$$

Multiply the first row by $\frac{1}{2}$:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 5 & 0 & 1 \end{array}\right)$$

Multiply the first row by -6 and add the result to the second row:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & -3 & 1 \end{array}\right)$$

Multiply the second row by $\frac{1}{2}$:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$$

Matrices

Matrices

Gauss-Jordan Elimination (Contd.)

Matrices

Gauss-Jordan Elimination (Contd.)

Multiply the second row by $-\frac{1}{2}$ and add the result to the first row:

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{5}{2} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$$

Matrices

Matrices

Rank of a Matrix

Matrices

Rank of a Matrix

The **rank** of an m x n matrix **A**, denoted as $r(\mathbf{A})$, is the maximum number of linearly independent columns (or rows) of **A**.

Matrices

Rank of a Matrix

The **rank** of an m x n matrix **A**, denoted as $r(\mathbf{A})$, is the maximum number of linearly independent columns (or rows) of **A**.

By definition, $r(\mathbf{A}) \leq \min\{m, n\}$.

Matrices

Rank of a Matrix

The **rank** of an m x n matrix **A**, denoted as $r(\mathbf{A})$, is the maximum number of linearly independent columns (or rows) of **A**.

By definition, $r(\mathbf{A}) \leq \min\{m, n\}$. If $r(\mathbf{A}) = \min\{m, n\}$, then **A** is said to be of **full rank**.

Matrices

Rank of a Matrix

The **rank** of an m x n matrix **A**, denoted as $r(\mathbf{A})$, is the maximum number of linearly independent columns (or rows) of **A**.

By definition, $r(\mathbf{A}) \leq \min\{m, n\}$. If $r(\mathbf{A}) = \min\{m, n\}$, then **A** is said to be of **full rank**.

There are several ways to get the rank, but the method used here will use elementary row operations to get

$$\left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{D} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

Matrices

Rank of a Matrix

The **rank** of an m x n matrix **A**, denoted as $r(\mathbf{A})$, is the maximum number of linearly independent columns (or rows) of **A**.

By definition, $r(\mathbf{A}) \leq \min\{m, n\}$. If $r(\mathbf{A}) = \min\{m, n\}$, then **A** is said to be of **full rank**.

There are several ways to get the rank, but the method used here will use elementary row operations to get

$$\left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{D} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

This shows that $r(\mathbf{A}) = k$.

Rank (example)

Rank (example)

Example

Rank (example)

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 3 & 1 & 9 & 5 \end{pmatrix}$$

Rank (example)

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 3 & 1 & 9 & 5 \end{pmatrix}$$

Reduced form of $\mathbf{A} =$

Rank (example)

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 3 & 1 & 9 & 5 \end{pmatrix}$$

$$\text{Reduced form of } \mathbf{A} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) =$$

Rank (example)

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 3 & 1 & 9 & 5 \end{pmatrix}$$

$$\text{Reduced form of } \mathbf{A} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I}_2 & \mathbf{D} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

Rank (example)

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 3 & 1 & 9 & 5 \end{pmatrix}$$

$$\text{Reduced form of } \mathbf{A} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I}_2 & \mathbf{D} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

This means that the rank of **A** is 2.

Outline

Linear Equations

Linear Equations

Equations

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Example

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Example

$$a_{1,1} \cdot x_1 \quad + \quad a_{1,2} \cdot x_2 \quad + \quad \cdots \quad + \quad a_{1,n} \cdot x_n \quad = \quad b_1$$

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Example

$$\begin{array}{ccccccccc} a_{1,1} \cdot x_1 & + & a_{1,2} \cdot x_2 & + & \cdots & + & a_{1,n} \cdot x_n & = & b_1 \\ a_{2,1} \cdot x_1 & + & a_{2,2} \cdot x_2 & + & \cdots & + & a_{2,n} \cdot x_n & = & b_2 \end{array}$$

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Example

$$\begin{array}{ccccccccc}
a_{1,1}\cdot x_1 & + & a_{1,2}\cdot x_2 & + & \cdots & + & a_{1,n}\cdot x_n & = & b_1 \\
a_{2,1}\cdot x_1 & + & a_{2,2}\cdot x_2 & + & \cdots & + & a_{2,n}\cdot x_n & = & b_2 \\
 & & & & & & & & \vdots \\
a_{m,1}\cdot x_1 & + & a_{m,2}\cdot x_2 & + & \cdots & + & a_{m,n}\cdot x_n & = & b_m
\end{array}$$

Linear Equations

Equations

One of the best known uses for matrices and determinants is for solving simultaneous linear equations.

Matrices and vectors give us a nice method for expressing the problem.

Example

$$\begin{array}{ccccccccc}
a_{1,1} \cdot x_1 & + & a_{1,2} \cdot x_2 & + & \cdots & + & a_{1,n} \cdot x_n & = & b_1 \\
a_{2,1} \cdot x_1 & + & a_{2,2} \cdot x_2 & + & \cdots & + & a_{2,n} \cdot x_n & = & b_2 \\
 & & & & & & & & \vdots \\
a_{m,1} \cdot x_1 & + & a_{m,2} \cdot x_2 & + & \cdots & + & a_{m,n} \cdot x_n & = & b_m
\end{array}$$

Linear Equations

Linear Equations

Example

Linear Equations

Example

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Linear Equations

Example

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear Equations

Example

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Linear Equations

Linear Equations

Solutions

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

1. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

1. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists. This is because if the addition of $\mathbf{b}$ to the columns of $\mathbf{A}$, increases the rank of the set by 1, then $\mathbf{b}$ cannot be written as a linear combination of the columns of $\mathbf{A}$.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that **b** can be written as a linear combination of the columns of **A**.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

1. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists. This is because if the addition of **b** to the columns of **A**, increases the rank of the set by 1, then **b** cannot be written as a linear combination of the columns of **A**.
2. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A})$, then there does exist a solution.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

1. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists. This is because if the addition of $\mathbf{b}$ to the columns of $\mathbf{A}$, increases the rank of the set by 1, then $\mathbf{b}$ cannot be written as a linear combination of the columns of $\mathbf{A}$.
2. If $r(\mathbf{A} \,|\, \mathbf{b}) = r(\mathbf{A})$, then there does exist a solution. This is because we can write $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$.

Linear Equations

Solutions

The set of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has either no solution, a unique solution, or an infinite number of solutions.

When determining if a solution exists, we are trying to find scalars $x_1, x_2, \ldots, x_n$ so that $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.

Conditions where a solution exists for $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$:

1. If $r(\mathbf{A} \mid \mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists. This is because if the addition of $\mathbf{b}$ to the columns of $\mathbf{A}$, increases the rank of the set by 1, then $\mathbf{b}$ cannot be written as a linear combination of the columns of $\mathbf{A}$.
2. If $r(\mathbf{A} \mid \mathbf{b}) = r(\mathbf{A})$, then there does exist a solution. This is because we can write $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$.

Linear Equations

Linear Equations

A Unique Solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

Linear Equations

A Unique Solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

There are several methods for solving for a unique solution, including Cramer's rule and Gaussian elimination.

Linear Equations

A Unique Solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

There are several methods for solving for a unique solution, including Cramer's rule and Gaussian elimination.

We will first use Cramer's rule; however, we should note that this is not an efficient approach computationally.

Linear Equations

A Unique Solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

There are several methods for solving for a unique solution, including Cramer's rule and Gaussian elimination.

We will first use Cramer's rule; however, we should note that this is not an efficient approach computationally. Let $\mathbf{A}_j$ be the matrix $\mathbf{A}$ where the jth column is replaced by $\mathbf{b}$.

Linear Equations

A Unique Solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

There are several methods for solving for a unique solution, including Cramer's rule and Gaussian elimination.

We will first use Cramer's rule; however, we should note that this is not an efficient approach computationally. Let $\mathbf{A}_j$ be the matrix $\mathbf{A}$ where the jth column is replaced by $\mathbf{b}$.

Cramer's rule states that the unique solution is given by $x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}$, for all $j = 1, \ldots, n$.

Linear Equations

Linear Equations

Using Cramer's Rule

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

Linear Equations

Using Cramer's Rule

$$\begin{aligned} 2 \cdot x_1 &+ & x_2 &+ & 2 \cdot x_3 &= & 6 \\ 2 \cdot x_1 &+ & 3 \cdot x_2 &+ & x_3 &= & 9 \\ x_1 &+ & x_2 &+ & x_3 &= & 3 \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2\cdot x_1 & + & x_2 & + & 2\cdot x_3 & = & 6 \\ 2\cdot x_1 & + & 3\cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcr} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_1 =$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcr} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 9 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{6}{1} = 6$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 9 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{6}{1} = 6 \quad x_2 =$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcr} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 9 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{6}{1} = 6 \quad x_2 = \frac{\begin{vmatrix} 2 & 6 & 2 \\ 2 & 9 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = 0$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 9 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{6}{1} = 6 \quad x_2 = \frac{\begin{vmatrix} 2 & 6 & 2 \\ 2 & 9 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = 0 \quad x_3 =$$

Linear Equations

Using Cramer's Rule

$$\begin{array}{rcrcrcl} 2 \cdot x_1 & + & x_2 & + & 2 \cdot x_3 & = & 6 \\ 2 \cdot x_1 & + & 3 \cdot x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 9 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{6}{1} = 6 \quad x_2 = \frac{\begin{vmatrix} 2 & 6 & 2 \\ 2 & 9 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = 0 \quad x_3 = \frac{\begin{vmatrix} 2 & 1 & 6 \\ 2 & 3 & 9 \\ 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = -3$$

Linear Equations

Linear Equations

Using Inverses

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$,

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$,

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} =$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix}$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\mathbf{x} =$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} =$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} =$$

Linear Equations

Using Inverses

Another approach to finding a unique solution is by using the inverse.

Given $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$, we can see that $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, which means that $\mathbf{I} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$, and hence, $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$$

Linear Equations

Linear Equations

Using Gauss-Jordan Elimination

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ to the form $(\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{b})$.

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ to the form $(\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{b})$.

Example

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ to the form $(\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} =$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} =$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

$$(\mathbf{A} \,|\, \mathbf{b}) =$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

$$(\mathbf{A} \,|\, \mathbf{b}) = \left(\begin{array}{ccc|c} 2 & 1 & 2 & 6 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

$$(\mathbf{A} \,|\, \mathbf{b}) = \left(\begin{array}{ccc|c} 2 & 1 & 2 & 6 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{array} \right)$$

Multiply the first row by $\frac{1}{2}$:

Linear Equations

Using Gauss-Jordan Elimination

Another approach is using the Gauss-Jordan elimination.

The idea is to use elementary row operations to reduce the augmented matrix $(\mathbf{A} \,|\, \mathbf{b})$ to the form $(\mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{b})$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$$

$$(\mathbf{A} \,|\, \mathbf{b}) = \left(\begin{array}{ccc|c} 2 & 1 & 2 & 6 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

Multiply the first row by $\frac{1}{2}$:

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 3 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

Linear Equations

Linear Equations

Example

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $\frac{1}{2}$:

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $\frac{1}{2}$:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $\frac{1}{2}$:

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $-\frac{1}{2}$ and add the result to both the first and third rows:

Linear Equations

Example

Multiply the first row by -2 and add the result to the second row, and multiply the first row by -1 and add the result to the third row:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $\frac{1}{2}$:

$$\left(\begin{array}{ccr|c} 1 & \frac{1}{2} & 1 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Multiply the second row by $-\frac{1}{2}$ and add the result to both the first and third rows:

$$\left(\begin{array}{ccr|r} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{1}{4} & -\frac{3}{4} \end{array}\right)$$

Linear Equations

Linear Equations

Example

Linear Equations

Example

Multiply the third row by 4:

Linear Equations

Example

Multiply the third row by 4:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Linear Equations

Example

Multiply the third row by 4:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Multiply the third row by $-\frac{5}{4}$ and add the result to the first row, and multiply the third row by $\frac{1}{2}$ and add the result to the second row:

Linear Equations

Example

Multiply the third row by 4:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Multiply the third row by $-\frac{5}{4}$ and add the result to the first row, and multiply the third row by $\frac{1}{2}$ and add the result to the second row:

$$(\mathbf{I} \,|\, \mathbf{A}^{-1} \cdot \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Linear Equations

Example

Multiply the third row by 4:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Multiply the third row by $-\frac{5}{4}$ and add the result to the first row, and multiply the third row by $\frac{1}{2}$ and add the result to the second row:

$$(\mathbf{I} \,|\, \mathbf{A}^{-1} \cdot \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$$

Linear Equations

Linear Equations

Infinite Number of Solutions

Linear Equations

Infinite Number of Solutions

This case is one of most interest since this scenario is the most likely to happen in linear programming.

Linear Equations

Infinite Number of Solutions

This case is one of most interest since this scenario is the most likely to happen in linear programming.

This happens when $r(\mathbf{A}) = r(\mathbf{A} \,|\, \mathbf{b}) < n$, where n is the number of variables.

Linear Equations

Infinite Number of Solutions

This case is one of most interest since this scenario is the most likely to happen in linear programming.

This happens when $r(\mathbf{A}) = r(\mathbf{A} \mid \mathbf{b}) < n$, where n is the number of variables.

Example

$$\begin{array}{rcrcrcr} 3 \cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

Linear Equations

Infinite Number of Solutions

This case is one of most interest since this scenario is the most likely to happen in linear programming.

This happens when $r(\mathbf{A}) = r(\mathbf{A} \,|\, \mathbf{b}) < n$, where n is the number of variables.

Example

$$\begin{aligned} 3 \cdot x_1 &+ x_2 - x_3 = 8 \\ x_1 &+ x_2 + x_3 = 4 \end{aligned}$$

We see that $r(\mathbf{A}) = r(\mathbf{A} \,|\, \mathbf{b}) = 2 < 3$, where

Linear Equations

Infinite Number of Solutions

This case is one of most interest since this scenario is the most likely to happen in linear programming.

This happens when $r(\mathbf{A}) = r(\mathbf{A}\,|\,\mathbf{b}) < n$, where n is the number of variables.

Example

$$\begin{array}{rcrcrcr} 3\cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

We see that $r(\mathbf{A}) = r(\mathbf{A}\,|\,\mathbf{b}) = 2 < 3$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and}\mathbf{b} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Linear Equations

Linear Equations

Infinite Number of Solutions (Contd.)

Linear Equations

Infinite Number of Solutions (Contd.)

For this case, we can choose r equations, where r is the rank, and find r of the variables in terms of the remaining $n - r$ variables.

Linear Equations

Infinite Number of Solutions (Contd.)

For this case, we can choose r equations, where r is the rank, and find r of the variables in terms of the remaining $n - r$ variables.

$$\begin{array}{rcrcrcl} 3 \cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

Linear Equations

Infinite Number of Solutions (Contd.)

For this case, we can choose r equations, where r is the rank, and find r of the variables in terms of the remaining $n - r$ variables.

$$\begin{array}{rcrcrcl} 3 \cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

Solving for x_1 and x_2 gets

$$\begin{array}{rcrcr} x_1 & = & 2 & + & x_3 \\ x_2 & = & 2 & - & 2 \cdot x_3 \end{array}$$

Linear Equations

Infinite Number of Solutions (Contd.)

For this case, we can choose r equations, where r is the rank, and find r of the variables in terms of the remaining $n - r$ variables.

$$\begin{array}{rcrcrcr} 3 \cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

Solving for x_1 and x_2 gets

$$\begin{array}{rcrcr} x_1 & = & 2 & + & x_3 \\ x_2 & = & 2 & - & 2 \cdot x_3 \end{array}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 + x_3 \\ 2 - 2 \cdot x_3 \\ x_3 \end{pmatrix}$$

Linear Equations

Infinite Number of Solutions (Contd.)

For this case, we can choose r equations, where r is the rank, and find r of the variables in terms of the remaining $n - r$ variables.

$$\begin{array}{rcrcrcr} 3 \cdot x_1 & + & x_2 & - & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

Solving for x_1 and x_2 gets

$$\begin{array}{rcrcr} x_1 & = & 2 & + & x_3 \\ x_2 & = & 2 & - & 2 \cdot x_3 \end{array}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 + x_3 \\ 2 - 2 \cdot x_3 \\ x_3 \end{pmatrix}$$

Although there are an infinite number of solutions, we will be concerned with only a finite number of them.