Duality

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

November 15, 2016

Outline

Linear Programming

A Motivating Example

A Motivating Example

Example

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$

$$x_1 - x_2 - x_3 + 3 \cdot x_4 \quad \leq \quad 1 \tag{2}$$

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \quad (1)$$

$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \quad (2)$$

$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \quad (3)$$

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$

$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$

$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$

$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$

$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$

$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$

$$- x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$

$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$
$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

Establishing bounds on z^*

A Motivating Example

Example

$$
\begin{aligned}
\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 & & & \quad (1)\\
x_1 - x_2 - x_3 + 3 \cdot x_4 &\leq & 1 & \quad (2)\\
5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 &\leq & 55 & \quad (3)\\
-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 &\leq & 3 & \quad (4)\\
x_1, x_2, x_3, x_4 &\geq & 0 & \quad (5)
\end{aligned}
$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$.

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \quad (1)$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \quad (2)$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \quad (3)$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \quad (4)$$
$$x_1, x_2, x_3, x_4 \geq 0 \quad (5)$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$
$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$
$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

How about an upper bound?

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$

$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$

$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$

$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$

$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

How about an upper bound? (3)+(4) gives

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \tag{1}$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \tag{2}$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \tag{3}$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \tag{4}$$
$$x_1, x_2, x_3, x_4 \geq 0 \tag{5}$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

How about an upper bound? (3)+(4) gives $4 \cdot x_1 + 3 \cdot x_2 + 6 \cdot x_3 + 3 \cdot x_4 \leq 58$.

A Motivating Example

Example

$$\max z = 4 \cdot x_1 + x_2 + 5 \cdot x_3 + 3 \cdot x_4 \quad (1)$$
$$x_1 - x_2 - x_3 + 3 \cdot x_4 \leq 1 \quad (2)$$
$$5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4 \leq 55 \quad (3)$$
$$-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4 \leq 3 \quad (4)$$
$$x_1, x_2, x_3, x_4 \geq 0 \quad (5)$$

Establishing bounds on z^*

Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

How about an upper bound? (3)+(4) gives $4 \cdot x_1 + 3 \cdot x_2 + 6 \cdot x_3 + 3 \cdot x_4 \leq 58$.

Can you conclude $z^* \leq 58$?

Finding bounds

Finding bounds

Establishing an upper bound

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) \quad \leq \quad y_1 \cdot (1)$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \end{aligned}$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \end{aligned}$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

$$(y_1 + 5 \cdot y_2 - y_3) \cdot x_1 +$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

$$(y_1 + 5 \cdot y_2 - y_3) \cdot x_1 + (-y_1 + y_2 + 2 \cdot y_3) \cdot x_2 +$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

$$(y_1 + 5 \cdot y_2 - y_3) \cdot x_1 + (-y_1 + y_2 + 2 \cdot y_3) \cdot x_2 +$$
$$(-y_1 + 3 \cdot y_2 + 3 \cdot y_3) \cdot x_3 +$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

$$(y_1 + 5 \cdot y_2 - y_3) \cdot x_1 + (-y_1 + y_2 + 2 \cdot y_3) \cdot x_2 + \\ (-y_1 + 3 \cdot y_2 + 3 \cdot y_3) \cdot x_3 + (3 \cdot y_1 + 8 \cdot y_2 - 5 \cdot y_3) \cdot x_4 \leq$$

Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

Formulate it as a linear program!

Multiplying the constraint equations by y_1, y_2, y_3, where the $y_i \geq 0$ (**Why?**), we get,

$$\begin{aligned} y_1 \cdot (x_1 - x_2 - x_3 + 3 \cdot x_4) &\leq y_1 \cdot (1) \\ y_2 \cdot (5 \cdot x_1 + x_2 + 3 \cdot x_3 + 8 \cdot x_4) &\leq y_2 \cdot (55) \\ y_3 \cdot (-x_1 + 2 \cdot x_2 + 3 \cdot x_3 - 5 \cdot x_4) &\leq y_3 \cdot (3) \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Summing up the above equations, we get,

$$\begin{aligned} (y_1 + 5 \cdot y_2 - y_3) \cdot x_1 + (-y_1 + y_2 + 2 \cdot y_3) \cdot x_2 + \\ (-y_1 + 3 \cdot y_2 + 3 \cdot y_3) \cdot x_3 + (3 \cdot y_1 + 8 \cdot y_2 - 5 \cdot y_3) \cdot x_4 \leq \quad & (y_1 + 55 \cdot y_2 + 3 \cdot y_3) \end{aligned}$$

Finding bounds (contd)

Finding bounds (contd)

Optimizing the bound

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z,

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

$$y_1 + 5 \cdot y_2 - y_3 \quad \geq \quad 4$$

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

$$\begin{aligned} y_1 + 5 \cdot y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2 \cdot y_3 &\geq 1 \end{aligned}$$

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

$$\begin{array}{rcl} y_1 + 5 \cdot y_2 - y_3 & \geq & 4 \\ -y_1 + y_2 + 2 \cdot y_3 & \geq & 1 \\ -y_1 + 3 \cdot y_2 + 3 \cdot y_3 & \geq & 5 \end{array}$$

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

$$
\begin{aligned}
y_1 + 5 \cdot y_2 - y_3 &\geq 4 \\
-y_1 + y_2 + 2 \cdot y_3 &\geq 1 \\
-y_1 + 3 \cdot y_2 + 3 \cdot y_3 &\geq 5 \\
3 \cdot y_1 + 8 \cdot y_2 - 5 \cdot y_3 &\geq 3
\end{aligned}
$$

Finding bounds (contd)

Optimizing the bound

In order to get the best bound on z, we must minimize $(y_1 + 55 \cdot y_2 + 3 \cdot y_3)$ so that,

$$\begin{aligned} y_1 + 5 \cdot y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2 \cdot y_3 &\geq 1 \\ -y_1 + 3 \cdot y_2 + 3 \cdot y_3 &\geq 5 \\ 3 \cdot y_1 + 8 \cdot y_2 - 5 \cdot y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Dual of the Canonical form

Linear Programming

Dual of the Canonical form

Dual

Dual of the Canonical form

Dual

Given the system

Dual of the Canonical form

Dual

Given the system (Primal)

Dual of the Canonical form

Dual

Given the system (Primal)

$$
\begin{aligned}
z = \max \mathbf{c} \cdot \mathbf{x} \\
\mathbf{A} \cdot \mathbf{x} &\leq \mathbf{b} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
$$

Dual of the Canonical form

Dual

Given the system (Primal)

$$z = \max \mathbf{c} \cdot \mathbf{x}$$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

the dual is defined as:

Dual of the Canonical form

Dual

Given the system (Primal)

$$
\begin{array}{rcl}
 & z = \max \mathbf{c} \cdot \mathbf{x} & \\
\mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\
\mathbf{x} & \geq & \mathbf{0}
\end{array}
$$

the dual is defined as:

$$
\begin{array}{rcl}
 & w = \min \mathbf{b} \cdot \mathbf{y} & \\
\mathbf{y} \cdot A & \geq & \mathbf{c} \\
\mathbf{y} & \geq & \mathbf{0}
\end{array}
$$

Dual of the Canonical form

Dual

Given the system (Primal)

$$\begin{array}{rcl} & z = \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

the dual is defined as:

$$\begin{array}{rcl} & w = \min \mathbf{b} \cdot \mathbf{y} & \\ \mathbf{y} \cdot A & \geq & \mathbf{c} \\ \mathbf{y} & \geq & \mathbf{0} \end{array}$$

The constraint system $\mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}$ can also be written as:

Dual of the Canonical form

Dual

Given the system (Primal)

$$
\begin{array}{rcc}
 & z = \max \mathbf{c} \cdot \mathbf{x} & \\
\mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\
\mathbf{x} & \geq & \mathbf{0}
\end{array}
$$

the dual is defined as:

$$
\begin{array}{rcc}
 & w = \min \mathbf{b} \cdot \mathbf{y} & \\
\mathbf{y} \cdot A & \geq & \mathbf{c} \\
\mathbf{y} & \geq & \mathbf{0}
\end{array}
$$

The constraint system $\mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}$ can also be written as: $\mathbf{A^T} \cdot \mathbf{y} \geq \mathbf{c}$.

Dual of the Standard Form

Linear Programming

Dual of the Standard Form

Dual (Standard)

Dual of the Standard Form

Dual (Standard)

Given the system

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

$$z = \max \mathbf{c} \cdot \mathbf{x}$$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

$$z = \max \mathbf{c} \cdot \mathbf{x}$$

$$\begin{array}{rcl} \mathbf{A} \cdot \mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

the dual is defined as:

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

$$
\begin{aligned}
& z = \max \mathbf{c} \cdot \mathbf{x} \\
\mathbf{A} \cdot \mathbf{x} & = \mathbf{b} \\
\mathbf{x} & \geq \mathbf{0}
\end{aligned}
$$

the dual is defined as:

$$
\begin{aligned}
& w = \min \mathbf{b} \cdot \mathbf{y} \\
\mathbf{y} \cdot \mathbf{A} & \geq \mathbf{c}
\end{aligned}
$$

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

$$
\begin{aligned}
z = \max \mathbf{c} \cdot \mathbf{x} \\
\mathbf{A} \cdot \mathbf{x} &= \mathbf{b} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
$$

the dual is defined as:

$$
\begin{aligned}
w = \min \mathbf{b} \cdot \mathbf{y} \\
\mathbf{y} \cdot \mathbf{A} &\geq \mathbf{c}
\end{aligned}
$$

The constraint system $\mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}$ can also be written as:

Dual of the Standard Form

Dual (Standard)

Given the system (Primal)

$$\begin{array}{rcl} & z = \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

the dual is defined as:

$$\begin{array}{rcl} & w = \min \mathbf{b} \cdot \mathbf{y} & \\ \mathbf{y} \cdot \mathbf{A} & \geq & \mathbf{c} \end{array}$$

The constraint system $\mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}$ can also be written as: $\mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c}$.

Dual of the Alternate Form

Dual of the Alternate Form

Dual (Alternate form)

Given the system

Dual of the Alternate Form

Dual (Alternate form)

Given the system (Primal)

Linear Programming

Dual of the Alternate Form

Dual (Alternate form)

Given the system (Primal)

$$
\begin{aligned}
z &= \max \mathbf{c} \cdot \mathbf{x} \\
\mathbf{A} \cdot \mathbf{x} &\leq \mathbf{b}
\end{aligned}
$$

Dual of the Alternate Form

Dual (Alternate form)

Given the system (Primal)

$$z = \max \mathbf{c} \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

the dual is defined as:

Dual of the Alternate Form

Dual (Alternate form)

Given the system (Primal)

$$
\begin{array}{rcl}
 & z = \max \mathbf{c} \cdot \mathbf{x} & \\
\mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b}
\end{array}
$$

the dual is defined as:

$$
\begin{array}{rcl}
 & w = \min \mathbf{b} \cdot \mathbf{y} & \\
\mathbf{y} \cdot \mathbf{A} & = & \mathbf{c} \\
\mathbf{y} & \geq & \mathbf{0}
\end{array}
$$

Example

Example

Example

Example

Example

Find the dual of:

Example

Example

Find the dual of:

$$\begin{aligned} \max 4 \cdot x_1 + 2 \cdot x_2 & & \\ x_1 + x_2 &\leq& 2 \\ x_1 + 2 \cdot x_2 &\leq& 15 \\ 2 \cdot x_1 - x_2 &\leq& 12 \\ x_1, x_2 &\geq& 0 \end{aligned}$$

Self-Involutoriness of Duality

Self-Involutoriness of Duality

Theorem

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

$$\begin{aligned} \min \mathbf{b} \cdot \mathbf{y} \\ \mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

$$\begin{aligned} \min \mathbf{b} \cdot \mathbf{y} \\ \mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

$$\begin{aligned} \min \mathbf{b} \cdot \mathbf{y} \\ \mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

The dual can be rewritten as:

$$\begin{aligned} -\max\ (-\mathbf{b}) \cdot \mathbf{y} \\ -\mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \leq -\mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The dual can be rewritten as:

$$\begin{aligned} -\max\,(-\mathbf{b}) \cdot \mathbf{y} \\ -\mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \leq -\mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

$$\begin{aligned} \min \mathbf{b} \cdot \mathbf{y} \\ \mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

As per definition, the dual of the dual is:

Self-Involutoriness of Duality

Theorem

The dual of the dual is the primal.

Proof.

Let the primal system be:

$$\begin{aligned} \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The dual can be rewritten as:

$$\begin{aligned} -\max\ (-\mathbf{b}) \cdot \mathbf{y} \\ -\mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \leq -\mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

As per definition, its dual is:

$$\begin{aligned} \min \mathbf{b} \cdot \mathbf{y} \\ \mathbf{A}^{\mathbf{T}} \cdot \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

As per definition, the dual of the dual is:

$$\begin{aligned} -\min\ -\mathbf{c} \cdot \mathbf{w} \\ (-\mathbf{A}^{\mathbf{T}})^{\mathbf{T}} \cdot \mathbf{w} \geq -\mathbf{b} \\ \mathbf{w} \geq \mathbf{0} \end{aligned}$$

□

Proof (contd.)

Proof (contd.)

Proof

Proof (contd.)

Proof

The dual of the dual is thus:

Proof (contd.)

Proof

The dual of the dual is thus:

$$\begin{aligned} \max\ \mathbf{c} \cdot \mathbf{w} \\ \mathbf{A} \cdot \mathbf{w} \leq \mathbf{b} \\ \mathbf{w} \geq \mathbf{0} \end{aligned}$$

Proof (contd.)

Proof

The dual of the dual is thus:

$$\begin{aligned} \max\ \mathbf{c} \cdot \mathbf{w} \\ \mathbf{A} \cdot \mathbf{w} \leq \mathbf{b} \\ \mathbf{w} \geq \mathbf{0} \end{aligned}$$

The theorem follows.

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}'$$

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ *and* $\mathbf{y}'$ *are any primal feasible and dual feasible solution respectively.*

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are any primal feasible and dual feasible solution respectively.

Proof

Since $\mathbf{x}'$ is primal feasible, we must have,

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are any primal feasible and dual feasible solution respectively.

Proof

Since $\mathbf{x}'$ is primal feasible, we must have, $\mathbf{A} \cdot \mathbf{x}' \leq \mathbf{b}$, $\mathbf{x}' \geq \mathbf{0}$.

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are any primal feasible and dual feasible solution respectively.

Proof

Since $\mathbf{x}'$ is primal feasible, we must have, $\mathbf{A} \cdot \mathbf{x}' \leq \mathbf{b}$, $\mathbf{x}' \geq \mathbf{0}$.

Since $\mathbf{y}'$ is dual feasible, we must have,

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ *and* $\mathbf{y}'$ *are any primal feasible and dual feasible solution respectively.*

Proof

Since $\mathbf{x}'$ is primal feasible, we must have, $\mathbf{A} \cdot \mathbf{x}' \leq \mathbf{b}$, $\mathbf{x}' \geq \mathbf{0}$.

Since $\mathbf{y}'$ is dual feasible, we must have, $\mathbf{y}' \cdot \mathbf{A} \geq \mathbf{c}$, $\mathbf{y}' \geq \mathbf{0}$.

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are any primal feasible and dual feasible solution respectively.

Proof

Since $\mathbf{x}'$ is primal feasible, we must have, $\mathbf{A} \cdot \mathbf{x}' \leq \mathbf{b}$, $\mathbf{x}' \geq \mathbf{0}$.

Since $\mathbf{y}'$ is dual feasible, we must have, $\mathbf{y}' \cdot \mathbf{A} \geq \mathbf{c}$, $\mathbf{y}' \geq \mathbf{0}$.

It follows that $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b}$ and $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \geq \mathbf{c} \cdot \mathbf{x}'$.

The Weak Duality theorem

Theorem

Given the primal and dual forms discussed above,

$$z = \mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b} = w$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are any primal feasible and dual feasible solution respectively.

Proof

Since $\mathbf{x}'$ is primal feasible, we must have, $\mathbf{A} \cdot \mathbf{x}' \leq \mathbf{b}$, $\mathbf{x}' \geq \mathbf{0}$.

Since $\mathbf{y}'$ is dual feasible, we must have, $\mathbf{y}' \cdot \mathbf{A} \geq \mathbf{c}$, $\mathbf{y}' \geq \mathbf{0}$.

It follows that $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b}$ and $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \geq \mathbf{c} \cdot \mathbf{x}'$.

The theorem follows.

Consequences of the weak duality theorem

Consequences of the weak duality theorem

Corollary

Linear Programming

Consequences of the weak duality theorem

Corollary

If the primal is unbounded,

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

If the dual is unbounded,

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

If the dual is unbounded, the primal is infeasible.

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

If the dual is unbounded, the primal is infeasible.

Example

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

If the dual is unbounded, the primal is infeasible.

Example

What is the primal dual relationship in the following linear program:

Consequences of the weak duality theorem

Corollary

If the primal is unbounded, the dual is infeasible.

Corollary

If the dual is unbounded, the primal is infeasible.

Example

What is the primal dual relationship in the following linear program:

$$\begin{array}{rcl} & \max x_1 + 2 \cdot x_2 & \\ -x_1 + 2 \cdot x_2 & \leq & -2 \\ x_1 - 2 \cdot x_2 & \leq & -2 \\ x_1, x_2 & \geq & 0 \end{array}$$

Optimality theorem from Weak duality

Optimality theorem from Weak duality

Corollary

Optimality theorem from Weak duality

Corollary

If $\mathbf{x}$ *is primal feasible and* $\mathbf{y}$ *is dual feasible, and* $\mathbf{c} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{b}$, *then* $\mathbf{x}$ *is primal optimal and* $\mathbf{y}$ *is dual optimal.*

The Strong Duality Theorem

The Strong Duality Theorem

Theorem

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

$$\begin{array}{rcl} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

$$\begin{array}{rcl} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

Let $\mathbf{B}$ denote the optimal basis of the primal in standard form.

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

$$\begin{array}{rcl} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

Let **B** denote the optimal basis of the primal in standard form.

Then the optimal point is $\mathbf{x} =$

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

$$\begin{aligned} & \max \mathbf{c} \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{x_s} &= \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} &\geq 0 \end{aligned}$$

Let $\mathbf{B}$ denote the optimal basis of the primal in standard form.

Then the optimal point is $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$ and the the optimal solution for the primal is $z =$

The Strong Duality Theorem

Theorem

Given the canonical forms of the primal and dual, if both the primal and the dual are feasible, then both have finite optimal solutions having the same value.

Proof

As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

Consider the standard form of the primal:

$$\begin{array}{rcl} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

Let $\mathbf{B}$ denote the optimal basis of the primal in standard form.

Then the optimal point is $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$ and the the optimal solution for the primal is $z = \mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Proof of strong duality (contd.)

Proof of strong duality (contd.)

Proof

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

The value of the dual at this point is:

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Since $\mathbf{B}$ is an optimal basis, we must have

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B^{-1}}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{b}$.

Since $\mathbf{B}$ is an optimal basis, we must have $(z_j - c_j) \geq 0$ for all the columns of $(\mathbf{A}, \mathbf{I})$.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B^{-1}}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{b}$.

Since $\mathbf{B}$ is an optimal basis, we must have $(z_j - c_j) \geq 0$ for all the columns of $(\mathbf{A}, \mathbf{I})$.

It follows that $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{A} - \mathbf{c} \geq \mathbf{0}$ and $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{I} \geq \mathbf{0}$.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Since $\mathbf{B}$ is an optimal basis, we must have $(z_j - c_j) \geq 0$ for all the columns of $(\mathbf{A}, \mathbf{I})$.

It follows that $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{A} - \mathbf{c} \geq \mathbf{0}$ and $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{I} \geq \mathbf{0}$.

In other words, $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{A} \geq \mathbf{c}$ and $\mathbf{c_B} \cdot \mathbf{B}^{-1} \geq \mathbf{0}$.

Proof of strong duality (contd.)

Proof

What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$.

The value of the dual at this point is: $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Since $\mathbf{B}$ is an optimal basis, we must have $(z_j - c_j) \geq 0$ for all the columns of $(\mathbf{A}, \mathbf{I})$.

It follows that $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{A} - \mathbf{c} \geq \mathbf{0}$ and $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{I} \geq \mathbf{0}$.

In other words, $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{A} \geq \mathbf{c}$ and $\mathbf{c_B} \cdot \mathbf{B}^{-1} \geq \mathbf{0}$.

In other words, the solution $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B}^{-1}$ is optimal for the dual.

Application of concepts

Application of concepts

Example

Application of concepts

Example

Solve the linear program

Application of concepts

Example

Solve the linear program

$$\begin{gathered}\max 10 \cdot x_1 + 6 \cdot x_2 - 4 \cdot x_3 + x_4 + 12 \cdot x_5 \\ 2 \cdot x_1 + x_2 + x_3 + 3 \cdot x_5 \leq 18 \\ x_1 + x_2 - x_3 + x_4 + 2 \cdot x_5 \leq 6 \\ x_1, x_2, x_3, x_4, x_5 \geq 0\end{gathered}$$

Duality for certificate generation

Duality for certificate generation

Certifying algorithm

A certifying algorithm can either produce $\mathbf{x} \in \mathbb{R}_+^n$, such that $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$,

Duality for certificate generation

Certifying algorithm

A certifying algorithm can either produce $\mathbf{x} \in \mathbb{R}_+^n$, such that $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, or $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y} \cdot \mathbf{A} \geq \mathbf{0}$ and $\mathbf{y} \cdot \mathbf{b} < 0$.