Foundations of the Simplex Method

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Outline

1. Geometric Interpretation of 2-dimensional linear programs

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2. Convexity and Polyhedral Sets

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Representing constraints as sections of a plane

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Geometric View of Constraints

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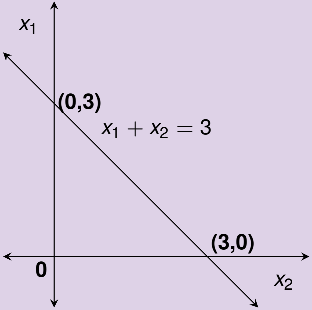

Linear Programming

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Similarly an inequality, such as $x_1 + x_2 \leq 3$, can be viewed as as the half plane above or below a line in the x_1,x_2 plane.

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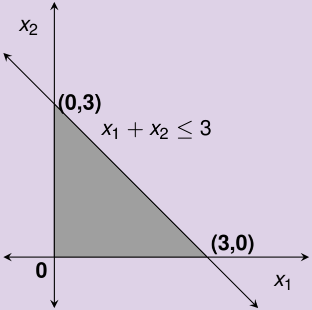

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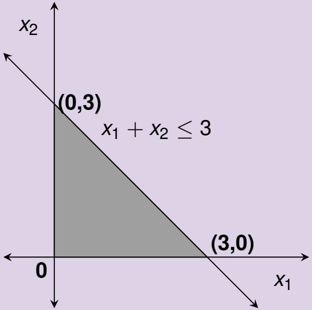

We are assuming non-negativity here.

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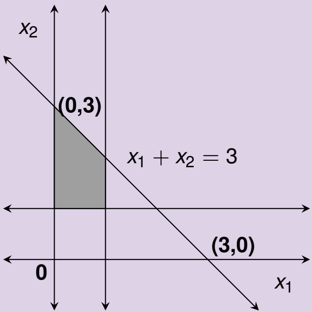

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As the objective function is of the form $z = c_1 \cdot x_1 + c_2 \cdot x_2$, the gradient is simply the vector (c_1, c_2).

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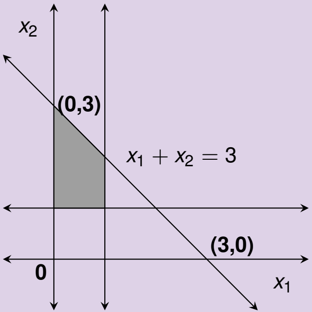

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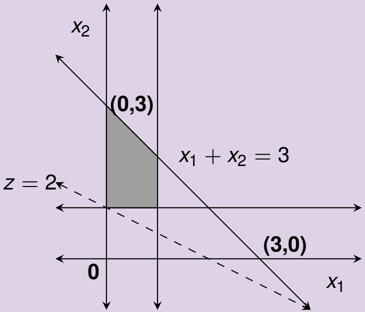

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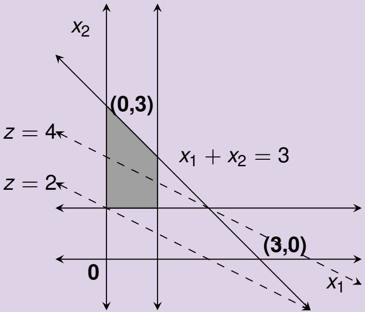

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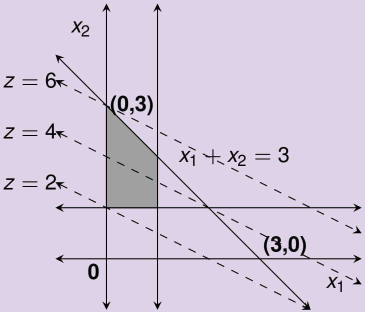

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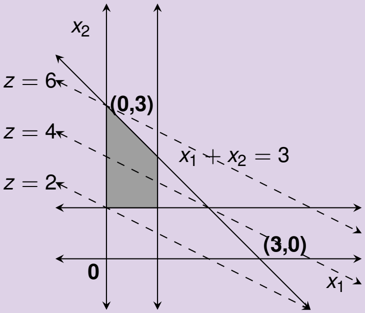

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Similarly trying to minimize z would yield $z = 2$, when $x_1 = 0$ and $x_2 = 1$.

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Solve the following linear program graphically

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$$\begin{aligned} 3 \cdot x_1 + 2 \cdot x_2 &\leq 24 \\ x_1 &\geq 5 \\ 3 \cdot x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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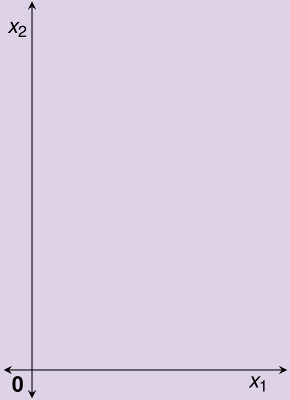

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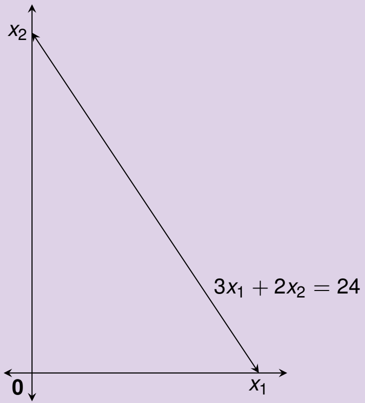

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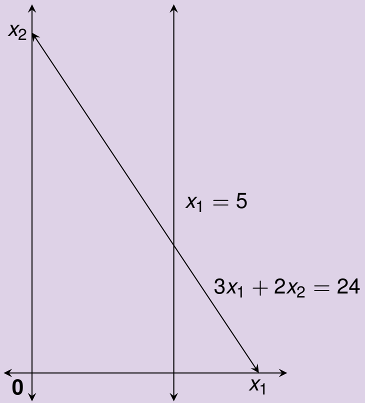

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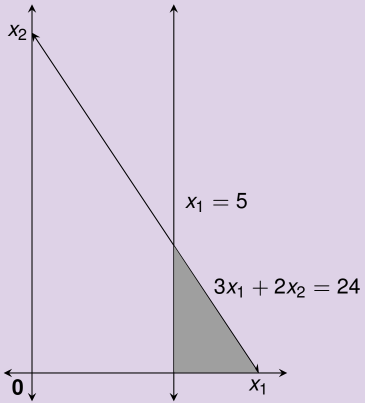

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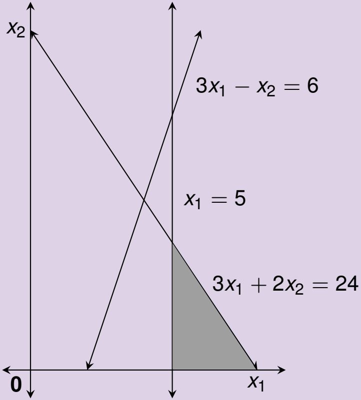

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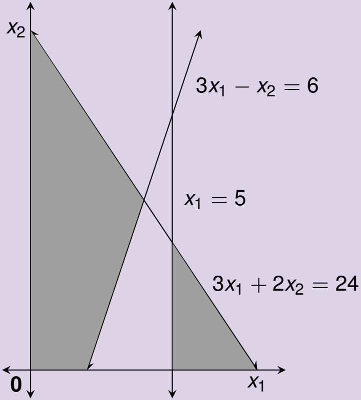

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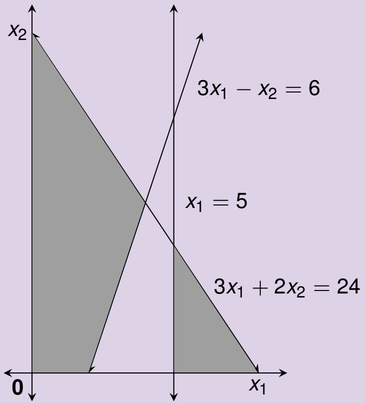

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$$\text{minimize } z = x_1 - 4 \cdot x_2$$

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$$\begin{aligned} x_1 + x_2 &\leq 12 \\ -2 \cdot x_1 + x_2 &\leq 4 \\ x_2 &\leq 8 \\ x_1 - 3 \cdot x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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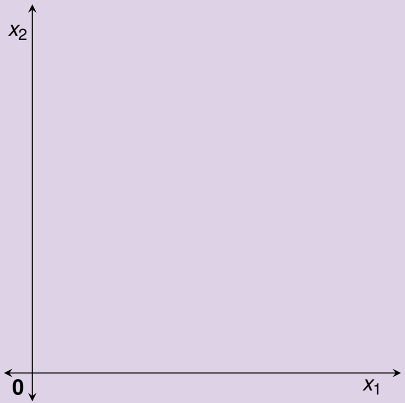

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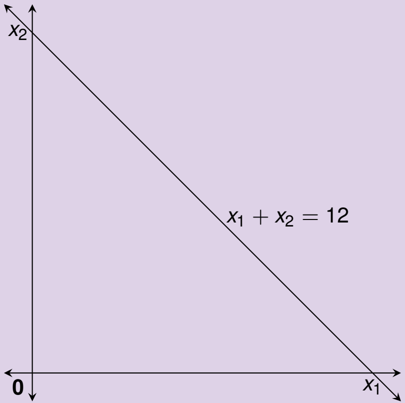

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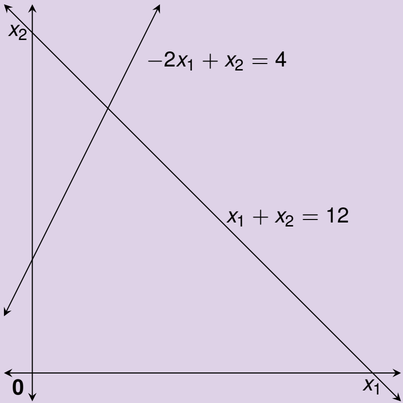

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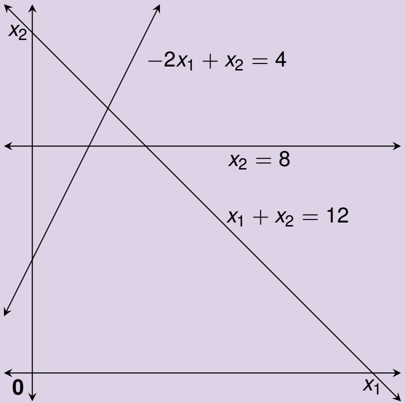

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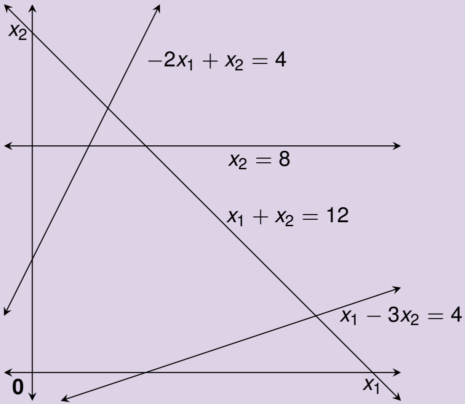

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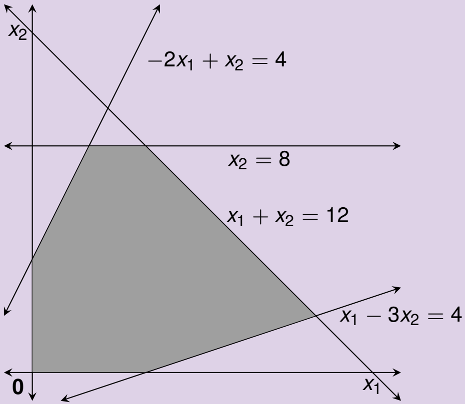

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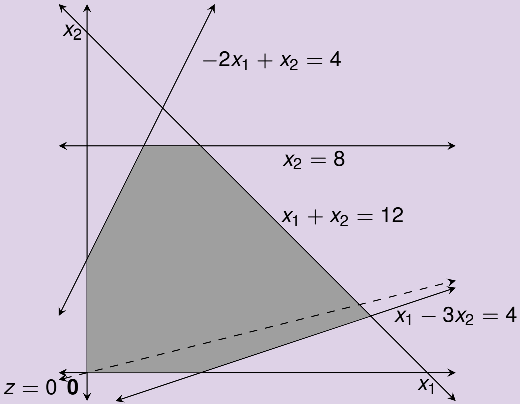

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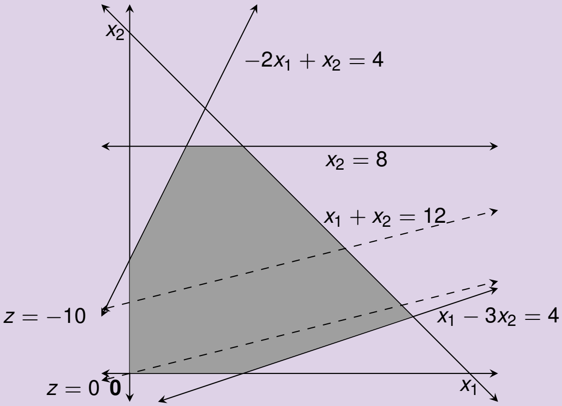

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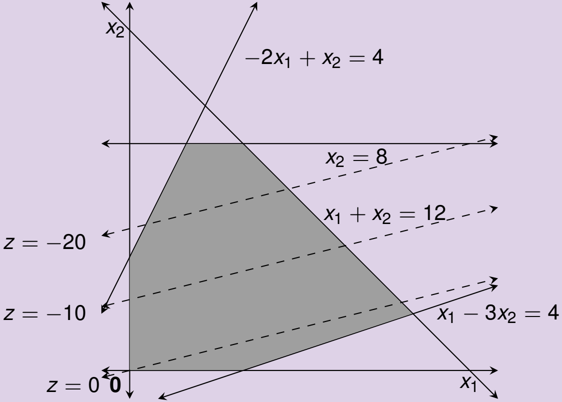

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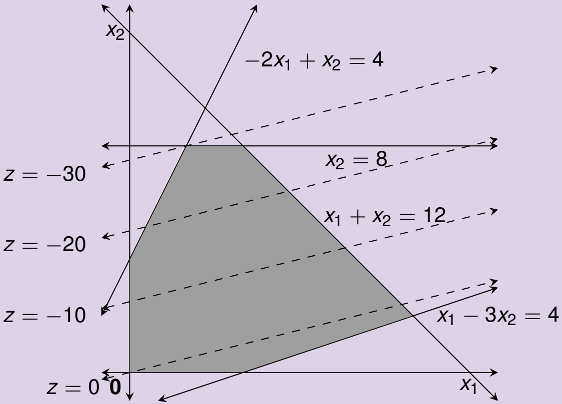

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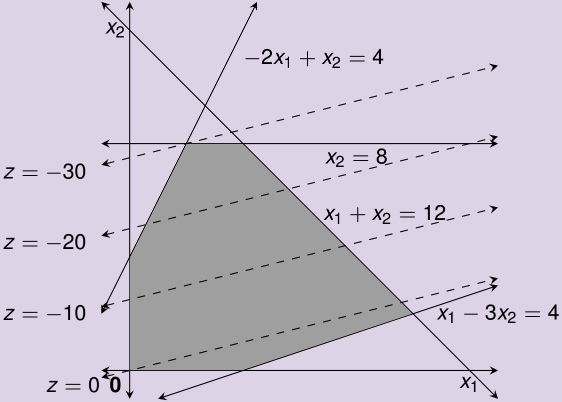

Linear Programming

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Thus, the minimum value of z is $z = -30$ and occurs at $(x_1, x_2) = (2, 8)$.

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$$\begin{aligned} -2 \cdot x_1 + x_2 &\leq 2 \\ 2 \cdot x_1 + 5 \cdot x_2 &\geq 10 \\ x_1 - 4 \cdot x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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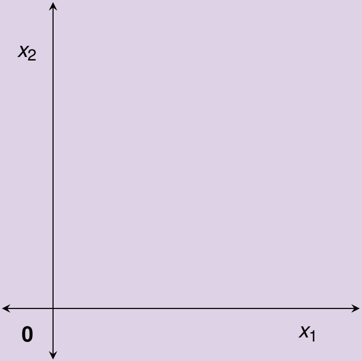

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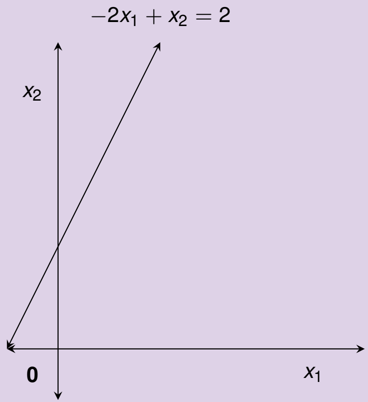

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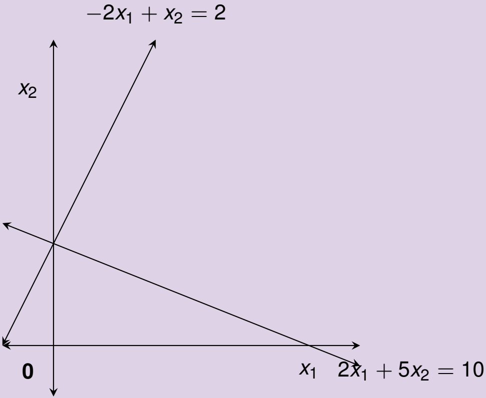

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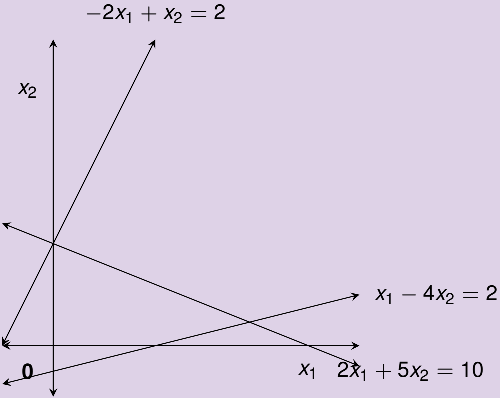

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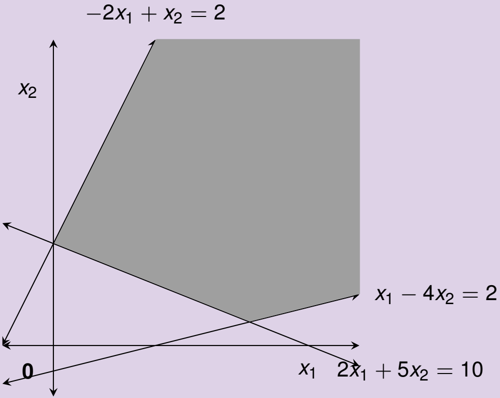

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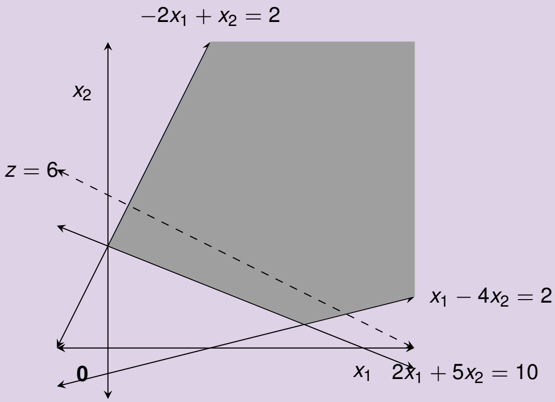

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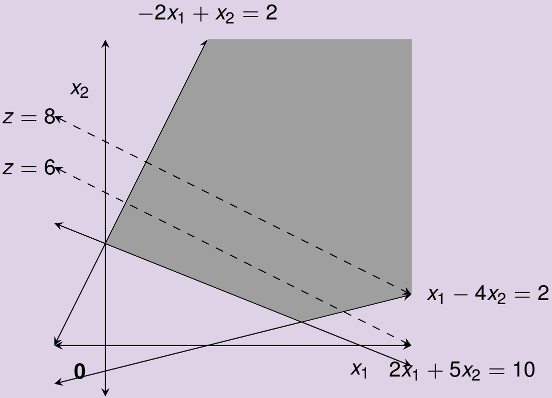

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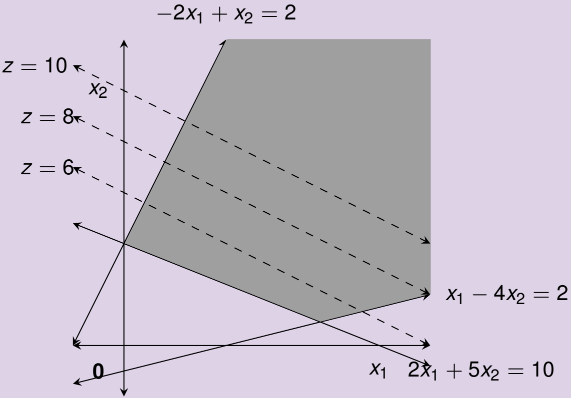

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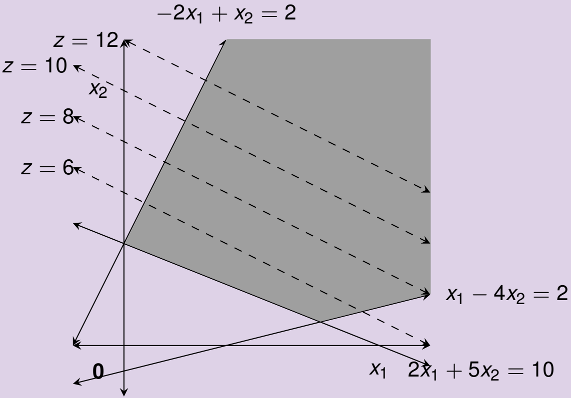

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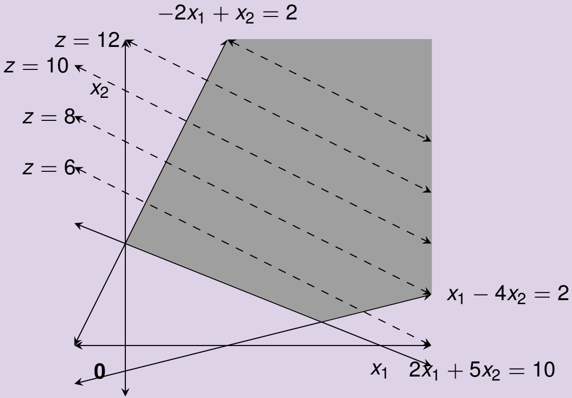

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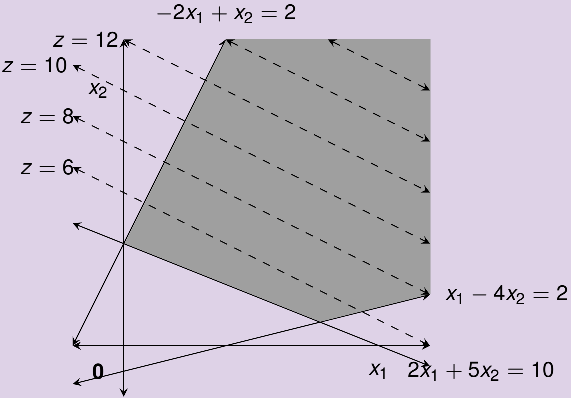

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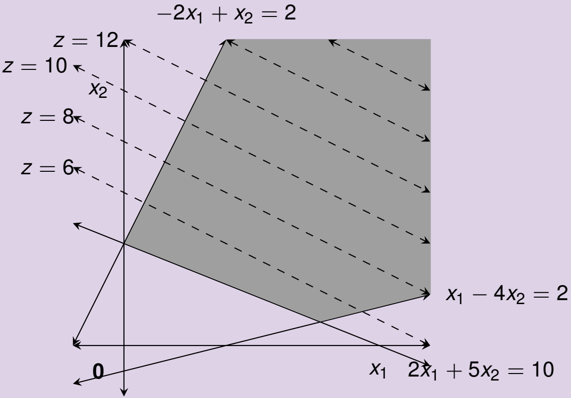

Observation

Linear Programming

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Thus, there is no maximum value of z as z can be increased indefinitely and the system will still be feasible.

Hyperplanes and Halfspaces

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A hyperplane is a set of points, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$, that satisfy $\mathbf{a} \cdot \mathbf{x} = b$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and b is a scalar.

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Definition (Halfspace)

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This means that $\forall \alpha \in [0, 1]$, $\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2 \in S$.

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Are linear functions convex?

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Exercise

Are linear functions convex? How about affine functions?

Local optimum and Global optimum

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Consider the following optimization problem:

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\begin{aligned}
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Local optimum and Global optimum

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If S is a convex set and f is a convex function of $\mathbf{x}$ *on S, then all local optima are also global optima.*

Polyhedra

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Systems of constraints as Polyhedral Sets

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Systems of constraints as Polyhedral Sets

A constraint system of the form $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is a polyhedral set as each constraint corresponds to a halfspace.

Linear Programming

Convexity of polyhedra

Convexity of polyhedra

Theorem

Convexity of polyhedra

Theorem

The set $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is convex.*

Extreme points

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Definition (Extreme Point)

Linear Programming

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A point $\mathbf{x}$ in a convex set S is said to be an extreme point if it does not lie on the interior of a line segment connecting two distinct points in S.

Linear Programming

Extreme points

Definition (Extreme Point)

A point $\mathbf{x}$ in a convex set S is said to be an extreme point if it does not lie on the interior of a line segment connecting two distinct points in S.

Mathematically there do not exist $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\mathbf{x}_1 \neq \mathbf{x}_2$, and $\alpha \in (0, 1)$ such that $\mathbf{x} = \alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2$.

Properties of Extreme points

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Goal

We want to develop a method of identifying the extreme points of a system of constraints in standard form.

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Extreme Point Theorem - *Let* $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$*, where* $\mathbf{A}$ *is* $m \times n$ *and* $rank(\mathbf{A}) = m < n$*.*

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$\bar{\mathbf{x}}$ *is an extreme point of* S *, if and only if* $\bar{\mathbf{x}}$ *is the intersection of* n *linearly independent hyperplanes.*

Proof of Extreme Point Theorem

Linear Programming

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Without loss of generality we can assume that $\overline{x}_i = 0$ for $i = 1, \ldots, p$ and $\overline{x}_i > 0$ for $i = p + 1, \ldots, n$.

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Thus, we can create a new system of constraints $\mathbf{Q} \cdot \mathbf{x} = \mathbf{h}$ formed by adding the constraints $x_i = 0$, for $i = 1, \ldots, p$ to $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.

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Thus, there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{Q} \cdot \mathbf{y} = \mathbf{0}$.

□

Proof (contd.)

Linear Programming

Proof (contd.)

Only If

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Only If

Now let us consider the points $\tilde{\mathbf{x}} = \overline{\mathbf{x}} + \lambda \cdot \mathbf{y}$ and $\hat{\mathbf{x}} = \overline{\mathbf{x}} - \lambda \cdot \mathbf{y}$ where $\lambda > 0$.

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Linear Programming

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$\tilde{x}_j = \overline{x}_j + \lambda \cdot y_j > 0$ and $\hat{x}_j = \overline{x}_j - \lambda \cdot y_j > 0$, for $j = p+1, \ldots, n$.

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However $\overline{\mathbf{x}} = \frac{1}{2} \cdot \tilde{\mathbf{x}} + \frac{1}{2} \cdot \hat{\mathbf{x}}$,

Proof (contd.)

Only If

Now let us consider the points $\tilde{\mathbf{x}} = \overline{\mathbf{x}} + \lambda \cdot \mathbf{y}$ and $\hat{\mathbf{x}} = \overline{\mathbf{x}} - \lambda \cdot \mathbf{y}$ where $\lambda > 0$.

We have that $\mathbf{Q} \cdot \tilde{\mathbf{x}} = \mathbf{Q} \cdot (\overline{\mathbf{x}} + \lambda \cdot \mathbf{y}) = \mathbf{h} + \lambda \cdot \mathbf{0} = \mathbf{h}$ and
$\mathbf{Q} \cdot \hat{\mathbf{x}} = \mathbf{Q}(\overline{\mathbf{x}} - \lambda \cdot \mathbf{y}) = \mathbf{h} - \lambda \cdot \mathbf{0} = \mathbf{h}$.

Thus, $\mathbf{A} \cdot \tilde{\mathbf{x}} = \mathbf{A} \cdot \hat{\mathbf{x}} = \mathbf{b}$ and $\tilde{x}_i = \hat{x}_i = 0$, for $i = 1, \ldots, p$.

Since $\overline{x}_j > 0$, for $j = p+1, \ldots, n$, there exists a λ such that
$\tilde{x}_j = \overline{x}_j + \lambda \cdot y_j > 0$ and $\hat{x}_j = \overline{x}_j - \lambda \cdot y_j > 0$, for $j = p+1, \ldots, n$.

Thus $\tilde{\mathbf{x}}, \hat{\mathbf{x}} \in S$.

However $\overline{\mathbf{x}} = \frac{1}{2} \cdot \tilde{\mathbf{x}} + \frac{1}{2} \cdot \hat{\mathbf{x}}$, contradicting the fact that $\overline{\mathbf{x}}$ is an extreme point of S.

Proof (contd.)

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Thus $\overline{\mathbf{x}}$ must lie on n linearly independent hyperplanes.

Proof (contd.)

Linear Programming

Proof (contd.)

If.

Proof (contd.)

If.

Let $\bar{\mathbf{x}}$ lie on the intersection of n linearly independent hyperplanes.

Linear Programming

Proof (contd.)

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Let $\bar{\mathbf{x}}$ lie on the intersection of n linearly independent hyperplanes.

Without loss of generality, we can define the n independent hyperplanes defining $\bar{\mathbf{x}}$ to be:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &= \mathbf{b} \qquad (1) \\ x_i &= 0, \; i = 1, 2, \ldots, (n-m) \end{aligned}$$

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$$\bar{\mathbf{x}} = \alpha \cdot \tilde{\mathbf{x}} + (1 - \alpha) \cdot \hat{\mathbf{x}},$$

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What do we need to show?

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where $\tilde{\mathbf{x}}, \hat{\mathbf{x}} \in S$ and $\alpha \in (0, 1)$.

What do we need to show? That $\tilde{\mathbf{x}} = \hat{\mathbf{x}} = \overline{\mathbf{x}}$.

$\square$

Proof (contd.)

If

Proof (contd.)

If

First, observe that,

Proof (contd.)

If

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$$\overline{x_i} = \alpha \cdot \tilde{x}_i + (1-\alpha) \cdot \hat{x}_i = 0, \text{ for } i = 1, 2, \ldots, (n-m)$$

Proof (contd.)

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We can immediately conclude that,

Proof (contd.)

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Let $\mathbf{A} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$.

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Since, $\mathbf{A} \cdot \overline{\mathbf{x}} = \mathbf{A} \cdot \tilde{\mathbf{x}} = \mathbf{A} \cdot \hat{\mathbf{x}} = \mathbf{b}$, we have,

Linear Programming

Proof (contd.)

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Since, $\mathbf{A} \cdot \overline{\mathbf{x}} = \mathbf{A} \cdot \tilde{\mathbf{x}} = \mathbf{A} \cdot \hat{\mathbf{x}} = \mathbf{b}$, we have,

$$\sum_{j=n-m+1}^{n} \overline{x_j} \cdot \mathbf{a_j} = \sum_{j=n-m+1}^{n} \tilde{x}_j \cdot \mathbf{a_j} = \sum_{j=n-m+1}^{n} \hat{x}_j \cdot \mathbf{a_j} = \mathbf{b}.$$

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Since $\overline{\mathbf{x}}$ is the unique solution to System (1),

Proof (contd.)

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Since $\overline{\mathbf{x}}$ is the unique solution to System (1), , it follows that the columns $\mathbf{a_{n-m+1}}, \mathbf{a_{n-m+2}}, \ldots \mathbf{a_n}$ are linearly independent.

Proof (contd.)

Linear Programming

Proof (contd.)

If

Proof (contd.)

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Hence,

$$\overline{x_i} = \tilde{x}_i = \hat{x}_i, \; i = (n - m + 1), \ldots, n$$

Proof (contd.)

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Hence,

$$\overline{x_i} = \tilde{x}_i = \hat{x}_i, \; i = (n - m + 1), \ldots, n$$

The claim follows.

Polytopes

Polytopes

Definition (Bounded Set)

Linear Programming

Polytopes

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A subset S of $\mathbb{R}^n$ is bounded if it can be contained within an n-dimensional ball.

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Linear Programming

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Note

We will only be dealing with bounded polyhedra for the rest of this topic.

Linear Programming

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Such polyhedra are called polytopes.

Representation theorem

Linear Programming

Representation theorem

Theorem

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Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *be non-empty, and let* E *be the set of extreme points of* S.

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Then,

1. S *has at least one extreme point and at most a finite number of extreme points, thus* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$.
2. *if* $\mathbf{x} \in S$, *then* $\mathbf{x}$ *can be written as a convex combination of extreme points*

Extreme point solutions

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Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *and consider the following linear program:*

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$$\begin{array}{c} \textit{maximize } z = \mathbf{c} \cdot \mathbf{x} \\ \textit{subject to } \mathbf{x} \in S. \end{array}$$

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Suppose S *is bounded and has extreme points* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$.

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If S *is bounded, a finite optimal solution exists.*

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Suppose S *is bounded and has extreme points* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$*.*

If S *is bounded, a finite optimal solution exists.*

Furthermore, an extreme point optimal solution exists.

Extreme points and basic feasible solutions

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Goal

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We have now shown that we can solve linear programs by restricting ourselves to the extreme points of the feasible space.

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Finding basic feasible solutions

Consider a linear system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix $\mathbf{b} = (b_1, \ldots, b_m)^t$, and $\mathbf{x} = (x_1, \ldots, x_n)^t$.

Linear Programming

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Assume that $rank(\mathbf{A}) = m \leq n$. That is we assume that the rows of $\mathbf{A}$ are linearly independent.

We also assume that the columns of $\mathbf{A}$ can be rearranged so that $\mathbf{A}$ can be written as $\mathbf{A} = (\mathbf{B} : \mathbf{N})$,

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We will refer to $\mathbf{B}$ as the *basis matrix*.

Finding basic feasible solutions

Finding basic feasible solutions

The Method

Linear Programming

Finding basic feasible solutions

The Method

We can rewrite $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ as $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{N} \cdot \mathbf{x_N} = \mathbf{b}$, where $\mathbf{x} = \binom{\mathbf{x_B}}{\mathbf{x_N}}$.

Linear Programming

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As **B** is non-singular, the inverse of **B** exists.

Finding basic feasible solutions

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As **B** is non-singular, the inverse of **B** exists.

Thus, $\mathbf{B}^{-1} \cdot \mathbf{B} \cdot \mathbf{x_B} + \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x_N} = \mathbf{B}^{-1} \cdot \mathbf{b}$.

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This is equivalent to stating that $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b} - \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x}_N$.

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As $\mathbf{B}$ is non-singular, the inverse of $\mathbf{B}$ exists.

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This is equivalent to stating that $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b} - \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x}_N$.

If we set $\mathbf{x_N} = \mathbf{0}$ then $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b}$ and $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$.

Finding basic feasible solutions

The Method

We can rewrite $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ as $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{N} \cdot \mathbf{x_N} = \mathbf{b}$, where $\mathbf{x} = \binom{\mathbf{x_B}}{\mathbf{x_N}}$.

As $\mathbf{B}$ is non-singular, the inverse of $\mathbf{B}$ exists.

Thus, $\mathbf{B}^{-1} \cdot \mathbf{B} \cdot \mathbf{x_B} + \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x_N} = \mathbf{B}^{-1} \cdot \mathbf{b}$.

This is equivalent to stating that $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b} - \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x}_N$.

If we set $\mathbf{x_N} = \mathbf{0}$ then $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b}$ and $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$.

This value is called a *basic solution*.

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We refer to the $\mathbf{x}_B$ as the vector of *basic variables* and we refer to $\mathbf{x_N}$ as the vector of *nonbasic variables*.

Finding basic feasible solutions

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Connecting extreme points and basic feasible soluionts

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Theorem

Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, *where* $\mathbf{A}$ *is* $m \times n$ *and* $rank(\mathbf{A}) = m < n$.

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$\mathbf{x}$ *is an extreme point of* S *if and only if* $\mathbf{x}$ *is a basic feasible solution.*

Proof

Linear Programming

Proof

Only If

Proof

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Let $\mathbf{x}$ be an extreme point of S.

Proof

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$\mathbf{x}$ lies on n independent hyperplanes.

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m of these hyperplanes come from $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ and $(n - m)$ of these hyperplanes must come from the non-negativity constraints.

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In other words, we must have at $x_i = 0$, for at least $(n - m)$ variables. Let us collectively refer to them as $\mathbf{x_N} = \mathbf{0}$.

Linear Programming

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Then, the extreme point $\mathbf{x}$ is the unique solution of the n linearly independent hyperplanes $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x_N} = \mathbf{0}$. (Why?)

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Then, $\mathbf{x}$ is the unique solution to $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{N} \cdot \mathbf{x_N} = \mathbf{b}, \mathbf{x_N} = \mathbf{0}$.

It follows that $\mathbf{x_B}$ is the unique solution to $\mathbf{B} \cdot \mathbf{x_B} = \mathbf{b}$.

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It follows that $\mathbf{x_B}$ is the unique solution to $\mathbf{B} \cdot \mathbf{x_B} = \mathbf{b}$. Hence, $\mathbf{B}$ is invertible and

therefore a basis matrix.

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Therefore, $\mathbf{x} = \binom{\mathbf{x_B}}{\mathbf{x_N}}$ is a basic solution.

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Therefore, $\mathbf{x} = \binom{\mathbf{x_B}}{\mathbf{x_N}}$ is a basic solution. Since it is an extreme point, it is also feasible.

Proof (contd.)

Linear Programming

Proof (contd.)

If

Proof (contd.)

If

Let **x** be a basic feasible solution of S.

Proof (contd.)

If

Let $\mathbf{x}$ be a basic feasible solution of S.

Since $\mathbf{x}$ is a bfs, there exists a basis matrix $\mathbf{B}$ such that,

Linear Programming

Proof (contd.)

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Proof (contd.)

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This implies that $\mathbf{x}$ is the unique solution to $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{B} \cdot \mathbf{x_N} = \mathbf{b}$, $\mathbf{x_N} = \mathbf{0}$,

Linear Programming

Proof (contd.)

If

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Since $\mathbf{x}$ is a bfs, there exists a basis matrix $\mathbf{B}$ such that,

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This implies that $\mathbf{x}$ is the unique solution to $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{B} \cdot \mathbf{x_N} = \mathbf{b}$, $\mathbf{x_N} = \mathbf{0}$, or equivalently, $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, $\mathbf{x_N} = \mathbf{0}$.

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Since $\mathbf{x}$ is unique, it follows that the hyperplanes are constituting the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, $\mathbf{x_N} = \mathbf{0}$ are linearly independent.

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There are clearly $m + (n - m) = n$ of these hyperplanes,

Proof (contd.)

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Proof (contd.)

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There are clearly $m + (n - m) = n$ of these hyperplanes, i.e., $\mathbf{x}$ lies at the intersection of n linearly independent hyperplanes.

It follows that $\mathbf{x}$ is an extreme point of S.

Example

Example

Example

Linear Programming

Example

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Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

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Example

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Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the basic feasible solutions to $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Solution

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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Solution

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

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$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$.

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Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the basic feasible solutions to $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

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$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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$\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$.

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Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the basic feasible solutions to $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Solution

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$.

Exercise

Linear Programming

Exercise

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Consider the linear program:

$$\begin{aligned} \max z = 2 \cdot x_1 + 3 \cdot x_2 & \\ x_1 - 2 \cdot x_2 & \leq 4 \\ 2 \cdot x_1 + x_2 & \leq 18 \\ x_2 & \leq 10 \\ x_1, x_2 & \geq 0 \end{aligned}$$

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Graphically map the extreme points and the corresponding basic feasible solutions.

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Identify the optimal solution, optimal extreme point and optimal basic feasible solution.

Summary

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Main ideas

Linear Programming

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1. Assuming that the linear program $S = \{\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is bounded, then any linear function $\mathbf{c} \cdot \mathbf{x}$ will reach its maximum at an extreme point of S.

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6. The Simplex Method will avoid exhaustive enumeration by using intelligent selection rules.