Mathematical Review

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Outline

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Order of magnitude of functions

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(iii) *We only care about functions from* $\Re_{\geq 0} \rightarrow \Re_{\geq 0}$.

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(i) Which function grows faster: $100 \cdot x^2$ or $\frac{1}{10^6} \cdot x^3$?

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Example

(i) Which function grows faster: $100 \cdot x^2$ or $\frac{1}{10^6} \cdot x^3$?

(ii) Which function grows faster: $x^2 - 10$ or $x + 10$?

Order of Magnitude (contd.)

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Definition

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Definition

Let f and g be functions mapping non-negative reals to non-negative reals.

Order of Magnitude (contd.)

Definition

Let f and g be functions mapping non-negative reals to non-negative reals.

Then $f = O(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) \leq c \cdot g(x)$.

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Let f and g be functions mapping non-negative reals to non-negative reals.

Then $f = \Omega(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) \geq c \cdot g(x)$.

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Let f and g be functions mapping non-negative reals to non-negative reals.

Then $f = o(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) < c \cdot g(x)$.

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Then $f = \Theta(g)$, if $f = O(g)$ and $g = O(f)$.

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Then $f = \Theta(g)$, if $f = O(g)$ and $g = O(f)$.

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(i) Let $f(x) = 2 \cdot x^2 - 2$ and $g(x) = \frac{1}{100} \cdot x^2 - 100$.

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(ii) Let $f(x) = 2 \cdot x^2 - 2$ and $g(x) = \frac{1}{100} \cdot x - 100$. $f = \Omega(g)$. Furthermore, $g = o(f)$.

Test to determine order

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The limit test

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Note

If $\lim_{x\to\infty} f(x) = \infty$ *and if* $\lim_{x\to\infty} g(x) = \infty$, *then,*

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Note

If $\lim_{x \to \infty} f(x) = \infty$ and if $\lim_{x \to \infty} g(x) = \infty$, then,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

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Note

If $\lim_{x\to\infty} f(x) = \infty$ *and if* $\lim_{x\to\infty} g(x) = \infty$, *then,*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

The above rule is called L'Hospital's rule.

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(i) Show that $x = o(x^2)$.

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(i) Show that $x = o(x^2)$.

(ii) Show that $x = o(x \cdot \log x)$.

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(ii) Show that $x = o(x \cdot \log x)$.

(iii) Show that $\log x = o(x)$.

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Well-Ordering Principle

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Every non-empty set of positive integers has a **least element**.

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Induction can only be applied to a well-ordered domain, where the concept of "next" is unambiguous,

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How about all integers?

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How about non-negative rationals?

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(i) Showing that $P(1)$ is **true** is called the basis step.

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Observations

(i) Showing that $P(1)$ is **true** is called the basis step.

(ii) Assuming that $P(k)$ is **true**, in order to show that $P(k+1)$ is **true** is called the inductive hypothesis.

First Example

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Show that the sum of the first n integers is $\frac{n \cdot (n+1)}{2}$.

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Show that the sum of the first n integers is $\frac{n\cdot(n+1)}{2}$.

Formally, $(\forall n)\ [\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}]$.

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$\textsc{Basis}$ $(P(1))$:

$$LHS = \sum_{i=1}^{1} i$$

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$$\begin{aligned} LHS &= \sum_{i=1}^{1} i \\ &= 1 \end{aligned}$$

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Thus, $LHS = RHS$ and $P(1)$ is true. □

Example

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$$\sum_{i=1}^{k} i =$$

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□

Proof (contd.)

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Observe that,

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$$LHS \quad =$$

Proof (contd.)

Proof.

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Proof (contd.)

Proof.

Observe that,

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1} i \\
&= 1 + 2 + 3 + \ldots + k + (k+1)
\end{aligned}
$$

Proof (contd.)

Proof.

Observe that,

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i \\ &= 1+2+3+\ldots+k+(k+1) \\ &= (1+2+3+\ldots+k)+(k+1) \end{aligned}$$

Proof (contd.)

Proof.

Observe that,

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1} i \\
&= 1+2+3+\ldots+k+(k+1) \\
&= (1+2+3+\ldots+k)+(k+1) \\
&= \frac{k\cdot(k+1)}{2}+(k+1), \text{ using the inductive hypothesis}
\end{aligned}
$$

Proof (contd.)

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LHS &= \sum_{i=1}^{k+1} i \\
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&= (k+1)\cdot(\frac{k}{2}+1)
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Proof (contd.)

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LHS &= \sum_{i=1}^{k+1} i \\
&= 1 + 2 + 3 + \ldots + k + (k+1) \\
&= (1 + 2 + 3 + \ldots + k) + (k+1) \\
&= \frac{k \cdot (k+1)}{2} + (k+1), \text{ using the inductive hypothesis} \\
&= (k+1) \cdot (\frac{k}{2} + 1) \\
&= (k+1) \cdot (\frac{k+2}{2})
\end{aligned}$$

Proof (contd.)

Proof.

Observe that,

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&= (k+1)\cdot(\frac{k}{2}+1) \\
&= (k+1)\cdot(\frac{k+2}{2}) \\
&= \frac{(k+1)\cdot(k+2)}{2}
\end{aligned}
$$

Proof (contd.)

Proof.

Observe that,

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1} i \\
&= 1+2+3+\ldots+k+(k+1) \\
&= (1+2+3+\ldots+k)+(k+1) \\
&= \frac{k\cdot(k+1)}{2}+(k+1), \text{ using the inductive hypothesis} \\
&= (k+1)\cdot(\frac{k}{2}+1) \\
&= (k+1)\cdot(\frac{k+2}{2}) \\
&= \frac{(k+1)\cdot(k+2)}{2} = RHS
\end{aligned}
$$

$\square$

Completing the proof

Completing the proof

Final Steps

Completing the proof

Final Steps

Since, $LHS = RHS$, we have shown that $P(k) \rightarrow P(k+1)$.

Completing the proof

Final Steps

Since, $LHS{=}RHS$, we have shown that $P(k) \rightarrow P(k+1)$.

Applying the first principle of mathematical induction, we conclude that the conjecture is **true**, i.e., $(\forall n)P(n)$ holds.

Central Theme

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Main Ideas

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(i) Mathematicize the conjecture.

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(ii) Prove the basis (usually $P(1)$ and usually easy.)

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Central Theme

Main Ideas

(i) Mathematicize the conjecture.

(ii) Prove the basis (usually $P(1)$ and usually easy.)

(iii) Assume $P(k)$.

(iv) Show $P(k+1)$. (The hard part. Use mathematical manipulation.)

(v) To show $P(k) \rightarrow P(k+1)$, you may use any of the proof techniques discussed, including exhaustive proof, direct proof, contraposition, contradiction, serendipity and induction!

Another Induction Example

Another Induction Example

Example

Another Induction Example

Example

Show that the sum of the squares of the first n integers is $\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$,

Another Induction Example

Example

Show that the sum of the squares of the first n integers is $\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$, i.e., show that $\sum_{i=1}^{n} i^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$.

Proving the Basis

Proving the Basis

Proof.

Proving the Basis

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BASIS ($P(1)$):

Proving the Basis

Proof.

BASIS ($P(1)$):

$$LHS =$$

Proving the Basis

Proof.

Basis $(P(1))$:

$$LHS = \sum_{i=1}^{1} i^2$$

Proving the Basis

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BASIS $(P(1))$:

$$\begin{aligned} LHS &= \sum_{i=1}^{1} i^2 \\ &= 1 \end{aligned}$$

Proving the Basis

Proof.

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$$\begin{aligned} LHS &= \sum_{i=1}^{1} i^2 \\ &= 1 \\ RHS &= \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} \end{aligned}$$

Proving the Basis

Proof.

Basis $(P(1))$:

$$\begin{aligned} LHS &= \sum_{i=1}^{1} i^2 \\ &= 1 \\ RHS &= \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} \\ &= \frac{1 \cdot (2) \cdot (3)}{6} \end{aligned}$$

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$$\begin{aligned} LHS &= \sum_{i=1}^{1} i^2 \\ &= 1 \\ RHS &= \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} \\ &= \frac{1 \cdot (2) \cdot (3)}{6} \\ &= \frac{6}{6} \end{aligned}$$

Proving the Basis

Proof.

BASIS ($P(1)$):

$$\begin{aligned}
LHS &= \sum_{i=1}^{1} i^2 \\
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&= \frac{6}{6} \\
&= 1
\end{aligned}$$

Proving the Basis

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Thus, $LHS = RHS$ and $P(1)$ is true. □

Induction example (contd.)

Induction example (contd.)

Proof.

Induction example (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

Induction example (contd.)

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Induction example (contd.)

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$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

Induction example (contd.)

Proof.

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$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true,

Induction example (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1} i^2 =$

Induction example (contd.)

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Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}$.

Induction example (contd.)

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Induction example (contd.)

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$$LHS \quad =$$

Induction example (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}$.

$$LHS = \sum_{i=1}^{k+1} i^2$$

Induction example (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}$.

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^2 \\ &= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \end{aligned}$$

Induction example (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}$.

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^2 \\ &= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \\ &= (1^2 + 2^2 + 3^2 + \ldots + k^2) + (k+1)^2 \end{aligned}$$

Induction example (contd.)

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Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k} i^2 = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}.$$

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$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^2 \\ &= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \\ &= (1^2 + 2^2 + 3^2 + \ldots + k^2) + (k+1)^2 \end{aligned}$$

$\square$

Induction proof (contd.)

Induction proof (contd.)

Proof.

Induction proof (contd.)

Proof.

$$= \quad \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2, \text{ using the inductive hypothesis}$$

Induction proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k \cdot (2 \cdot k + 1) + 6 \cdot (k+1))
\end{aligned}
$$

Induction proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{k \cdot (k+1) \cdot (2 \cdot k+1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k \cdot (2 \cdot k+1) + 6 \cdot (k+1)) \\
&= \frac{k+1}{6}(2 \cdot k^2 + k + 6 \cdot k + 6)
\end{aligned}
$$

Induction proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k \cdot (2 \cdot k + 1) + 6 \cdot (k+1)) \\
&= \frac{k+1}{6}(2 \cdot k^2 + k + 6 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k^2 + 7 \cdot k + 6)
\end{aligned}
$$

Induction proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k \cdot (2 \cdot k + 1) + 6 \cdot (k+1)) \\
&= \frac{k+1}{6}(2 \cdot k^2 + k + 6 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k^2 + 7 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k^2 + 4 \cdot k + 3 \cdot k + 6)
\end{aligned}
$$

Induction proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k \cdot (2 \cdot k + 1) + 6 \cdot (k+1)) \\
&= \frac{k+1}{6}(2 \cdot k^2 + k + 6 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k^2 + 7 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k^2 + 4 \cdot k + 3 \cdot k + 6) \\
&= \frac{k+1}{6}(2 \cdot k \cdot (k+2) + 3 \cdot (k+2))
\end{aligned}
$$

Induction proof (contd.)

Proof.

$$\begin{aligned}
&= \frac{k\cdot(k+1)\cdot(2\cdot k+1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k\cdot(2\cdot k+1) + 6\cdot(k+1)) \\
&= \frac{k+1}{6}(2\cdot k^2 + k + 6\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k^2 + 7\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k^2 + 4\cdot k + 3\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k\cdot(k+2) + 3\cdot(k+2)) \\
&= \frac{k+1}{6}(2\cdot k+3)\cdot(k+2))
\end{aligned}$$

Induction proof (contd.)

Proof.

$$\begin{aligned}
&= \frac{k\cdot(k+1)\cdot(2\cdot k+1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{k+1}{6}(k\cdot(2\cdot k+1) + 6\cdot(k+1)) \\
&= \frac{k+1}{6}(2\cdot k^2 + k + 6\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k^2 + 7\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k^2 + 4\cdot k + 3\cdot k + 6) \\
&= \frac{k+1}{6}(2\cdot k\cdot(k+2) + 3\cdot(k+2)) \\
&= \frac{k+1}{6}(2\cdot k+3)\cdot(k+2))
\end{aligned}$$

$\square$

Induction Proof (contd.)

Induction Proof (contd.)

Proof.

Induction Proof (contd.)

Proof.

$$= \quad \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}$$

Induction Proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6} \\
&= RHS.
\end{aligned}
$$

Induction Proof (contd.)

Proof.

$$
\begin{aligned}
&= \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6} \\
&= RHS.
\end{aligned}
$$

Since, $LHS{=}RHS$, we have shown that $P(k) \rightarrow P(k+1)$.

Induction Proof (contd.)

Proof.

$$\begin{aligned} &= \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6} \\ &= RHS. \end{aligned}$$

Since, $LHS = RHS$, we have shown that $P(k) \rightarrow P(k+1)$.
Applying the first principle of mathematical induction, we conclude that the conjecture is true. □

Induction Example

Induction Example

Example

Induction Example

Example

Show that the sum of the first n odd integers is n^2,

Induction Example

Example

Show that the sum of the first n odd integers is n^2, i.e., show that $\sum_{i=1}^{n}(2 \cdot i - 1) = n^2$.

Proving the conjecture

Proving the conjecture

Proof.

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Proof.

BASIS $(P(1))$:

Proving the conjecture

Proof.

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Proving the conjecture

Proof.

BASIS ($P(1)$):

$$LHS =$$

Proving the conjecture

Proof.

BASIS $(P(1))$:

$$LHS = \sum_{i=1}^{1} (2 \cdot i - 1)$$

Proving the conjecture

Proof.

BASIS $(P(1))$:

$$\begin{aligned} LHS &= \sum_{i=1}^{1}(2 \cdot i - 1) \\ &= 2 \cdot 1 - 1 \end{aligned}$$

Proving the conjecture

Proof.

BASIS ($P(1)$):

$$\begin{aligned} LHS &= \sum_{i=1}^{1}(2 \cdot i - 1) \\ &= 2 \cdot 1 - 1 \\ &= 1 \end{aligned}$$

Proving the conjecture

Proof.

BASIS $(P(1))$:

$$
\begin{aligned}
LHS &= \sum_{i=1}^{1}(2 \cdot i - 1) \\
&= 2 \cdot 1 - 1 \\
&= 1 \\
RHS &= 1^2
\end{aligned}
$$

Proving the conjecture

Proof.

BASIS ($P(1)$):

$$\begin{aligned} LHS &= \sum_{i=1}^{1}(2 \cdot i - 1) \\ &= 2 \cdot 1 - 1 \\ &= 1 \\ RHS &= 1^2 \\ &= 1 \end{aligned}$$

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BASIS $(P(1))$:

$$\begin{aligned} LHS &= \sum_{i=1}^{1}(2 \cdot i - 1) \\ &= 2 \cdot 1 - 1 \\ &= 1 \\ RHS &= 1^2 \\ &= 1 \end{aligned}$$

Thus, $LHS = RHS$ and $P(1)$ is true. □

Proof (contd.)

Proof (contd.)

Proof.

Proof (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

Proof (contd.)

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Proof (contd.)

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Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k}(2 \cdot i - 1) = k^2$$

Proof (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k}(2 \cdot i - 1) = k^2$$

We need to show that $P(k+1)$ is true,

Proof (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k}(2 \cdot i - 1) = k^2$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1}(2 \cdot i - 1) = (k+1)^2$.

Proof (contd.)

Proof.

Let us assume that $P(k)$ is true, i.e., assume that

$$\sum_{i=1}^{k}(2 \cdot i - 1) = k^2$$

We need to show that $P(k+1)$ is true, i.e., we need to show that $\sum_{i=1}^{k+1}(2 \cdot i - 1) = (k+1)^2$.

□

Completing the proof

Completing the proof

Proof.

Completing the proof

Proof.

Completing the proof

Proof.

$$LHS =$$

Completing the proof

Proof.

$$LHS = \sum_{i=1}^{k+1}(2 \cdot i - 1)$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1)
\end{aligned}
$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1} (2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1) \\
&= (1 + 3 + 5 + \ldots (2 \cdot k - 1)) + (2 \cdot k + 1)
\end{aligned}
$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1) \\
&= (1 + 3 + 5 + \ldots (2 \cdot k - 1)) + (2 \cdot k + 1) \\
&= k^2 + (2 \cdot k + 1), \ \text{using the inductive hypothesis}
\end{aligned}
$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1) \\
&= (1 + 3 + 5 + \ldots (2 \cdot k - 1)) + (2 \cdot k + 1) \\
&= k^2 + (2 \cdot k + 1), \text{ using the inductive hypothesis} \\
&= (k+1)^2
\end{aligned}
$$

Completing the proof

Proof.

$$\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2 \cdot i - 1) \\
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&= k^2 + (2 \cdot k + 1), \text{ using the inductive hypothesis} \\
&= (k+1)^2 \\
&= RHS
\end{aligned}$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1) \\
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&= k^2 + (2 \cdot k + 1), \text{ using the inductive hypothesis} \\
&= (k+1)^2 \\
&= RHS
\end{aligned}
$$

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1}(2\cdot i-1) \\
&= 1+3+5+\ldots(2\cdot k-1)+(2\cdot(k+1)-1) \\
&= (1+3+5+\ldots(2\cdot k-1))+(2\cdot k+1) \\
&= k^2+(2\cdot k+1),\ \text{using the inductive hypothesis} \\
&= (k+1)^2 \\
&= RHS
\end{aligned}
$$

Since $LHS = RHS$, we have shown that $P(k) \rightarrow P(k+1)$.

Completing the proof

Proof.

$$
\begin{aligned}
LHS &= \sum_{i=1}^{k+1} (2 \cdot i - 1) \\
&= 1 + 3 + 5 + \ldots (2 \cdot k - 1) + (2 \cdot (k+1) - 1) \\
&= (1 + 3 + 5 + \ldots (2 \cdot k - 1)) + (2 \cdot k + 1) \\
&= k^2 + (2 \cdot k + 1), \text{ using the inductive hypothesis} \\
&= (k+1)^2 \\
&= RHS
\end{aligned}
$$

Since $LHS = RHS$, we have shown that $P(k) \rightarrow P(k+1)$. Applying the first principle of mathematical induction, we conclude that the conjecture is true. $\square$

One Final Example

One Final Example

Example

One Final Example

Example

Show that $7^n - 5^n$ is always an even number for $n \geq 0$,

One Final Example

Example

Show that $7^n - 5^n$ is always an even number for $n \geq 0$, i.e., show that $2 \mid (7^n - 5^n)$, $\forall n \geq 0$.

One Final Example

Example

Show that $7^n - 5^n$ is always an even number for $n \geq 0$, i.e., show that $2 \mid (7^n - 5^n)$, $\forall n \geq 0$.

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Since the LHS is even, we have proven the basis $P(0)$. □

Proof (contd.)

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Observe that,

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Observe that,

$$\begin{aligned} 7^{k+1} - 5^{k+1} &= 7 \cdot 7^k - 5 \cdot 5^k \\ &= 7 \cdot (2 \cdot m + 5^k) - 5 \cdot 5^k, \text{ using the inductive hypothesis} \end{aligned}$$

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We have thus shown that $P(k) \rightarrow P(k+1)$.

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We have thus shown that $P(k) \rightarrow P(k+1)$. Applying the first principle of mathematical induction, we conclude that the conjecture is true. $\square$

Second Principle of Induction

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Note

Also called Strong Induction. Is necessary, when the first principle does not help us.

Example of Strong Induction

Example of Strong Induction

Example

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Example

Show that every number greater than or equal to 8 can be expressed in the form $5 \cdot a + 3 \cdot b$, for suitably chosen a and b.

Proving the conjecture

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Proof.

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(i) The conjecture is clearly true for 8, 9 and 10.

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(iii) Consider the integer $(k + 1)$.

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(iv) Without loss of generality, we assume that $(k+1) \geq 11$.

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(iii) Consider the integer $(k+1)$.

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(v) Observe that $(k+1) - 3 = (k-2)$ is at least 8 and less than k.

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(v) Observe that $(k+1) - 3 = (k-2)$ is at least 8 and less than k.

(vi) As per the inductive hypothesis, $(k-2)$ can be expressed in the form $3 \cdot a + 5 \cdot b$, for suitably chosen a and b.

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(vii) It follows that $(k+1) = (k-2)+3 = 3 \cdot a + 5 \cdot b + 3 = 3 \cdot (a+1) + 5 \cdot b$ can also be so expressed.

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(vii) It follows that $(k+1) = (k-2) + 3 = 3 \cdot a + 5 \cdot b + 3 = 3 \cdot (a+1) + 5 \cdot b$ can also be so expressed.

(viii) Applying the second principle of mathematical induction, we conclude that the conjecture is true, for all $n \geq 8$.

□

Another Example

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Show that every element in the set $S = \{2, 3, \ldots, \}$ is either a prime number or a product of primes.

Proving the conjecture

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Proof.

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1. For the basis, observe that 2 is a prime.

Proving the conjecture

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2. Assume that the conjecture holds for all r, $2 \leq r \leq k$.

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1. For the basis, observe that 2 is a prime.
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   In other words, assume that every number in the set $S_k = \{2, 3, \ldots, k\}$ is either a prime or can be expressed as a product of primes.

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3. Now consider the number $(k+1)$.

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3. Now consider the number $(k + 1)$. If $(k + 1)$ is a prime, then

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   In other words, assume that every number in the set $S_k = \{2, 3, \ldots, k\}$ is either a prime or can be expressed as a product of primes.
3. Now consider the number $(k+1)$. If $(k+1)$ is a prime, then we are done.
4. If $(k+1)$ is composite, then $(k+1) = a \cdot b$, where $a, b < (k+1)$.

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5. As per the inductive hypothesis, both a and b are either primes themselves or can be expressed as products of primes.

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5. As per the inductive hypothesis, both a and b are either primes themselves or can be expressed as products of primes.
6. In either case, it follows that $(k+1)$ can be expressed as a product of primes.
7. Applying the second principle of mathematical induction, we conclude that the conjecture is true for all elements in the domain.

$\square$

Logarithms

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4. $\log_a b = \frac{\log_c b}{\log_c a}$, $c > 1$.
5. $\log_a b^n = n \cdot \log_a b$.

Logarithms

Definition

If $x = a^b$, where $a > 0$, then $b = \log_a x$.

Laws

Let $a, b, c > 0$.

1. $\log_a(x \cdot y) = \log_a x + \log_a y$.
2. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$.
3. $\log_a 1 = 0$.
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8. $a^{\log_b c} = c^{\log_b a}$.
9. $\log^b n = o(n^a)$.

Motivating Examples

Motivating Examples

Example

Motivating Examples

Example

How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Motivating Examples

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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In how many ways can 6 people be seated in a row?

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In how many ways can 6 people be seated in a row?

Example

In how many ways can 6 people be seated around a circular table with 6 chairs?

Motivating Examples

Example

How many 4 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

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How many 2 digit numbers can you create using the digits 1, 2, 3, and 4, assuming no digit repeats?

Example

In how many ways can 6 people be seated in a row?

Example

In how many ways can 6 people be seated around a circular table with 6 chairs? (Only relative positions can be distinguished.)

Permutations

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A permutation is an ordered arrangement of objects.

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Definition

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ & \end{cases}$$

Permutations

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A permutation is an ordered arrangement of objects.

The number of distinct permutations of r distinct objects chosen from n distinct objects is denoted by $P(n, r)$.

Definition

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

Computing the number of permutations

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Computing $P(n, r)$

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Computing $P(n, r)$

Using the multiplication principle,

Computing the number of permutations

Computing $P(n, r)$

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$$P(n,r) \quad =$$

Computing the number of permutations

Computing $P(n, r)$

Using the multiplication principle,

$$P(n, r) = n$$

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Using the multiplication principle,

$$P(n, r) \quad = \quad n \cdot (n-1)$$

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Computing $P(n, r)$

Using the multiplication principle,

$$P(n, r) \quad = \quad n \cdot (n-1) \cdot \ldots (n-r+1)$$

Computing the number of permutations

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$$\begin{aligned} P(n, r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\ &= \end{aligned}$$

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Using the multiplication principle,

$$\begin{aligned} P(n,r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\ &= n \cdot (n-1) \cdot \ldots (n-r+1) \cdot \frac{(n-r) \cdot (n-r-1) \cdot \ldots 1}{(n-r) \cdot (n-r-1) \cdot \ldots 1} \end{aligned}$$

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$$
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P(n,r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\
&= n \cdot (n-1) \cdot \ldots (n-r+1) \cdot \frac{(n-r) \cdot (n-r-1) \cdot \ldots 1}{(n-r) \cdot (n-r-1) \cdot \ldots 1} \\
&= \frac{n!}{(n-r)!},\ 0 \leq r \leq n
\end{aligned}
$$

Permutations (contd.)

Permutations (contd.)

Example

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution:

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210,

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1,

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution: 210, 1, n,

Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Solution: 210, 1, n, and $n!$.

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution:

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8,3)$.

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8,3)$.

Example

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8,3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8,3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Solution:

Permutations (contd.)

Example

Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.

Solution: 210, 1, n, and $n!$.

Example

How many 3 letter words can be formed using the letters in the word "compiler"?

Solution: $P(8,3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?

Solution: $P(20,2)$.

One more example

One more example

Example

One more example

Example

A library has 4 books on programming, 7 on algorithms and 3 on complexity.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

Provided that the books of a subject are required to be together?

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Solution

If there is no restriction, the number of arrangements is

One more example

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Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

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If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects.

One more example

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

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Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

One more example

Example

A library has 4 books on programming, 7 on algorithms and 3 on complexity.

In how many ways can the books be ordered on a shelf?

Provided that the books of a subject are required to be together?

Solution

If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

Corresponding to each such arrangement,

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If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Corresponding to each such arrangement,

the programming books can be permuted in $P(4, 4) = 4!$ ways,

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A library has 4 books on programming, 7 on algorithms and 3 on complexity.

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If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

Corresponding to each such arrangement,

the programming books can be permuted in $P(4, 4) = 4!$ ways,

the algorithms books can be permuted in $P(7, 7) = 7!$ ways,

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the programming books can be permuted in $P(4, 4) = 4!$ ways,

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the programming books can be permuted in $P(4, 4) = 4!$ ways,

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Now consider the case in which the books of a given subject are required to be together.

First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

Corresponding to each such arrangement,

the programming books can be permuted in $P(4, 4) = 4!$ ways,

the algorithms books can be permuted in $P(7, 7) = 7!$ ways,

and the complexity books can be permuted in $P(3, 3) = 3!$ ways.

Using the multiplication principle, the total number of arrangements is $3! \cdot 4! \cdot 7! \cdot 3!$.

More Examples

More Examples

Example

More Examples

Example

Solve the motivating examples.

Motivating Examples

Motivating Examples

Example

Motivating Examples

Example

How many 5-card hands are possible with a 52 card deck?

Motivating Examples

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How many 5-card hands are possible with a 52 card deck?

Example

Ten athletes compete in an Olympic event.

Motivating Examples

Example

How many 5-card hands are possible with a 52 card deck?

Example

Ten athletes compete in an Olympic event. Three will be declared winners.

Motivating Examples

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How many 5-card hands are possible with a 52 card deck?

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Ten athletes compete in an Olympic event. Three will be declared winners.

In how many ways can the winners be selected?

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A committee of 3 is to be formed from 5 men and 2 women.

Motivating Examples

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In how many ways can the committee be formed, if

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In how many ways can the winners be selected?

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In how many ways can the committee be formed, if

1. The committee must include at least one woman.

Motivating Examples

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In how many ways can the winners be selected?

Example

A committee of 3 is to be formed from 5 men and 2 women.

In how many ways can the committee be formed, if

1. The committee must include at least one woman.
2. There cannot be more than two men on the committee.

Combinations

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A combination is an (unordered) selection of objects.

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The number of distinct combinations of r distinct objects chosen from n distinct objects is denoted by $C(n, r)$.

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Computing $C(n, r)$

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The objects in this combination can be permuted in $r!$ different ways to get $r!$ distinct permutations.

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It follows that $C(n, r) \cdot r! = P(n, r)$, i.e., $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$, $0 \leq r \leq n$.

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Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

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Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution:

Combinations

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The objects in this combination can be permuted in $r!$ different ways to get $r!$ distinct permutations.

It follows that $C(n,r)\cdot r! = P(n,r)$, i.e., $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\cdot(n-r)!}$, $0 \le r \le n$.

Example

Compute $C(7,3)$, $C(n,0)$, $C(n,1)$ and $C(n,n)$.

Solution: 35,

Combinations

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The objects in this combination can be permuted in $r!$ different ways to get $r!$ distinct permutations.

It follows that $C(n, r) \cdot r! = P(n, r)$, i.e., $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$, $0 \leq r \leq n$.

Example

Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

Solution: 35, 1,

Combinations

Definition

A combination is an (unordered) selection of objects.

The number of distinct combinations of r distinct objects chosen from n distinct objects is denoted by $C(n, r)$.

Computing $C(n, r)$

Focus on a given combination of r objects chosen from n objects.

The objects in this combination can be permuted in $r!$ different ways to get $r!$ distinct permutations.

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4. it contains at least one freshman. **Solution:** $C(53,8) - C(34,8)$.

More examples

More examples

Example

More examples

Example

Solve the motivating examples.

Exercises

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Identities

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1. In how many ways, can you seat 11 men and 8 women in a row, so that no two women sit together?
2. A committee of three has to be chosen from five Democrats, three Republicans and four independents.

   In how many ways can the committee be chosen, if it cannot include both Democrats and Republicans?

Motivation

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(i) $(a+b)^1 =$

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We want a general formula that permits us to write down the terms of $(a+b)^n$ without actual multiplication.

Pascal's Triangle

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The coefficient table

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Consider the following table:

Pascal's Triangle

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Row 0: $C(0,0)$

Pascal's Triangle

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Consider the following table:

Row 0:			$C(0, 0)$		
Row 1:		$C(1, 0)$		$C(1, 1)$	

Pascal's Triangle

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Consider the following table:

Row 0:			$C(0,0)$			
Row 1:		$C(1,0)$		$C(1,1)$		
Row 2:	$C(2,0)$		$C(2,1)$		$C(2,2)$	

Pascal's Triangle

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Consider the following table:

Row 0:				$C(0,0)$				
Row 1:			$C(1,0)$		$C(1,1)$			
Row 2:		$C(2,0)$		$C(2,1)$		$C(2,2)$		
Row 3:	$C(3,0)$		$C(3,1)$		$C(3,2)$		$C(3,3)$	

Pascal's Triangle

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$$\begin{array}{lccccccc}
\text{Row 0:} & & & & C(0,0) & & & \\
\text{Row 1:} & & & C(1,0) & & C(1,1) & & \\
\text{Row 2:} & & C(2,0) & & C(2,1) & & C(2,2) & \\
\text{Row 3:} & C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) \\
\vdots & & & & & & &
\end{array}$$

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Row 1:				$C(1,0)$		$C(1,1)$			
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Row 3:		$C(3,0)$		$C(3,1)$		$C(3,2)$		$C(3,3)$	
$\vdots$									
Row n:	$C(n,0)$		$C(n,1)$	$\ldots$	$\ldots$		$C(n,n-1)$		$C(n,n)$

Pascal's triangle (contd.)

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The Value Table

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Row 0: 1

Pascal's triangle (contd.)

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Row 1:	1		1

Pascal's triangle (contd.)

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Row 1:		1		1		
Row 2:	1		2		1	

Pascal's triangle (contd.)

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Row 0:				1				
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Row 3:	1		3		3		1	

Pascal's triangle (contd.)

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\text{Row } 0: & & & & 1 & & & \\
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\text{Row } 3: & 1 & & 3 & & 3 & & 1 \\
\vdots & & & & & & &
\end{array}
$$

Pascal's triangle (contd.)

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\text{Row } 3: & & 1 & & 3 & & 3 & & 1 & \\
\vdots & & & & & & & & & \\
\text{Row } n: & 1 & & n & & \ldots & & n & & 1
\end{array}
$$

Pascal's formula

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Theorem

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$C(n, k) = C(n-1, k-1) + C(n-1, k)$, $1 \leq k \leq n-1$.

Proving Pascal's formula

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Proof.

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Observe that,

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$$C(n-1, k-1) + C(n-1, k) =$$

Proving Pascal's formula

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Observe that,

$$C(n-1, k-1) + C(n-1, k) = \frac{(n-1)!}{(k-1)! \cdot [(n-1-(k-1))!]} + \frac{(n-1)!}{k! \cdot (n-1-k)!}$$

Proving Pascal's formula

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Observe that,

$$\begin{aligned} C(n-1, k-1) + C(n-1, k) &= \frac{(n-1)!}{(k-1)! \cdot [(n-1-(k-1))!]} + \frac{(n-1)!}{k! \cdot (n-1-k)!} \\ &= \frac{(n-1)!}{(k-1)! \cdot (n-k)!} + \frac{(n-1)!}{k! \cdot (n-1-k)!} \end{aligned}$$

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Observe that,

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Proving Pascal's formula

Proof.

Observe that,

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C(n-1, k-1) + C(n-1, k) &= \frac{(n-1)!}{(k-1)! \cdot [(n-1-(k-1))!]} + \frac{(n-1)!}{k! \cdot (n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)! \cdot (n-k)!} + \frac{(n-1)!}{k! \cdot (n-1-k)!} \\
&= \frac{k \cdot (n-1)!}{k! \cdot (n-k)!} + \frac{(n-k) \cdot (n-1)!}{k! \cdot (n-k)!} \\
&= \frac{(n-1)!}{k! \cdot (n-k)!}[k + (n-k)]
\end{aligned}$$

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&= \frac{(n-1)!}{k! \cdot (n-k)!}[k + (n-k)] \\
&= \frac{n \cdot (n-1)!}{k! \cdot (n-k)!}
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&= \frac{(n-1)!}{k! \cdot (n-k)!}[k + (n-k)] \\
&= \frac{n \cdot (n-1)!}{k! \cdot (n-k)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \\
&= C(n, k).
\end{aligned}$$

$\square$

Alternative Proof

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A second Proof

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(iv) Let T_1 denote the number of ways in which k objects are selected from the n objects, with o definitely included.

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(viii) Using the addition principle, $C(n, k) = T_1 + T_2 = C(n-1, k-1) + C(n-1, k)$.

Note on Proof Techniques

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The above proof is called a combinatorial proof and is always preferred on account of its elegance.

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The above proof is called a combinatorial proof and is always preferred on account of its elegance.

Recall the combinatorial proof for proving that $C(n, r) = C(n, n - r)$.

The Theorem

The Theorem

Theorem

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Theorem

$$(a+b)^n =$$

The Theorem

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Exercise

Find bounds on H_n, the n^{th} Harmonic number.