Complexity

Theorem

The **k-COLORABILITY***-problem is* NP*-complete for any fixed* $k \geq 3$.
The **2-COLORABILITY***-problem is in* P.

Proof

NP-Membership of **k-COLORABILITY**:
1. Guess an assignment $f : V \to \{1, \ldots, k\}$
2. Check for every edge $[i, j] \in E$ that $f(i) \neq f(j)$.

P-Membership of **2-COLORABILITY**: (w.l.o.g., G is connected)
1. Start by assigning an arbitrary color to an arbitrary vertex $v \in V$.
2. Suppose that the vertices in $S \subset V$ have already been assigned a color. Choose $x \in S$ and assign to all vertices adjacent to x the opposite color.

G is 2-colorable iff step 2 never leads to a contradiction.

NP-Hardness Proof of **3-COLORABILITY**

By reduction from **NAESAT**: Let an arbitrary instance of **NAESAT** be given by a Boolean formula $\varphi = c_1 \wedge \ldots \wedge c_m$ in 3-CNF with variables $x_1, \ldots, x_n$. We construct the following graph $G(\varphi)$:

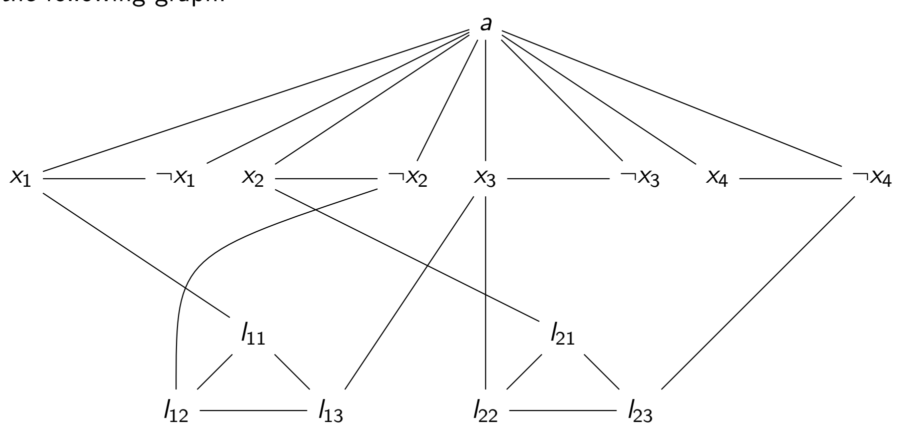
Let $V = \{a\} \cup \{x_i, \neg x_i \mid 1 \leq i \leq n\} \cup \{l_{i1}, l_{i2}, l_{i3} \mid 1 \leq i \leq m\}$, i.e. $|V| = 1 + 2n + 3m$.

For each variable x_i in φ, we introduce a triangle $[a, x_i, \neg x_i]$, i.e. all these triangles share the node a.

For each clause c_i in φ, we introduce a triangle $[l_{i1}, l_{i2}, l_{i3}]$. Moreover, each of these vertices l_{ij} is further connected to the node corresponding to this literal, i.e.: if the j-th literal in c_i is of the form x_α (resp. $\neg x_\alpha$) then we introduce an edge between l_{ij} and x_α (resp. $\neg x_\alpha$)

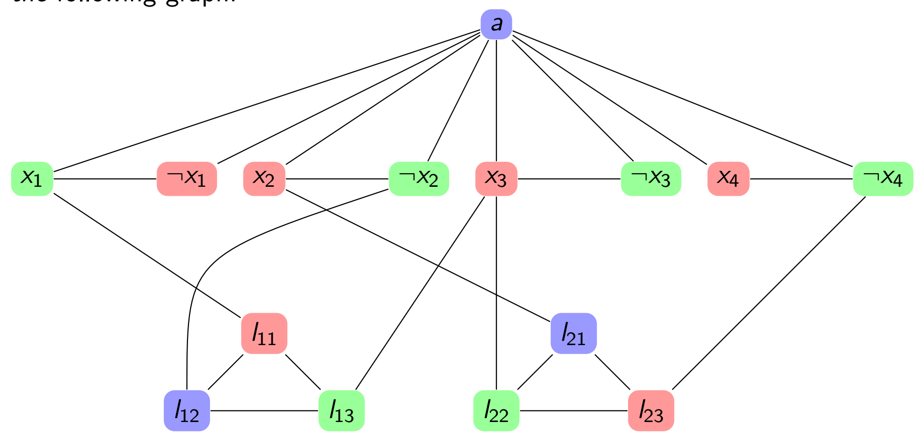
Example

The 3-CNF formula $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$ is reduced to the following graph:



Example

The 3-CNF formula $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$ is reduced to the following graph:



Let red = **false** and green = **true**. The above 3-coloring corresponds to $T(x_1) = T(\neg x_2) = T(\neg x_3) = T(\neg x_4) =$ **true**.

Correctness of the Problem Reduction

Proof (continued)

"$\Leftarrow$" Suppose that G has a 3-coloring with colors $\{0, 1, 2\}$. W.l.o.g., the node a has the color 2. This induces a truth assignment T via the colors of the nodes x_i: if the color is 1, then $T(x_i) = \mathbf{true}$ else $T(x_i) = \mathbf{false}$. We claim that T is a legal **NAESAT**-assignment. Indeed, if in some clause, all literals had the value **false** (resp. **true**), then we could not use the color 0 (resp. 1) for coloring the triangle $[l_{i1}, l_{i2}, l_{i3}]$, a contradiction.

"$\Rightarrow$" Suppose that there exists an **NAESAT**-assignment T of φ. Then we can extract a 3-coloring for G from T as follows:

(i) Node a is colored with color 2.
(ii) If $T(x_i) = \mathbf{true}$, then color x_i with 1 and $\neg x_i$ with 0 else vice versa.
(iii) From each $[l_{i1}, l_{i2}, l_{i3}]$, color two literals having opposite truth values with 0 (**true**) and 1 (**false**). Color the third with 2.

HAMILTON-PATH

INSTANCE: (directed or undirected) graph $G = (V, E)$

QUESTION: Does G have a *Hamilton path*?
i.e., a path visiting all vertices of G exactly once.

HAMILTON-CYCLE

INSTANCE: (directed or undirected) graph $G = (V, E)$

QUESTION: Does G have a *Hamilton cycle*?
i.e., a cycle visiting all vertices of G exactly once.

TSP(D)

INSTANCE: n cities $1, \dots, n$ and a nonnegative integer distance d_{ij} between any two cities i and j (such that $d_{ij} = d_{ji}$), and an integer B.

QUESTION: Is there a tour through all cities of length at most B?
i.e., a permutation π s.t. $\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$ with $\pi(n+1) = \pi(1)$.

Complexity

Theorem

HAMILTON-PATH, **HAMILTON-CYCLE**, *and* **TSP(D)** *are* NP-*complete.*

Proof

We shall show the following chain of reductions:

$$\textbf{HAMILTON-PATH} \leq_{\mathrm{L}} \textbf{HAMILTON-CYCLE} \leq_{\mathrm{L}} \textbf{TSP(D)}$$

It suffices to show NP-membership for the *hardest* problem:
1. Guess a tour π through the n cities.
2. Check that $\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$ with $\pi(n+1) = \pi(1)$.

Likewise, it suffices to prove the NP-hardness of the *easiest* problem.
The NP-hardness of **HAMILTON-PATH** (by a reduction from **3-SAT**) is quite involved and is therefore omitted here (see Papadimitriou's book).

HAMILTON-PATH vs. HAMILTON-CYCLE

HAMILTON-PATH $\leq_{\mathrm{L}}$ HAMILTON-CYCLE

(We only consider undirected graphs). Let an arbitrary instance of **HAMILTON-PATH** be given by the graph $G = (V, E)$. We construct an equivalent instance $G' = (V', E')$ of **HAMILTON-CYCLE** as follows:

Let $V' := V \cup \{z\}$ for some new vertex z and $E' := E \cup \{[v, z] \mid v \in V\}$.
G has a Hamilton path $\Leftrightarrow$ G' has a Hamilton cycle

"$\Rightarrow$" Suppose that G has a Hamilton path π starting at vertex a and ending at b. Then $\pi \cup \{z\}$ is clearly a Hamilton cycle in G'.

"$\Leftarrow$" Let C be a Hamilton cycle in G'. In particular, C goes through z. Let a and b be the two neighboring nodes of z in this cycle. Then $C \setminus \{z\}$ is a Hamilton path (starting at vertex a and ending at b) in G.