Probability and Expectation

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Outline

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Note

Discrete Probability combines aspects of logic, set theory, combinatorics and inference.

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(i) Suppose that the experiment consists of tossing a coin. Then, $S = \{H, T\}$.

(ii) Suppose that the experiment consists of tossing a die.
Then, $S = \{1, 2, 3, 4, 5, 6\}$.

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Then, $S = [0, \infty)$.

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(iii) Suppose that the experiment consists of tossing two coins.
Then, $S = \{HH, \ HT, \ TH, \ TT\}$.

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Any subset of the sample space S is called an event.

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Remember that events are sets when reasoning about them.

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The above three conditions are called the axioms of probability theory.

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In this case,

$$P(E) = \sum_{x_i \in E} p(x_i).$$

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Assume that all 4 outcomes are equally likely.

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Let E denote the event that the first coin turns up heads and F denote the event that the second coin turns up heads.

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Exercise

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What is the probability that either the first or the second coin turns up heads?

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The conditional probability of E, given that the event F has occurred is denoted by $P(E \mid F)$.

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$P(E \mid F)$ is equal to $\frac{P(EF)}{P(F)}$, assuming $P(F) > 0$.

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Hence, $P(E \mid F) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

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Observe that $P(F) = \frac{1}{2}$ and $P(EF) = \frac{1}{4}$.

Hence, $P(E \mid F) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

Notice that $P(E) = \frac{1}{4} \neq P(E \mid F)$.

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Assume that the sample space is $S = \{(b, g),\ (b, b),\ (g, b),\ (g, g)\}$

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A family has two children.

What is the conditional probability that both are boys given that at least one of them is a boy?

Assume that the sample space is $S = \{(b, g),\ (b, b),\ (g, b),\ (g, g)\}$ and that all outcomes are equally likely.

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Assume that an urn contains 7 *black balls and* 5 *white balls.*

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Assume that an urn contains 7 black balls and 5 white balls.

Two balls are chosen from this urn, one after the other, without replacement and at random.

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Solution

Let E denote the event that the first ball is black and F denote the event that the second ball is black.

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Assume that an urn contains 7 black balls and 5 white balls.

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Let X denote the event that the patient responded positively to compound A. Let Y denote the event that the patient responded positively to compound B. We are interested in the event $Y \mid X$.

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Are E_1 and F independent?

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Are E_1 and F independent?

How about E_2 and F?

Bayes' Formula

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$$P(E) = P(EF) + P(EF^c)$$

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Thus, the probability of an event E occurring can be computed as the weighted average of its conditional probability on an arbitrary event F.

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Urn 1 has 2 white balls and 7 black balls.

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A fair coin is tossed.

If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2.

Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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Let W denote the event that a white ball was drawn and let H denote the event that the coin turned up heads.

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Accordingly, we have, $P(H) = \frac{1}{2}$ and $P(W \mid H) = \frac{2}{9}$.

We are interested in the quantity $P(H \mid W)$.

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As per Bayes' formula,

$$P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^c)(1 - P(H))$$

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As per Bayes' formula,

$$\begin{aligned} P(W) &= P(W \mid H) \cdot P(H) + P(W \mid H^c)(1 - P(H)) \\ &= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2} \end{aligned}$$

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$,

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From conditional probability, we know that, $P(H \mid W) = \frac{P(HW)}{P(W)}$.

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.

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We may not care whether the actual outcome is $(1, 6)$, $(6, 1)$, or . . . .

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In other words, a random variable is a function $X : S \to E$, where S is the sample space and E is a measurable space.

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$$\begin{aligned} P\{X = 1\} &= 0 \\ P\{X = 2\} &= \frac{1}{36} \end{aligned}$$

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&\vdots \\
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$$P\{Y = 0\} \quad =$$

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$$P\{Y = 0\} = \frac{1}{4}$$

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\begin{aligned}
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P\{Y = 1\} &=
\end{aligned}
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$$\begin{aligned} P\{Y = 0\} &= \frac{1}{4} \\ P\{Y = 1\} &= \frac{1}{2} \end{aligned}$$

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$$
\begin{aligned}
P\{Y=0\} &= \frac{1}{4} \\
P\{Y=1\} &= \frac{1}{2} \\
P\{Y=2\} &=
\end{aligned}
$$

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\begin{aligned}
P\{Y=0\} &= \frac{1}{4} \\
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\end{aligned}
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Discrete Random Variable

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Definition

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A random variable that can take on only a countable number of possible values is said to be *discrete*.

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For a discrete random variable X, the probability mass function (pmf) $p(a)$ is defined as:

$$p(a) = P\{X = a\}.$$

The Bernoulli Random Variable

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Main idea

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Consider an experiment which has exactly two outcomes;

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If we let the random variable X assume the value 1, if the experiment was a success and 0, if the experiment was a failure, then X is said to be a Bernoulli random variable.

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Assume that the probability that the experiment results in a success is p.

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$$p(1) = P\{X = 1\} = p.$$

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The probability mass function of X is given by:

$$\begin{aligned} p(1) &= P\{X = 1\} = p. \\ p(0) &= P\{X = 0\} = 1 - p. \end{aligned}$$

The Binomial Random Variable

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Consider an experiment which consists of n independent Bernoulli trials, with the probability of success in each trial being p.

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The probability mass function of X is given by:

$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1 - p)^{n-i}, \; i = 0, 1, 2, \ldots n.$$

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What is the probability that you will get two heads and two tails?

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Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials.

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$$p(2) \quad =$$

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As discussed previously,

$$\begin{aligned} p(2) &= C(4,2) \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2 \\ &= \frac{3}{8}. \end{aligned}$$

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If X is the random variable that counts the number of trials until the first success, then X is said to be a geometric random variable.

The probability mass function of X is given by:

$$p(i) = P\{X = i\} = (1-p)^{i-1} \cdot p, \ i = 1, 2, \ldots$$

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Let X denote a discrete random variable with probability mass function $p(x)$.

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Let X denote the random variable that records the outcome of tossing a fair die.

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$E[X]$ is the weighted average of the possible values that X can assume, each value being weighted by the probability that X assumes that value.

Example

Let X denote the random variable that records the outcome of tossing a fair die.

What is $E[X]$?

Expectation of a Bernoulli Random Variable

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Bernoulli Random Variable

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Let X denote a Bernoulli Random Variable with p denoting the probability of success.

Expectation of a Bernoulli Random Variable

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Let X denote a Bernoulli Random Variable with p denoting the probability of success. Compute $E[X]$.

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Let X denote a Bernoulli Random Variable with p denoting the probability of success. Compute $E[X]$.

Solution

$$E[X] \quad =$$

Expectation of a Bernoulli Random Variable

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Let X denote a Bernoulli Random Variable with p denoting the probability of success. Compute $E[X]$.

Solution

$$E[X] = 1 \cdot p$$

Expectation of a Bernoulli Random Variable

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Let X denote a Bernoulli Random Variable with p denoting the probability of success. Compute $E[X]$.

Solution

$$E[X] = 1 \cdot p + 0 \cdot (1 - p)$$

Expectation of a Bernoulli Random Variable

Bernoulli Random Variable

Let X denote a Bernoulli Random Variable with p denoting the probability of success. Compute $E[X]$.

Solution

$$\begin{aligned} E[X] &= 1 \cdot p + 0 \cdot (1 - p) \\ &= p. \end{aligned}$$

Expectation of a Binomial Random Variable

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Binomial Random Variable

Expectation of a Binomial Random Variable

Binomial Random Variable

Let X denote a Binomial Random Variable with parameters n and p.

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Expectation of a Binomial Random Variable

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Let X denote a Binomial Random Variable with parameters n and p.

Compute $E[X]$.

Solution

$$E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}$$

Expectation of a Binomial Random Variable

Binomial Random Variable

Let X denote a Binomial Random Variable with parameters n and p.
Compute $E[X]$.

Solution

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{n} i \cdot p(i), \text{ by definition} \\
&= \sum_{i=0}^{n} i \cdot C(n,i) \cdot p^i \cdot (1-p)^{n-i}
\end{aligned}
$$

Expectation of a Binomial Random Variable

Binomial Random Variable

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Compute $E[X]$.

Solution

$$\begin{aligned} E[X] &= \sum_{i=0}^{n} i \cdot p(i), \text{ by definition} \\ &= \sum_{i=0}^{n} i \cdot C(n,i) \cdot p^{i} \cdot (1-p)^{n-i} \\ &= \sum_{i=0}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^{i} \cdot (1-p)^{n-i} \end{aligned}$$

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Expectation of a Binomial Random Variable (contd.)

Expectation of a Binomial Random Variable (contd.)

Solution

Expectation of a Binomial Random Variable (contd.)

Solution

Expectation of a Binomial Random Variable (contd.)

Solution

$$E[X] = \sum_{i=1}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i}$$

Expectation of a Binomial Random Variable (contd.)

Solution

$$E[X] = \sum_{i=1}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i}$$

$$=$$

Expectation of a Binomial Random Variable (contd.)

Solution

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\ &= \sum_{i=1}^{n} \frac{n!}{(i-1)! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \end{aligned}$$

Expectation of a Binomial Random Variable (contd.)

Solution

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\ &= \sum_{i=1}^{n} \frac{n!}{(i-1)! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\ &= n \cdot p \cdot \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)! \cdot (n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i} \end{aligned}$$

Expectation of a Binomial Random Variable (contd.)

Solution

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n} i \cdot \frac{n!}{i! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\
&= \sum_{i=1}^{n} \frac{n!}{(i-1)! \cdot (n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\
&= n \cdot p \cdot \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)! \cdot (n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}
\end{aligned}
$$

Expectation of a Binomial Random Variable (contd.)

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Solution

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$E[X] =$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$E[X] = n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i-1)$, we get,

$$E[X] = n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$

$$= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k}$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

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$$=$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$
\begin{aligned}
E[X] &= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1} \\
&= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k} \\
&= n \cdot p \cdot \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k}
\end{aligned}
$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$
\begin{aligned}
E[X] &= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1} \\
&= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k} \\
&= n \cdot p \cdot \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k} \\
&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ recall the Binomial theorem}
\end{aligned}
$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$\begin{aligned}
E[X] &= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1} \\
&= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k} \\
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&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ recall the Binomial theorem} \\
&= n \cdot p \cdot 1
\end{aligned}$$

Expectation of a Binomial Random Variable (contd.)

Solution

Substituting $k = (i - 1)$, we get,

$$
\begin{aligned}
E[X] &= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1} \\
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&= n \cdot p \cdot \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k} \\
&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ recall the Binomial theorem} \\
&= n \cdot p \cdot 1 \\
&= n \cdot p. \qquad \square
\end{aligned}
$$

Expectation of a Geometric Random Variable

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Compute $E[X]$.

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Expectation of a Geometric Random Variable

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Compute $E[X]$.

Solution

$$E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}$$

Expectation of a Geometric Random Variable

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E[X] &= \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition} \\
&= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p
\end{aligned}
$$

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Expectation of a Geometric Random Variable (contd.)

Expectation of a Geometric Random Variable (contd.)

Solution

Expectation of a Geometric Random Variable (contd.)

Solution

$$E[X] = p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1}$$

Expectation of a Geometric Random Variable (contd.)

Solution

$$E[X] = p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1}$$

$$=$$

Expectation of a Geometric Random Variable (contd.)

Solution

$$
\begin{aligned}
E[X] &= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} \\
&= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq}[q^i]
\end{aligned}
$$

Expectation of a Geometric Random Variable (contd.)

Solution

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Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

Solution

$$
\begin{aligned}
E[X] &= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} \\
&= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq}[q^i] \\
&= p \cdot \frac{d}{dq}[\sum_{i=1}^{\infty} q^i] \\
&= p \cdot \frac{d}{dq}[\frac{q}{1-q}] \\
&= p \cdot \frac{(1-q) \cdot \frac{d}{dq}[q] - q \cdot \frac{d}{dq}[1-q]}{(1-q)^2}
\end{aligned}
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Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

Solution

$$\begin{aligned} E[X] &= p \cdot \frac{1}{(1-q)^2} \\ &= p \cdot \frac{1}{p^2} \\ &= \frac{1}{p}. \end{aligned}$$

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Let X be a random variable, with the following pmf:

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Compute $E[X^2]$.

Expectation of functions of random variables (contd.)

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Solution

Expectation of functions of random variables (contd.)

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Let $Y = X^2$.

Expectation of functions of random variables (contd.)

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Observe that Y is also a random variable.

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Expectation of functions - The Direct Approach

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Note that linearity of expectation holds even when the random variables are **not** *independent.*

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Example

What is the expected value of the sum of the upturned faces, when two fair dice are tossed?

Another Application

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Compute the expected value of the Binomial random variable.

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□

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Note that $E[X \mid Y]$ is itself a random variable.

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Example

Computing the mean of the geometric random variable

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\Rightarrow E[X] &= \frac{1}{p}.
\end{aligned}$$