### Recursion and Recurrence Relations

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Recursive Definitions

- Recursive Definitions
- Solving Recurrences

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- Formula (including Master Theorem)
  - The Master Method

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- Expand-Guess-Verify
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- 5 The Recurrence Tree Method

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- (ii) An inductive or recursive step, where new cases of the item being defined are given in terms of previous cases.

#### Note

Strong connection between induction and recursion.

### Recursive Objects

(i) Sequences.

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- (ii) Sets.

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- (ii) Sets.
- (iii) Operations.

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- (iii) Operations.
- (iv) Algorithms.

# Sequences

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Write down the first 5 elements of the following recursively defined sequence:

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The second part of the definition is called a recurrence relation.

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$$T(1) = 1$$
  
 $T(n) = T(n-1) + 3, n \ge 2.$ 

# Sequences (contd.)

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Fibonacci Sequence

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Enumerate the first 5 elements of the Fibonacci sequence. Show that

$$F(n+4) = 3 \cdot F(n+2) - F(n)$$
, for all  $n \ge 1$ 

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Observe that,

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Applying the second principle of mathematical induction, we conclude that the conjecture is true for all  $n \geq 1$ .

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# Recursively Defined Sets

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- (i) The empty string  $\lambda$  is a word.
- (ii) {a, b, c, ..., z} are words.
- (iii) If x and y are words, then so is  $x \cdot y$ .



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- (ii) 0 and 1 are palindromes.
- (iii) If x is a palindrome, then so are  $0 \cdot x \cdot 0$  and  $1 \cdot x \cdot 1$ .

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Define multiplication in terms of addition.

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$$x \cdot y = x + x \cdot (y - 1), y \ge 1.$$

#### Function Max(a, b)

- 1: if  $(a \ge b)$  then
- 2: **return**(*a*).
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#### The Find-Max Algorithm

#### Function FIND-MAX(A, n)

- 1: **if** (n = 1) **then**
- 2: **return**(A[1]).
- 3: else
- 4: **return**(MAX(A[n], FIND-MAX(A, n 1))).
- 5: end if

#### **Function** Max(a, b)

- 1: if (a > b) then
- 2: **return**(a).
- 3: else
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#### Function FIND-MAX(A, n)

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- 2: return(A[1]).
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Can you prove the correctness of the above algorithm?

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(i) Expand-Guess-Verify (EGV).

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- (ii) Formula (including Master Theorem).

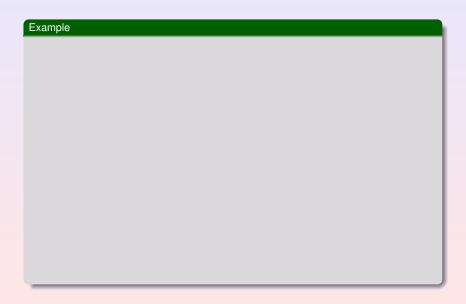
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- (iii) Recursion Tree.



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$$\text{(i)} \ \ \mathsf{Expand:} \ S(1) = 1, \, S(2) = S(1) + 1 = 2, \, S(3) = S(2) + 1 = 3,$$

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- (i) Expand: S(1) = 1, S(2) = S(1) + 1 = 2, S(3) = S(2) + 1 = 3, ....
- (ii) Guess: S(n) =

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$$S(1) = 1$$
  
 $S(n) = S(n-1) + 1, n \ge 2.$ 

- (i) Expand: S(1) = 1, S(2) = S(1) + 1 = 2, S(3) = S(2) + 1 = 3, ....
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$$LHS = 1$$

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Since LHS=RHS, the basis is proven.

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$$S(k+1) = S(k) + 1$$
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 =  $S(k) + 1$ , by definition  
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Expand-Guess-Verify

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- (ii) Guess: S(n) = n.
- (iii) Verify: Using Induction! BASIS: n = 1

$$LHS = 1$$
  
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Since LHS=RHS, the basis is proven.

INDUCTIVE STEP: Assume that S(k)=k. We need to show that S(k+1)=(k+1). Observe that,

$$S(k + 1)$$
 =  $S(k) + 1$ , by definition  
 =  $k + 1$ , by inductive hypothesis

Applying the first principle of mathematical induction, we conclude that S(n) = n.

# Example

### Example

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$$S(1) = 2$$

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$$S(n) = 2 \cdot S(n-1), n \geq 2.$$

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$$S(1) = 2$$

$$S(n) = 2 \cdot S(n-1), n \geq 2.$$

(i) Expand: 
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,  $S(2) =$ 

### Example

$$S(1) = 2$$
  
 $S(n) = 2 \cdot S(n-1), n \ge 2.$ 

(i) Expand: 
$$S(1) = 2$$
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 $S(n) = 2 \cdot S(n-1), n \ge 2.$ 

(i) Expand: 
$$S(1) = 2$$
,  $S(2) = 2 \cdot S(2) = 4$ ,  $S(3) = 1$ 

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 $S(n) = 2 \cdot S(n-1), n \ge 2.$ 

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- (i) Expand: S(1) = 2,  $S(2) = 2 \cdot S(2) = 4$ ,  $S(3) = 2 \cdot S(2) = 8$ , ....
- (ii) Guess:  $S(n) = 2^n$ .

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- (iii) Verify: Using Induction! BASIS: n = 1

#### Example

$$S(1) = 2$$
  
 $S(n) = 2 \cdot S(n-1), n > 2.$ 

- (i) Expand: S(1) = 2,  $S(2) = 2 \cdot S(2) = 4$ ,  $S(3) = 2 \cdot S(2) = 8$ , ....
- (ii) Guess:  $S(n) = 2^n$ .
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$$LHS = 2$$

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Since LHS=RHS, the basis is proven.

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Since LHS=RHS, the basis is proven.

INDUCTIVE STEP: Assume that  $S(k) = 2^k$ . We need to show that  $S(k+1) = 2^{k+1}$ . Observe that,

$$S(k + 1)$$
 =  $2 \cdot S(k)$ , by definition  
=  $2 \cdot 2^k$ , by inductive hypothesis  
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Applying the first principle of mathematical induction, we conclude that  $S(n) = 2^n$ .

Example	

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### Example

$$T(1) = 1$$

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$$T(1) = 1$$
  
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- (ii) Guess:  $T(n) = 3 \cdot n 2$ .

## EGV (contd.)

#### Example

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- (ii) Guess:  $T(n) = 3 \cdot n 2$ .
- (iii) Verify: Somebody from class!

Definition

### Definition

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$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \dots + f_k(n) \cdot S(n-k) + g(n).$$

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The above formula is called linear, because the S() terms occur only in the first power.

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#### Note

The above formula is called linear, because the S() terms occur only in the first power. It is called first-order, if S(n) depends only on S(n-1).

For example,  $S(n) = c \cdot S(n-1) + g(n)$ .

The recurrence is called homogeneous, if g(n) = 0, for all n.

$$S(1) = k_0$$

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$$S(n) = c \cdot S(n-1) + g(n)$$

$$S(1) = k_0$$
  

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$$\Rightarrow S(n) =$$

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$$S(n) = c \cdot S(n-1) + g(n)$$

$$\Rightarrow S(n) = c^{n-1} \cdot k_0 +$$

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$$S(n) = c \cdot S(n-1) + g(n)$$

$$\Rightarrow S(n) = c^{n-1} \cdot k_0 + \sum_{i=2}^n c^{n-i} \cdot g(i).$$

Formula (including Master Theorem)

xample	

$$S(1) = 2$$

$$S(1) = 2$$
  
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### Example

$$S(1) = 2$$
  
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As per the formula,

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$$S(1) = 2$$
  
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$$S(1) = 2$$
  
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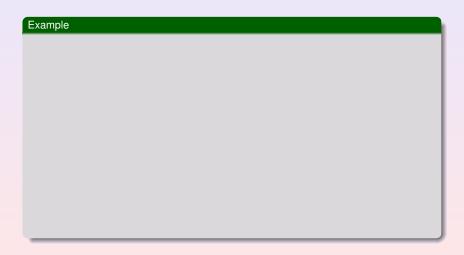
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Solve the recurrence:

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$$= 2^{n+1} + 3 \cdot [2^{n-1} - 1].$$

Formula (including Master Theorem)

# Second Order homogeneous Linear Recurrence with constant coefficients

Formula		

Formula		

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(i) Form:  $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$ , subject to some initial conditions

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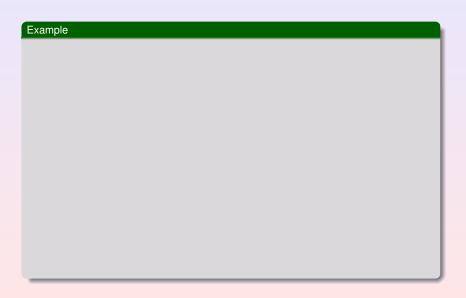
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$$T(1) = 5$$

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$$T(2) = 13$$

## Example

Formula (including Master Theorem)

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$$T(n) = 6 \cdot T(n-1) - 5 \cdot T(n-2), \ n \ge 3.$$

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We get p = 3 and q = 2.

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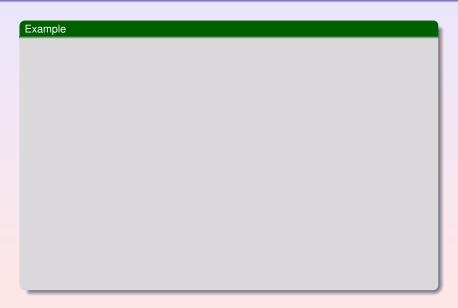
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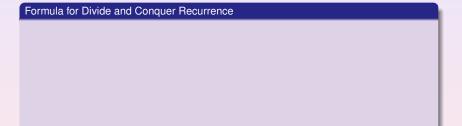
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(iii) Accordingly, the solution is  $S(n) = 4^{n-1} + 2 \cdot (n-1) \cdot 4^{n-1} = (2 \cdot n - 1) \cdot 4^{n-1}$ .



$$S(1) = k_0$$

$$S(1) = k_0$$
  
 $S(n) = c \cdot S(\frac{n}{2}) + g(n), n \ge 2, n = 2^m.$ 

$$\begin{split} & \mathcal{S}(1) & = & k_0 \\ & \mathcal{S}(n) & = & c \cdot \mathcal{S}(\frac{n}{2}) + g(n), \ n \geq 2, \ n = 2^m. \end{split}$$

$$\Rightarrow S(n) =$$

$$S(1) = k_0 S(n) = c \cdot S(\frac{n}{2}) + g(n), \ n \ge 2, \ n = 2^m.$$

$$\Rightarrow S(n) = c^{\log n} \cdot k_0 +$$

# Divide and Conquer Recurrences

### Formula for Divide and Conquer Recurrence

$$S(1) = k_0$$
  
 $S(n) = c \cdot S(\frac{n}{2}) + g(n), n \ge 2, n = 2^m.$ 

$$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i).$$

## Divide and Conquer Recurrences

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Note that  $c^{\log n - i}$  in the expression above stands for  $\frac{c^{\log n}}{c^i}$ .

Example	

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$$C(1) = 1$$

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$$C(1) = 1$$
  
 $C(n) = 1 + C(\frac{n}{2}), n \ge 2, n = 2^{m}.$ 

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Note that

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Note that  $k_0 =$ 

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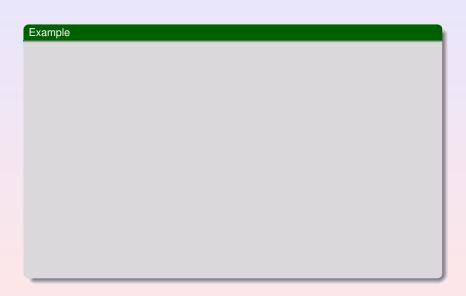
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$$\begin{split} T(n) & = & 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ & = & 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ & = & 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \\ & = & 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\ & = & 3 \cdot n + n \cdot 2 \cdot \log n, \end{split}$$

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### Outline

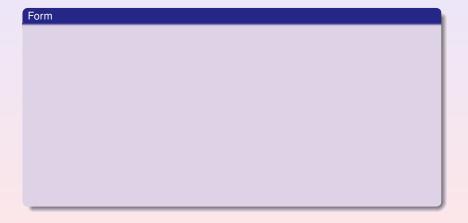
- Recursive Definitions
- Solving Recurrences
- 3 Expand-Guess-Verify
- Formula (including Master Theorem)The Master Method
- 5 The Recurrence Tree Method

Formula (including Master Theorem)

The Master Method

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Recursion

Formula (including Master Theorem)

The Master Method

# Examples

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### Methodology

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## Function Max(a, b)

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#### Note

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#### Note

How many element-to-element comparisons are performed by the FIND-MAX() algorithm on an array of size n?