Recursion and Recurrence Relations

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Outline

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Recursive Definitions

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Note

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Parts of a recursive definition

(i) A basis, where some simple cases of the object being defined are explicitly provided,

(ii) An inductive or recursive step, where new cases of the item being defined are given in terms of previous cases.

Note

Strong connection between induction and recursion.

Types of objects defined recursively

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Recursive Objects

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Recursive Objects

(i) Sequences.

Types of objects defined recursively

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(i) Sequences.
(ii) Sets.

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(i) Sequences.
(ii) Sets.
(iii) Operations.

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Recursive Objects

(i) Sequences.
(ii) Sets.
(iii) Operations.
(iv) Algorithms.

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Write down the first 5 elements of the following recursively defined sequence:

$$S(1) = 2$$

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Write down the first 5 elements of the following recursively defined sequence:

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\begin{aligned}
S(1) &= 2 \\
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Write down the first 5 elements of the following recursively defined sequence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

The second part of the definition is called a recurrence relation.

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Example

Write down the first 5 elements of the following recursively defined sequence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3, \; n \geq 2. \end{aligned}$$

Sequences (contd.)

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Fibonacci Sequence

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Fibonacci Sequence

$$F(1) = 1$$

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Fibonacci Sequence

$$\begin{aligned} F(1) &= 1 \\ F(2) &= 1 \end{aligned}$$

Sequences (contd.)

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\begin{aligned}
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Enumerate the first 5 elements of the Fibonacci sequence.

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Show that

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$$\begin{aligned} F(1) &= 1 \\ F(2) &= 1 \\ F(n) &= F(n-1) + F(n-2),\ n \geq 3 \end{aligned}$$

Example

Enumerate the first 5 elements of the Fibonacci sequence.
Show that

$$F(n+4) = 3 \cdot F(n+2) - F(n),\ \text{for all } n \geq 1$$

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Observe that,

$$F(k + 1 + 4) = F(k + 5)$$

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&= 3 \cdot (F(k+2) + F(k+1)) - (F(k) + F(k-1)) \\
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\end{aligned}$$

Applying the second principle of mathematical induction, we conclude that the conjecture is true for all $n \geq 1$. □

Direct proof

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$$F(n+4) = F(n+3) + F(n+2)$$

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\begin{aligned}
F(n+4) &= F(n+3) + F(n+2) \\
&= F(n+2) + F(n+1) + F(n+2) \\
&= 2 \cdot F(n+2) + F(n+1)
\end{aligned}
$$

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&= F(n+2) + F(n+1) + F(n+2) \\
&= 2 \cdot F(n+2) + F(n+1) \\
&= 2 \cdot F(n+2) + (F(n+2) - F(n))
\end{aligned}$$

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&= 2 \cdot F(n+2) + F(n+1) \\
&= 2 \cdot F(n+2) + (F(n+2) - F(n)) \\
&= 3 \cdot F(n+2) - F(n)
\end{aligned}
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□

Recursively Defined Sets

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Example

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Define the set of ancestors of John.

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(i) John's parents are his ancestors.

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(ii) $\{$**a, b, c, . . . , z**$\}$ are words.
(iii) If x and y are words, then so is $x \cdot y$.

Recursively Defined Sets (contd.)

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Recursively Defined Sets (contd.)

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Define the set of binary palindromes.

(i) The empty string λ is a palindrome.
(ii) 0 and 1 are palindromes.

Recursively Defined Sets (contd.)

Example

Define the set of binary palindromes.

(i) The empty string λ is a palindrome.
(ii) 0 and 1 are palindromes.
(iii) If x is a palindrome, then so are $0 \cdot x \cdot 0$ and $1 \cdot x \cdot 1$.

Recursively Defined Operations

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$$a^0 = 1$$

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Define exponentiation in terms of multiplication.

$$\begin{aligned} a^0 &= 1 \\ a^m &= a \cdot (a^{m-1}), \; m \geq 1. \end{aligned}$$

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Define multiplication in terms of addition.

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Define multiplication in terms of addition.

$$x \cdot 0 = 0$$

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Define exponentiation in terms of multiplication.

$$\begin{aligned} a^0 &= 1 \\ a^m &= a \cdot (a^{m-1}),\ m \geq 1. \end{aligned}$$

Example

Define multiplication in terms of addition.

$$\begin{aligned} x \cdot 0 &= 0 \\ x \cdot y &= x + x \cdot (y-1),\ y \geq 1. \end{aligned}$$

Recursively Defined Algorithms

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Function MAX(a, b)

```
if (a ≥ b) then
    return(a).
else
    return(b).
end if
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The Find-Max Algorithm

Recursively Defined Algorithms

Function MAX(a, b)

if $(a \geq b)$ **then**
    return(a).
else
    return(b).
end if

The Find-Max Algorithm

Function FIND-MAX($\mathbf{A}, n$)

if $(n = 1)$ **then**
    return($A[1]$).
else
    return(MAX($A[n]$, FIND-MAX($\mathbf{A}, n - 1$))).
end if

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The Find-Max Algorithm

Function FIND-MAX($\mathbf{A}$, n)
```
if (n = 1) then
    return(A[1]).
else
    return(MAX(A[n], FIND-MAX(A, n − 1))).
end if
```

Can you prove the correctness of the above algorithm?

Solving recurrences

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Theww methods

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Theww methods

(i) Expand-Guess-Verify (EGV).

Solving recurrences

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Given a recurrence relation describing a function, say $T(n)$,
we want to find a closed form expression which exactly describes $T(n)$,
i.e., an expression not involving calls to the function $T()$.

Theww methods

(i) Expand-Guess-Verify (EGV).
(ii) Formula (including Master Theorem).

Solving recurrences

Problem definition

Given a recurrence relation describing a function, say $T(n)$,
we want to find a closed form expression which exactly describes $T(n)$,
i.e., an expression not involving calls to the function $T()$.

Theww methods

(i) Expand-Guess-Verify (EGV).
(ii) Formula (including Master Theorem).
(iii) Recursion Tree.

Expand-Guess-Verify

Expand-Guess-Verify

Example

Expand-Guess-Verify

Example

Consider the recurrence:

Expand-Guess-Verify

Example

Consider the recurrence:

$$S(1) = 1$$

Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1) + 1, \;\; n \geq 2. \end{aligned}$$

Expand-Guess-Verify

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 1 \\
S(n) &= S(n-1)+1, \;\; n \geq 2.
\end{aligned}
$$

(i) Expand: $S(1) = 1$,

Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1)+1, \ \ n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 1$, $S(2) =$

Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1)+1, \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 1$, $S(2) = S(1) + 1 = 2$,

Expand-Guess-Verify

Example

Consider the recurrence:

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(i) Expand: $S(1) = 1$, $S(2) = S(1) + 1 = 2$, $S(3) =$

Expand-Guess-Verify

Example

Consider the recurrence:

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\begin{aligned}
S(1) &= 1 \\
S(n) &= S(n-1) + 1, \;\; n \geq 2.
\end{aligned}
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(i) Expand: $S(1) = 1$, $S(2) = S(1) + 1 = 2$, $S(3) = S(2) + 1 = 3$,

Expand-Guess-Verify

Example

Consider the recurrence:

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Expand-Guess-Verify

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BASIS: $n = 1$

Expand-Guess-Verify

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$$LHS =$$

Expand-Guess-Verify

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Consider the recurrence:

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BASIS: $n = 1$

$$LHS = 1$$

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$$\begin{aligned} LHS &= 1 \\ RHS &= \end{aligned}$$

Expand-Guess-Verify

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BASIS: $n = 1$

$$\begin{aligned} LHS &= 1 \\ RHS &= 1 \end{aligned}$$

Since LHS=RHS, the basis is proven.

Expand-Guess-Verify

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INDUCTIVE STEP: Assume that $S(k) = k$.

Expand-Guess-Verify

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Consider the recurrence:

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Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = k$. We need to show that $S(k+1) = (k+1)$.

Expand-Guess-Verify

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Expand-Guess-Verify

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BASIS: $n = 1$

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Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = k$. We need to show that $S(k+1) = (k+1)$. Observe that,

$$
S(k+1) = S(k) + 1, \quad \text{by definition}
$$

Expand-Guess-Verify

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Consider the recurrence:

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(ii) Guess: $S(n) = n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

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\begin{aligned}
LHS &= 1 \\
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\end{aligned}
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Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = k$. We need to show that $S(k+1) = (k+1)$. Observe that,

$$
\begin{aligned}
S(k+1) &= S(k) + 1, \quad \text{by definition} \\
&= k + 1, \quad \text{by inductive hypothesis}
\end{aligned}
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Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1)+1, \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 1$, $S(2) = S(1) + 1 = 2$, $S(3) = S(2) + 1 = 3, \ldots.$

(ii) Guess: $S(n) = n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 1 \\ RHS &= 1 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = k$. We need to show that $S(k+1) = (k+1)$. Observe that,

$$\begin{aligned} S(k+1) &= S(k)+1, \quad \text{by definition} \\ &= k+1, \quad \text{by inductive hypothesis} \end{aligned}$$

Expand-Guess-Verify

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 1 \\ S(n) &= S(n-1)+1, \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 1$, $S(2) = S(1) + 1 = 2$, $S(3) = S(2) + 1 = 3, \ldots$.

(ii) Guess: $S(n) = n$.

(iii) Verify: Using Induction!
Basis: $n = 1$

$$\begin{aligned} LHS &= 1 \\ RHS &= 1 \end{aligned}$$

Since LHS=RHS, the basis is proven.
Inductive Step: Assume that $S(k) = k$. We need to show that $S(k+1) = (k+1)$. Observe that,

$$\begin{aligned} S(k+1) &= S(k)+1, \quad \text{by definition} \\ &= k+1, \quad \text{by inductive hypothesis} \end{aligned}$$

Applying the first principle of mathematical induction, we conclude that $S(n) = n$.

EGV (contd.)

EGV (contd.)

Example

EGV (contd.)

Example

Consider the recurrence:

EGV (contd.)

Example

Consider the recurrence:

$$S(1) = 2$$

EGV (contd.)

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 2 \\
S(n) &= 2 \cdot S(n-1), \; n \geq 2.
\end{aligned}
$$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \ n \geq 2. \end{aligned}$$

(i) Expand: $S(1) =$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$,

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) =$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \ n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$,

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) =$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8$,

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \ n \geq 2. \end{aligned}$$

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Consider the recurrence:

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(ii) Guess:

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.
(ii) Guess: $S(n) = 2^n$.

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.
(ii) Guess: $S(n) = 2^n$.
(iii) Verify:

EGV (contd.)

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(iii) Verify: Using Induction!

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Consider the recurrence:

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(ii) Guess: $S(n) = 2^n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

EGV (contd.)

Example

Consider the recurrence:

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(ii) Guess: $S(n) = 2^n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$LHS = 2$$

EGV (contd.)

Example

Consider the recurrence:

$$S(1) = 2$$
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(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$LHS = 2$$
$$RHS = 2^1$$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

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(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

EGV (contd.)

Example

Consider the recurrence:

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Since LHS=RHS, the basis is proven.

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Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$.

EGV (contd.)

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Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$.

EGV (contd.)

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INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

EGV (contd.)

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INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$S(k+1) \quad =$$

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INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$S(k+1) = 2 \cdot S(k), \quad \text{by definition}$$

EGV (contd.)

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$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \end{aligned}$$

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Consider the recurrence:

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$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \\ &= 2^{k+1}. \end{aligned}$$

EGV (contd.)

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot S(2) = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$.

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

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Applying the first principle of mathematical induction, we conclude that $S(n) = 2^n$.

EGV (contd.)

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Example

EGV (contd.)

Example

Solve the recurrence:

EGV (contd.)

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Solve the recurrence:

$$T(1) = 1$$

EGV (contd.)

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Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1)+3, \ n \geq 2. \end{aligned}$$

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EGV (contd.)

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EGV (contd.)

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(iii) Verify: Somebody from class!

Formula approach

Formula approach

Definition

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A general linear recurrence has the form:

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It is called first-order*, if $S(n)$ depends only on $S(n-1)$.*
For example, $S(n) = c \cdot S(n-1) + g(n)$.

The recurrence is called homogeneous, if $g(n) = 0$, for all n.

Linear first-order recurrence with constant coefficients

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Formula for Linear first-order recurrence

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$$S(1) = k_0$$

Linear first-order recurrence with constant coefficients

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\begin{aligned}
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Another Example

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Second Order homogeneous Linear Recurrence with constant coefficients

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Examples of second order recurrences

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$$T(1) = 5$$

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Solve the recurrence relation

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\begin{aligned}
T(1) &= 5 \\
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We get $p = 3$ and $q = 2$.

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(iii) Accordingly, the solution is $T(n) =$

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3. \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.
Solution is: $r_1 = 1$, $r_2 = 5$.

(ii) Solve the equations:

$$\begin{aligned} p + q &= T(1) = 5 \\ p \cdot 1 + q \cdot 5 &= T(2) = 13 \end{aligned}$$

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□

One More Example

One More Example

Example

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Solve the recurrence relation:

One More Example

Example

Solve the recurrence relation:

$$S(1) = 1$$

One More Example

Example

Solve the recurrence relation:

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\begin{aligned}
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One More Example

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Solve the recurrence relation:

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\begin{aligned}
S(1) &= 1 \\
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One More Example

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(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$.

One More Example

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(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

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$$\begin{aligned} p &= 1 \\ p \cdot 4 + q \cdot 4 &= 12 \end{aligned}$$

One More Example

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We get $p = 1$ and $q = 2$.

(iii) Accordingly, the solution is $S(n) = 4^{n-1} + 2 \cdot (n-1) \cdot 4^{n-1} = (2 \cdot n - 1) \cdot 4^{n-1}$.

□

Divide and Conquer Recurrences

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Formula for Divide and Conquer Recurrence

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$$S(1) = k_0$$

Divide and Conquer Recurrences

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m. \end{aligned}$$

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$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i).$

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$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i)$. (All logarithms are to base 2).

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Formula for Divide and Conquer Recurrence

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$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{\log n - i} \cdot g(2^i)$. (All logarithms are to base 2).

Note that $c^{\log n - i}$ in the expression above stands for $\frac{c^{\log n}}{c^i}$.

Example

Example

Example

Example

Example

Solve the recurrence:

Example

Example

Solve the recurrence:

$$C(1) = 1$$

Example

Example

Solve the recurrence:

$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m. \end{aligned}$$

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Example

Example

Solve the recurrence:

$$
\begin{aligned}
T(1) &= 3 \\
T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m.
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&= 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \\
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$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n), \\ &= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\ &= 3 \cdot n + n \cdot 2 \cdot \log n, \;\; \text{since } (a^{\log_a n} = n, \;\; a \neq 0) \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m. \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. It follows that $g(2^i) = 2 \cdot 2^i, \forall i$.

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Outline

The Master Method

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where, $a > 0$, $c > 0$, $b > 1$, d is an integer constant, and $f(n)$ is a complexity function.

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(i) If there is a small constant $\epsilon > 0$, such that $f(n) \in O(n^{r-\epsilon})$, then $T(n) \in \Theta(n^r)$.

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(ii) If there is a constant $k \geq 0$, such that $f(n) \in \Theta(n^r \cdot \log^k n)$, then $T(n) \in \Theta(n^r \log^{k+1} n)$.

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(i) If there is a small constant $\epsilon > 0$, such that $f(n) \in O(n^{r-\epsilon})$, then $T(n) \in \Theta(n^r)$.

(ii) If there is a constant $k \geq 0$, such that $f(n) \in \Theta(n^r \cdot \log^k n)$, then $T(n) \in \Theta(n^r \log^{k+1} n)$.

(iii) If there are small constants $\epsilon > 0$ and $\delta > 1$, such that $f(n) \in \Omega(n^{r+\epsilon})$, and $a \cdot f(\frac{n}{b}) \leq \delta \cdot f(n)$, for $n \geq d$, then $T(n) \in \Theta(f(n))$.

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1. $T(n) = 4 \cdot T(\frac{n}{2}) + n.$

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2. $T(n) = 2 \cdot T(\frac{n}{2}) + n \cdot \log n$.

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3. $T(n) = T(\frac{n}{3}) + n$.

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5. $T(n) = 2 \cdot T(\sqrt{n}) + \log n$.

The Recurrence Tree Method

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1. $T(n) = 2 \cdot T(\frac{n}{2}) + c \cdot n.$
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2. $T(n) = 3 \cdot T(\frac{n}{4}) + c \cdot n^2$.
3. $T(n) = T(\frac{n}{3}) + T(\frac{2 \cdot n}{3}) + c \cdot n$.

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Function MAX(a, b)
if (a ≥ b) then
    return(a).
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Note

How many element-to-element comparisons are performed by the FIND-MAX() *algorithm on an array of size n?*