

# Advanced Analysis of Algorithms - Final

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## 1 Instructions

1. You are required to turn in the exam by 10 am.
2. You are permitted to use class notes.
3. Each question is worth 4 points.

## 2 Problems

1. The Maximum Subarray problem is defined as follows: Given an array  $\mathbf{A}[1 \dots n]$  of  $n$  integers (with at least one positive element) find a contiguous sub-array within  $\mathbf{A}$  which has the largest sum.  
For instance, in the array  $\mathbf{A}$  defined by  $A[1] = -2, A[2] = 1, A[3] = -3, A[4] = 4, A[5] = -1, A[6] = 2, A[7] = 1, A[8] = -5, A[9] = 4$ , the contiguous subarray with the largest sum is  $A[4 \dots 7]$  with sum  $4 + (-1) + 2 + 1 = 6$ .  
Design a Divide-and-Conquer algorithm for the Maximum Subarray problem and analyze its running time.
2. A *matching* in a graph  $\mathbf{G} = \langle V, E \rangle$  is a collection of vertex-disjoint edges. The size of a matching is the number of edges in the disjoint collection. A *perfect matching* is a matching of size  $\frac{|V|}{2}$ . Observe that in a perfect matching, every vertex is matched to another vertex. Likewise, a graph with an odd number of vertices cannot have a perfect matching. Design a *linear-time* algorithm to check whether  $\mathbf{G}$  has a perfect matching, under the assumption that  $\mathbf{G}$  has no cycles.
3. (a) Consider the following linear program in canonical form:

$$\begin{array}{rcll} & & \max \mathbf{c} \cdot \mathbf{x} & \\ (\mathbf{P}) : & \mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

As discussed in class, its dual is:

$$\begin{array}{rcll} & & \min \mathbf{b} \cdot \mathbf{y} & \\ (\mathbf{D}) : & \mathbf{y} \cdot \mathbf{A} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

Let  $\mathbf{x}^*$  be an optimal solution to  $\mathbf{P}$  and let  $\mathbf{y}^*$  be an optimal solution to  $\mathbf{D}$ .

Let  $\mathbf{s}^* = \mathbf{b} - \mathbf{A} \cdot \mathbf{x}^*$  and let  $\mathbf{t}^* = \mathbf{y}^* \cdot \mathbf{A} - \mathbf{c}$ .

Show that  $\mathbf{x}^* \cdot \mathbf{t}^* = \mathbf{0}$  and that  $\mathbf{y}^* \cdot \mathbf{s}^* = \mathbf{0}$ .

(b) Use the above theorem to solve the following linear programming problem:

$$\begin{array}{rcl} & \max & 10 \cdot x_1 + 6 \cdot x_2 - 4 \cdot x_3 + x_4 + 12 \cdot x_5 \\ & & 2 \cdot x_1 + x_2 + x_3 + 3 \cdot x_5 & \leq 18 \\ & & x_1 + x_2 - x_3 + x_4 + 2 \cdot x_5 & \leq 6 \\ & & & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

4. (a) Let  $A$  be an **NP-complete** set and let  $B$  be a set in **P**. Assume that the  $A \cap B = \emptyset$ . What is the complexity of the set  $A \cup B$ ? If  $A \cap B \neq \emptyset$ , what can you say about the complexity of  $A \cup B$ ?
- (b) Let  $A$  and  $B$  be two languages (sets) in **NP**. Establish that the languages (sets)  $A \cup B$ ,  $A \cap B$  and  $A^*$  are also in **NP**.
5. The Minimal SAT problem is defined as follows: Given a 3CNF formula  $\phi$ , defined over the variables  $\{x_1, x_2, \dots, x_n\}$  and the clauses  $C_1, C_2, \dots, C_m$ , is there a satisfying assignment for  $\phi$ , such that exactly one literal in each clause is set to **true**? Prove that the Minimal SAT problem is **NP-complete**. (*Hint: 3SAT or 4SAT.*)