

Advanced Analysis of Algorithms - Homework I

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Instructions

1. The homework is due on September 15, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. Combinatorics:

Prove the following combinatorial identities:

(a)

$$\binom{2 \cdot n}{2} = 2 \cdot \binom{n}{2} + n^2, n \geq 0.$$

(b)

$$\sum_{k=0}^n k \cdot \binom{n}{k} = n \cdot 2^{n-1}, n \geq 0.$$

2. Summation:

- (a) Show that $\sum_{i=1}^n \frac{1}{i^2}$ is bounded above by a constant.
- (b) Give an asymptotically tight upper bound on the sum $\sum_{k=1}^n k \cdot \log k$.

3. Induction:

- (a) Show that $\sum_{k=1}^n k \cdot (k!) = (n+1)! - 1$.
- (b) In class, we discussed the MERGE-SORT() algorithm. Write down the algorithm formally and provide a formal proof of both the MERGE() procedure and the MERGE-SORT() algorithm.

4. Asymptotics:

- (a) Show that $\log n! \in \Theta(n \cdot \log n)$.
- (b) Let $f(n)$ and $g(n)$ be non-negative functions. Show that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

5. Discrete Probability and Random Variables:

- (a) A laboratory blood test is 95% effective in detecting a disease, when it is present. However, the test also yields a false positive for 1% of the healthy persons tested, i.e., if a healthy person is tested, with probability 0.01, the test result will imply that he has the disease. If 0.5% of the population actually has the disease, what is the probability that a person has the disease, given that his test result is positive?
- (b) Suppose that an airplane engine will fail in flight with probability $1 - p$, independently from engine to engine. A flight will be successful if at least 50% of its engines remain operative in flight. For what values of p is a four-engine airplane preferable to a two-engine airplane?