Advanced Analysis of Algorithms - Homework III

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1 Instructions

1. The homework is due on November 7.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. **Greedy Algorithms**

   A **matroid** is an ordered pair $M = (S, I)$ satisfying the following conditions:

   (a) S is a finite set.

   (b) I is a non-empty family of subsets of S, called the **independent** subsets of S, such that if $B \in I$ and $A \subseteq B$, then $A \in I$.

   (c) If $A, B \in I$ and $|A| < |B|$, then there exists an element $x \in B - A$, such that $A \cup \{x\} \in I$.

   Matroids exhibit the greedy choice property.

   (a) Let $M = (S, I)$ be a matroid. Argue that (S, I_k) is a matroid, where I_k is the set of all subsets of S of size at most k, where $k \leq |S|$.

   (b) Show that if (S, I) is a matroid, then so is (S, I'), where,

   $$I' = \{A' : S - A' \text{ contains some maximal } A \in I\}$$

   (c) Let T denote an $m \times n$ matrix with entries in the set $\Re$ (real numbers). Let S denote the set of columns of T. A set $A \subseteq S$ belongs to I, if and only if the columns in A are linearly independent. Argue that (S, I) is a matroid.

2. **Dynamic Programming**

   (a) In class, we discussed a table-filling algorithm for the matrix chain multiplication problem. We also showed that this algorithm runs in $O(n^3)$ time, where n is the number of matrices in the chain. Argue that the algorithm runs in $\Theta(n^3)$ time.

   (b) Suppose that we are given a directed acyclic graph $\mathbf{G} = \langle V, E \rangle$ with real-valued edge weights and two distinguished vertices s and t. Describe a dynamic programming approach for finding a longest weighted simple path from s to t. Establish the correctness of your algorithm and give an asymptotic bound on its running time.

3. **Numerical problems**

   (a) Compute the product of the two matrices below, using Strassen's matrix multiplication algorithm.

$$\mathbf{X} = \begin{pmatrix} 9 & 3 \\ 2 & -1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

   (b) Compute the optimal parenthesization of the following matrix chain: $\langle A_{10\times 15} \cdot B_{15\times 9} \cdot C_{9\times 7} \cdot D_{7\times 10}\rangle$.

   In both problems, you are required to show all the intermediate steps (and tables, if necessary).

4. **Amortized Analysis**

   (a) Assuming that a DECREMENT() operation is added to the binary counter example discussed in class. What is the cost of a sequence of n operations?

   (b) Assume that we have a potential function Φ, such that $\Phi(D_0) \neq 0$, but that $\Phi(D_i) \geq \Phi(D_0$, for all $i \geq 1$. Argue that there exists a potential function Φ', such that

      i. $\Phi'(D_i \geq 0, \forall i \geq 1$.

      ii. $\Phi'(D_0) = 0$.

      iii. The amortized costs under Φ' are the same as the amortized costs under Φ.

   (c) In class, we discussed a strategy for dynamic table insertion and deletion, where the load of the table is always between $\frac{1}{4}$ and 1. Consider the following strategy for table contraction: Shrink the table to $\frac{2}{3}^{rd}$ its size, when the load drops below $\frac{1}{3}$. Argue that the amortized cost of deleting an item from this table is bounded above by a constant.

5. **Linear Programming**

   (a) Solve the following problems using the Simplex procedure:

      i.

$$\begin{aligned} \min z = 4 \cdot x_1 + 3 \cdot x_2 & \\ \text{subject to} & \\ -x_1 + x_2 &\leq 6 \\ 2 \cdot x_1 + x_2 &\leq 20 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

      ii.

$$\begin{aligned} \max z = 2 \cdot x_1 + x_2 & \\ \text{subject to} & \\ x_1 - x_2 &\geq 8 \\ 2 \cdot x_1 + 3 \cdot x_2 &\leq 24 \\ 2 \cdot x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   For each problem, identify the basis matrix $\mathbf{B}$, the basic variables and the non-basic variables in each iteration.

   (b) Let L denote the linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{A}$ has dimensions $m \times n$ and $\mathbf{b}$ has dimensions $m \times 1$. Prove that either L is non-empty or (mutually exclusively) $\exists \mathbf{y} \in \Re_{+}^{m}\ \mathbf{y} \cdot \mathbf{A} \geq \mathbf{0}, \mathbf{y} \cdot \mathbf{b} < 0$.