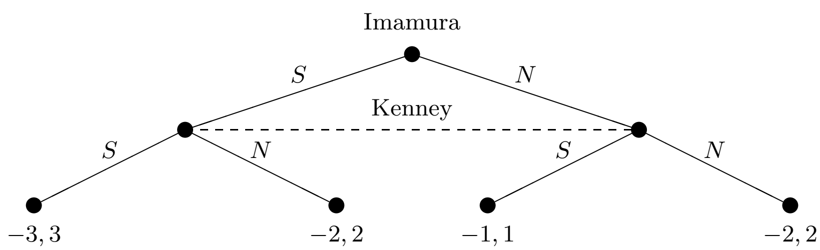
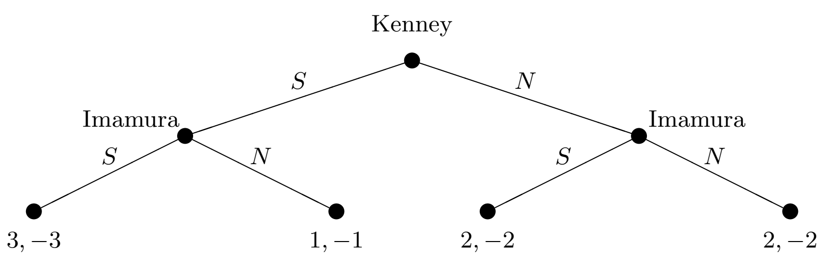
Complete Solutions Edition 2015

Problems of Chapter 1

1.1 *Battle of the Bismarck Sea*
(a) See the following diagram, with payoffs: Imamura,Kenney.



(b) Diagram as in (a) but without the dashed line. Backward induction: after S Kenney chooses S, and after N Kenny chooses N. Therefore, Imamura chooses N, which is followed by N of Kenney.
(c) See the following diagram, with payoffs: Kenney,Imamura.

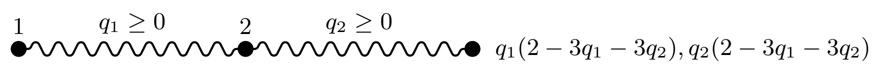


Backward induction: after S Imamura chooses N and after N Imamura chooses S or N. Hence, Kenney chooses N.

1.2 *Variant of Matching Pennies*
If $x = -1$ then there are saddlepoints at $i = j = 1$ and at $i = 2$, $j = 1$. If $x < -1$ then there is a saddlepoint at $i = 2$, $j = 1$. If $x > -1$ then there is no saddlepoint.

1.3 *Mixed Strategies*
(a) The maximum in the first column is 3, but this is not minimal in its row. The maximum in the second column is 4, but this is not minimal in its row either.
(b) With probability p on T and $1-p$ on B, we should have that the expected payoff to player 1 is independent of whether player 2 plays L or R, hence $3p+(1-p) = 2p+4(1-p)$, hence $p = 3/4$, so the mixed strategy is $(3/4, 1/4)$.
(c) Analogous to (b): if player 2 plays L with probability q, then we must have $3q + 2(1-q) = q + 4(1-q)$, hence $(q, 1-q) = (1/2, 1/2)$.
(d) By playing $(3/4, 1/4)$ player 1 obtains $10/4 = 2.5$ for sure (independent of what player 2 does). Similarly, by playing $(1/2, 1/2)$, player 2 is sure to pay 2.5. So 2.5 is the value of this game. Given a rational opponent, no player can hope to do better by playing differently.

1.4 *Sequential Cournot*



(a) See picture. (b) Given $q_1 \geq 0$, player 2's best reply (profit maximizing quantity) is obtained by maximizing $q_2(2-3q_1-3q_2)$ with respect to q_2, which yields $q_2 = 1/3-(1/2)q_1$. Hence, player 1 maximizes $q_1(2-3q_1-1+(3/2)q_1) = q_1(1-(3/2)q_1)$, which yields $q_1 = 1/3$ and, thus, $q_2 = 1/3-(1/2)(1/3) = 1/6$.

1.5 *Three Cooperating Cities*
(a) The vector of contributions according to the ordering 1,2,3 is $(0, 90, 130)$. For the ordering 1,3,2, it is $(v(\{1\}), v(N) - v(\{1,3\}), v(\{1,3\}) - v(\{1\})) = (0, 120, 100)$. Similarly, for 2,1,3 we obtain $(90, 0, 130)$, for 2,3,1 it is $(100, 0, 120)$, for 3,1,2 it is $(100, 120, 0)$, and for 3,2,1 it is $(100, 120, 0)$. The average of these six vectors is $(65, 75, 80)$, which is indeed the Shaply value as given in the text. Observe that $65, 75, 80 \geq 0$, $65 + 75 \geq 90$, $65 + 80 \geq 100$, $75 + 80 \geq 120$, and $65 + 75 + 80 = 220$. So the Shapley value is in the core of the game.
(b) The argument for the nucleolus $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$ is analogous.

1.6 *Glove Game*
(a) If (x_1, x_2, x_3) is in the core of the glove game, then $x_1 + x_3 \geq 1$. Since $x_1 + x_2 + x_3 = 1$ and all coordinates are nonnegative, we have $x_2 = 0$. In the same way we derive $x_1 = 0$. Hence $x_3 = 1$, so that $(x_1, x_2, x_3) = (0, 0, 1)$. Indeed, $(0, 0, 1)$ is in the core of the glove game, so that it is the unique vector in the core of this game.
(b) Since the Shapley value is $(1/6, 1/6, 4/6)$ (can be computed similarly as in Problem 1.5), it follows from (a) that the Shapley value of this game is not in the core.

1.7 *Dentist Appointments*
(a) A vector (x_1, x_2, x_3) is in the core of the dentist game if and only if the following constraints are satisfied: $x_1 \geq 2$, $x_2 \geq 5$, $x_3 \geq 4$, $x_1 + x_2 \geq 14$, $x_1 + x_3 \geq 18$, $x_2 + x_3 \geq 9$, $x_1 + x_2 + x_3 = 24$. By making a picture it can be seen that the core is the convex hull of the vectors $(15, 5, 4)$, $(14, 6, 4)$, $(8, 6, 10)$, and $(9, 5, 10)$, i.e., the quadrangle with these vectors as vertices, plus its inside (cf. Chap. 9). The Shapley value $(9\frac{1}{2}, 6\frac{1}{2}, 8)$ is *not* in the core of this game: $9\frac{1}{2} + 8 = 17\frac{1}{2} < 18$. The Shapley value can be computed as in Problem 1.5. The vectors associated with the orderings 1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2; and 3,2,1, are now respectively: $(2, 12, 10)$, $(2, 6, 16)$, $(9, 5, 10)$, $(15, 5, 4)$, $(14, 6, 4)$, and $(15, 5, 4)$. Taking the average yields $(9\frac{1}{2}, 6\frac{1}{2}, 8)$.
(b) The nucleolus $(11\frac{1}{2}, 5\frac{1}{2}, 7)$ is in the core of the game, as follows easily by checking the core constraints.

1.8 *Nash Bargaining*
(a) The problem to solve is $\max_{0 \leq \alpha \leq 1} \alpha\sqrt{1-\alpha}$. Obviously, the solution must be interior: $0 < \alpha < 1$. The first derivative is $\sqrt{1-\alpha} - \alpha/(2\sqrt{1-\alpha})$, and setting this equal to 0 yields

$$\sqrt{1-\alpha} - \frac{\alpha}{2\sqrt{1-\alpha}} = 0$$

hence (multiply by $2\sqrt{1-\alpha}$)

$$2(1-\alpha) - \alpha = 0$$

hence $\alpha = 2/3$. The second derivative is

$$-1/(2\sqrt{1-\alpha}) - \left(2\sqrt{1-\alpha} + \alpha/(\sqrt{1-\alpha})\right)/4(1-\alpha) \ ,$$

which is negative. So we have a maximum indeed.
(b) The problem to solve is now $\max_{0 \leq \alpha \leq 1}(2\alpha - \alpha^2)(1-\alpha)$. The derivative of this function is $3\alpha^2 - 6\alpha + 2$, which is equal to zero for $\alpha = 1 - (1/3)\sqrt{3}$ (the other root is larger than 1). At this value, the second derivative is negative. So the Nash bargaining solution in terms of utilities is $(2/3, (1/3)\sqrt{3})$, and in terms of distribution of the good it is $(1 - (1/3)\sqrt{3}, (1/3)\sqrt{3})$.

1.9 *Variant of Glove Game*
$v(S) = \min\{|S \cap L|, |S \cap R|\}$, where L is the set of left-hand players and R is the set of right-hand players. In words: a coalition S can make a number of glove pairs equal to the minimum of the numbers of left-hand and right-hand gloves its members possess.

Problems of Chapter 2

2.1 *Solving Matrix Games*

(a) The optimal strategies are $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2. The value of the game is $30/11$. In the original game the optimal strategies are $(5/11, 6/11, 0)$ for player 1 and $(5/11, 6/11, 0)$ for player 2.
(b) Columns 1 and 4 are strictly dominated by column 3. After deletion of these columns we are left with the game

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} .$$

For an arbitrary strategy $\mathbf{q} = (q, 1-q)$ the payoffs to player 1 of the three rows are given by

$$\begin{aligned} \mathbf{e}^1 A\mathbf{q} &= -q \\ \mathbf{e}^2 A\mathbf{q} &= 0 \\ \mathbf{e}^3 A\mathbf{q} &= q - 1 \ . \end{aligned}$$

By making a diagram with $0 \leq q \leq 1$ on the horizontal axis, we see that the maximum payments that player 2 has to make are equal to 0 for any value of q. Hence, any $(q, 1-q)$ is a minimax strategy for player 2 in this game, and the value of the game is 0. For player 1, putting any probability on the first row or the third row guarantees less than 0. Hence, the unique maximin strategy is $(0, 1, 0)$. The minimax strategies in the original game are equal to $(0, q, 1-q, 0)$ for any $0 \leq q \leq 1$.

Alternatively, one can start with observing that the game has two saddlepoints, namely $(2, 2)$ and $(2, 3)$. So the value is 0. After eliminating columns 1 and 4 it is then straightforward to determine the maximin and minimax strategies: for player 1, only the second row guarantees a payoff of 0, whereas for player 2 both remaining columns (and mixtures) guarantee a payment of at most 0.
(c) In this game the second column is strictly dominated by the third one. Solving the remaining game as in (b) yields: the value of the game is 1, the unique minimax strategy is $(1/2, 0, 1/2)$, and the maximin strategies are: $(p, (1-p)/2, (1-p)/2)$ for $0 \leq p \leq 1$.
(d) In this game the last two rows are strictly dominated by the mixed strategy that puts probability 1/2 on each of the first two rows. The remaining 2×3-game can be solved by plotting the payoffs from the three columns as functions of $0 \leq p \leq 1$, where p is the probability put by player 1 on the first row. The three lines in the resulting diagram all cross through the point $(1/2, 9)$. This is also the highest point of the lower envelope, so that the value of the game is 9 and player 1's maximin strategy is $(1/2, 1/2, 0, 0)$ (expressed in the original game). The minimax strategies are all (q_1, q_2, q_3) that give an expected payoff of 9. Hence, they are all solutions of the system: $16q_1 + 12q_2 + 2q_3 = 9$, $2q_1 + 6q_2 + 16q_3 = 9$, $q_1 + q_2 + q_3 = 1$, $q_1, q_2, q_3 \geq 0$. This results in the set $\{(\alpha, (7-14\alpha)/10, (3+4\alpha)/10) \in \mathbb{R}^3 \mid 0 \leq \alpha \leq 1/2\}$.

(e) In this game the first column is strictly dominated by (e.g.) putting probability $9/20$ on the second column and $11/20$ on the third column. The remaining game, consisting of the last three columns, is a 2×3 game. This game can be solved graphically. The value is $8/5$. The unique maximin strategy is $(2/5, 3/5)$ and the unique minimax strategy (in the original game) is $(0, 4/5, 1/5, 0)$.
(f) In this game the first row is strictly dominated by the third row, and the first column by the second column. The remaining 2×2 game can be solved graphically. As an alternative, observe that player 2 would possibly put positive probability on the second column in the remaining game only if player 1 plays the first row in that game with probability 1; however, player 1 would not do that if player 2 puts probability on the second column. Hence, player 2 plays the first column in the remaining game with probability 1, and player 1 can play any strategy in this game. Consequently, in the original game the value is equal to 1, player 2 has a unique minimax strategy namely $(0, 1, 0)$, and the set of maximin strategies is $\{(0, p, 1-p) \mid 0 \leq p \leq 1\}$.

2.2 *Saddlepoints*
(a) By the definition of a saddlepoint, $a_{kl} \geq a_{il} \geq a_{ij} \geq a_{kj} \geq a_{kl}$. Hence, all inequalities must be equalities, so $a_{kl} = a_{ij}$.
(b) First, $a_{11} \leq a_{14} \leq a_{44}$. Since, by (a), $a_{11} = a_{44}$ it follows that $a_{14} = a_{11} = a_{44}$. But then a_{14} is also minimal in its row and maximal in its column, hence there is a saddlepoint at $(1, 4)$. In the same way one shows that there is a saddlepoint at $(4, 1)$.
(c) In light of (b) these saddlepoints have to be in the same row or in the same column. E.g., take a 4×4-matrix A with $a_{11} = a_{12} = a_{13} = 1$, $a_{14} = 2$, and $a_{ij} = 0$ otherwise.

2.3 *Maximin Rows and Minimax Columns*
(a) The maximin row is the second row; the first and second columns are minimax columns. From this we can conclude that the value of the game must be between 1 and 3, i.e., $1 \leq v(A) \leq 3$.
(b) All payoffs in the second column are larger than $\frac{12}{7}$, the value of the game. This implies that in a minimax strategy with positive probability on the second column, the combination of the first and third column must guarantee a payment below $\frac{12}{7}$, but then player 2 would be better off transferring the probability put on the second column to the other columns. (Alternatively, the second column is strictly dominated by for instance $7/12$ times the first column plus $5/12$ times the third column.)
(c) The game can be further reduced by eliminating the second and fourth rows. The resulting 2×2 game is easily solved. In A, the unique maximin strategy is $(\frac{4}{7}, 0, \frac{3}{7}, 0)$ and the unique minimax strategy is $(\frac{4}{7}, 0, \frac{3}{7})$.

2.4 *Subgames of Matrix Games*
(a) Both rows are maximin. The second column is minimax. This implies $0 \leq v(A) \leq 2$.

(b) The values of the 2×2 games are, respectively: $5/3$, 3, $15/7$, $8/5$, 2, $20/9$. Since player 2 can choose which columns to play, the value of A must be equal to the minimum of these numbers, which is $v(A_4) = 8/5$.
(c) The unique minimax strategy is $(0, 4/5, 1/5, 0)$ and the unique maximin strategy is $(2/5, 3/5)$.

2.5 *Rock-Paper-Scissors*
The associated matrix game is:

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array} .$$

From considerations of symmetry, the optimal strategies are guessed to be $(1/3, 1/3, 1/3)$ for each. It is easy to compute that, independent of player 2's strategy, playing $(1/3, 1/3, 1/3)$ yields to player 1 an expected payoff of 0. By the same reasoning, player 2 expects to pay at most 0 by playing $(1/3, 1/3, 1/3)$, independent of the strategy of player 1. Hence, the value of the game is 0 and $(1/3, 1/3, 1/3)$ is an optimal strategy for both.

Check that any other strategy for player 1 does not guarantee at least 0. For instance, suppose that $p_1 \geq p_2 \geq p_3$ for some strategy $\mathbf{p} \in \Delta^3$, with at least one inequality strict. If player 2 plays the second column the payoff to player 1 is $-p_1 + p_3 < 0$, so that $\mathbf{p}$ cannot be optimal.

Similarly for player 2. Hence, the optimal strategies are unique.

Problems of Chapter 3

3.1 *Some Applications*
(a) Let Smith be the row player and Brown the column player, then the bimatrix game is:

$$\begin{array}{c} \\ L \\ S \end{array} \begin{array}{c} \begin{array}{cc} L & S \end{array} \\ \begin{pmatrix} 2,2 & -1,-1 \\ -1,-1 & 1,1 \end{pmatrix} \end{array} .$$

The Nash equilibria are: (L, L), (S, S), and $((2/5, 3/5), (2/5, 3/5))$.
(b) Let the government be the row player and the pauper the column player. The bimatrix game is:

$$\begin{array}{c} \\ \text{aid} \\ \text{not} \end{array} \begin{array}{c} \begin{array}{cc} \text{work} & \text{not} \end{array} \\ \begin{pmatrix} 3,2 & -1,3 \\ -1,1 & 0,0 \end{pmatrix} \end{array} .$$

The unique Nash equilibrium is: $((1/2, 1/2), (1/5, 4/5))$.
(c) Let worker 1 be the row player and worker 2 the column player. The bimatrix game is:

$$\begin{array}{r c} & \begin{array}{cc} \text{apply to firm 1} & \text{apply to firm 2} \end{array} \\ \begin{array}{r} \text{apply to firm 1} \\ \text{apply to firm 2} \end{array} & \left(\begin{array}{cc} \frac{w_1}{2}, \frac{w_1}{2} & w_1, w_2 \\ w_2, w_1 & \frac{w_2}{2}, \frac{w_2}{2} \end{array} \right) \end{array} .$$

There are two pure Nash equilibria, namely where both workers apply to different firms, and one mixed Nash equilibrium, namely: $(((2w_1 - w_2)/(w_1 + w_2), (2w_2 - w_1)/(w_1 + w_2)), ((2w_1 - w_2)/(w_1 + w_2), (2w_2 - w_1)/(w_1 + w_2)))$. This can be seen by plotting the best replies in a diagram, or by arguing that each player must be indifferent between his two pure strategies.
(d) The bimatrix game is:

$$\begin{array}{r c} & \begin{array}{cc} A & NA \end{array} \\ \begin{array}{r} A \\ NA \end{array} & \begin{pmatrix} 40,40 & 60,30 \\ 30,60 & 50,50 \end{pmatrix} \end{array} .$$

This is a prisoners' dilemma, Nash equilibrium: (A, A).
(e) The bimatrix game is:

$$\begin{array}{r c} & \begin{array}{ccc} (3,0,0) & (2,1,0) & (1,1,1) \end{array} \\ \begin{array}{r} (3,0,0) \\ (2,1,0) \\ (1,1,1) \end{array} & \begin{pmatrix} 3/2,3/2 & 1,2 & 1,2 \\ 2,1 & 3/2,3/2 & 3/2,3/2 \\ 2,1 & 3/2,3/2 & 3/2,3/2 \end{pmatrix} \end{array} .$$

The first row and the first column are strictly dominated. Hence, the set of all Nash equilibria is: $\{((0,p,1-p),(0,q,1-q)) \mid 0 \le p \le 1,\ 0 \le q \le 1\}$.
(f) The bimatrix game is:

$$\begin{array}{r c} & \begin{array}{ccc} (3,0,0) & (2,1,0) & (1,1,1) \end{array} \\ \begin{array}{r} (3,0,0) \\ (2,1,0) \\ (1,1,1) \end{array} & \begin{pmatrix} a,a & 1,\sqrt{2} & 1,\sqrt{2} \\ \sqrt{2},1 & a,a & a,a \\ \sqrt{2},1 & a,a & b,b \end{pmatrix} \end{array} ,$$

where $a = \frac{1}{2}(1+\sqrt{2})$ and $b = \frac{9}{20}(1+\sqrt{2}) + \frac{1}{20}\sqrt{3}$. The first row and the first column are strictly dominated. The set of all Nash equilibria is: $\{((0,p,1-p),(0,q,1-q)) \mid 0 \le p,q \le 1, p = 1 \text{ or } q = 1\}$.

3.2 *Matrix Games*
(a) You should find the same solution, namely $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2, as the unique Nash equilibrium.
(b) If player 2 plays a minimax strategy then 2's payoff is at least $-v$, where v is the value of the game. Hence, any strategy that gives player 1 at least v is a best reply. So a maximin strategy is a best reply. Similarly, a minimax strategy is a best reply against a maximin strategy, so any pair consisting of a maximin and a minimax strategy is a Nash equilibrium.
Conversely, in a Nash equilibrium the payoffs must be $(v, -v)$ otherwise one of the players could improve by playing an optimal (maximin or minimax)

strategy. But then player 1's strategy must be a maximin strategy since otherwise player 2 would have a better reply, and player 2's strategy must be a minimax strategy since otherwise player 1 would have a better reply.
(c) The appropriate definition for player 2 would be: a maximin strategy *for player 2* in B, since now B represents the payoffs to player 2, and not what player 2 has to pay to player 1.
The Nash equilibrium of Problem 3.1(b), for instance, does not consist of maximin strategies of the players. The maximin strategy of player 1 in A is $(1/5, 4/5)$, which is not part of a (the) Nash equilibrium. The maximin strategy of player 2 (!) in B is $(1, 0)$, which is not part of a (the) Nash equilibrium.

3.3 *Strict Domination*
(a) Z is strictly dominated by W.
(b) Put probability α on W and $1-\alpha$ on Y. Then it should hold that $6\alpha + 2(1-\alpha) > 4$ and $5\alpha + 8(1-\alpha) > 6$. Simplifying yields $1/2 < \alpha < 2/3$.
(c) In view of (b) and (c) we are left with the game

$$\begin{array}{c c} & \begin{array}{cc} W & Y \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 6,6 & 1,2 \\ 4,5 & 2,8 \end{pmatrix} \end{array},$$

which three Nash equilibria: $((1,0),(1,0))$, $((0,1),(0,1))$, and $((3/7,4/7),(1/3,2/3))$. Hence the original game has the three Nash equilibria $((1,0),(1,0,0,0))$, $((0,1),(0,0,1,0))$, and $((3/7,4/7),(1/3,0,2/3,0))$.

3.4 *Iterated Elimination (1)*
(a) There are many different ways, e.g.: first Z then X (or conversely), then C or D (or conversely), then W, then A. One can also start with C or D, then W, X, or Z.
(b) By (a), the unique equilibrium is (B, Y).

3.5 *Iterated Elimination (2)*
First delete the third row and next the second column. Solve the remaining 2×2 game, which has three Nash equilibria. The Nash equilibria in the original game are $((1/3,2/3,0),(2/3,0,1/3))$, $((0,1,0),(1,0,0))$, and $((1,0,0),(0,0,1))$.

3.6 *Weakly Dominated Strategies*
(a) The unique pure Nash equilibrium of this game is (B, Y). We can delete B and Y and obtain a 2×2 game with unique Nash equilibrium $((1/2,1/2),(1/2,1/2))$, hence $((1/2,0,1/2),(1/2,0,1/2))$ in the original game.
(b) Consecutive deletion of Z, C, A results in the Nash equilibria (B, X) and (B, Y). Consecutive deletion of C, Y, B, Z results in the Nash equilibrium (A, X).

3.7 *A Parameter Game*
For $a > 2$: $\{((1,0),(1,0))\}$.
For $a = 2$: $\{((1,0),(1,0))\} \cup \{((p,1-p),(0,1)) \mid 0 \le p \le \frac{1}{2}\}$.
For $a < 2$: $\{((\frac{1}{2},\frac{1}{2}),(\frac{2-a}{3-a},\frac{1}{3-a}))\} \cup \{((1,0),(1,0)),((0,1),(0,1))\}$.

3.8 *Equalizing Property of Mixed Equilibrium Strategies*
(a) Check by substitution.
(b) Suppose the expected payoff (computed by using $\mathbf{q}^*$) of row i played with positive probability (p_i^*) in a Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$, hence the number $\mathbf{e}^i A\mathbf{q}^*$, would not be maximal. Then player 1 would improve by adding the probability p_i^* to some row j with higher expected payoff $\mathbf{e}^j A\mathbf{q}^* > \mathbf{e}^i A\mathbf{q}^*$, and in this way increase his payoff, a contradiction. A similar argument can be made for player 2 and the columns.

3.9 *Voting*
(a,b,c) Set the total number of voters equal to 10 (in order to avoid fractions). Then the bimatrix game and best replies are given by:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 5,5 & 1,\underline{9} & 2,8 & 3,7 & 4,6 & 5,5 \\ \underline{9},1 & 5,5 & 3,\underline{7} & 4,6 & 5,5 & 6,4 \\ 8,2 & \underline{7},3 & \underline{5},\underline{5} & \underline{5},\underline{5} & 6,4 & 7,3 \\ 7,3 & 6,4 & \underline{5},\underline{5} & \underline{5},\underline{5} & \underline{7},3 & 8,4 \\ 6,4 & 5,5 & 4,6 & 3,\underline{7} & 5,5 & \underline{9},1 \\ 5,5 & 4,6 & 3,7 & 4,8 & 1,\underline{9} & 5,5 \end{pmatrix} \end{array}$$

So the game has four Nash equilibria in pure strategies.
(d) Now we have:

$$\begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 2 & 4 \end{array} \\ \begin{pmatrix} \underline{9},1 & 3,\underline{7} & 5,5 \\ 7,3 & \underline{5},\underline{5} & \underline{7},3 \\ 5,5 & 3,7 & 1,\underline{9} \end{pmatrix} \end{array}$$

So there is a unique Nash equilibrium in pure strategies.
(e) In both games, subtract 5 from all payoffs. The value is 0 in each case, and the pure Nash equilibrium strategies are the pure optimal strategies.

3.10 *Guessing Numbers*
(a) Suppose player 2 plays each pure strategy with equal probability $\frac{1}{K}$. Then the expected payoff to player 1 is the same for every pure strategy, namely $\frac{1}{K}$. Hence any mixed strategy of player 1 is a best reply, in particular the strategy in which player 1 plays every pure strategy with probability $\frac{1}{K}$. The argument for player 2 is similar.
(b) Suppose, in a Nash equilibrium, player 1 plays some numbers with zero probability. Then any best reply of player 2 would put positive probability only on those numbers. Then, in turn, in any best reply player 1 would put positive probability only on those numbers, a contradiction.
(c) Suppose, in a Nash equilibrium, player 2 plays some numbers with zero probability. Then in any best reply player 1 would zero probability on those numbers. But then, in any best reply, player 2 would but positive probability only on those numbers, a contradiction.

(d) Suppose player 1 would play some pure strategy with probability less than $\frac{1}{K}$. A best reply of player 2 would be to play a pure strategy on which player 1 puts minimal probability, resulting in player 2 paying less than $\frac{1}{K}$. But then player 1 can improve by playing each pure strategy with equal probability, see (a). By an analogous argument for player 2, we obtain that the equilibrium in (a) is the unique Nash equilibrium.
(e) The value of this game is $\frac{1}{K}$, and the unique optimal strategy for each player is to choose every number with equal probability.

3.11 *Bimatrix Games*
(a) For instance the game

$$\begin{pmatrix} 1,1 & 1,0 \\ 0,1 & 1,1 \end{pmatrix} .$$

The best reply curve of player 1 consists of the lower and right edges of the square, the best reply curve of player 2 consists of the upper and left edges.
(b) $e < a$, $b < d$, $c < g$, $h < f$. The unique Nash equilibrium is

$$\left((\frac{f-h}{f-h+d-b}, \frac{d-b}{f-h+d-b}), (\frac{g-c}{g-c+a-e}, \frac{a-e}{g-c+a-e}) \right) .$$

Problems of Chapter 4

4.1 *Counting strategies*
White has 20 possible opening moves, and therefore also 20 possible strategies. Black can choose from 20 moves after each opening move of White. Hence Black has $20 \times 20 \times \ldots \times 20 = 20^{20}$ different strategies.

4.2 *Extensive versus strategic form*
For the game with perfect information, start with a decision node of player 1 (the root of the tree) and let player 1 have two actions/strategies. Player 2 observes these actions and at each of his two decision nodes has two actions. So player 2 has four strategies.
For the game with imperfect information, start with player 2 and let player 2 have four actions/strategies. Next, player 1 moves, but player 1 does not observe the move of player 2. So player 1 has one, nontrivial, information set with four nodes, and two actions at this information set. Consequently, player 1 has two strategies.

4.3 *Entry deterrence*
(a) The strategic form is:

$$\begin{array}{c c} & \begin{array}{cc} C & F \end{array} \\ \begin{array}{c} E \\ O \end{array} & \begin{pmatrix} 40,50 & -10,0 \\ 0,100 & 0,100 \end{pmatrix} \end{array} .$$

(b) The Nash equilibria in pure strategies are: (E, C) and (O, F). The backward induction (or subgame perfect) equilibrium is (E, C). The equilibrium (O, F) is based on the 'incredible threat' that the incumbent would actually fight after entry of the entrant: this is not in the own interest of the incumbent.

4.4 *Choosing objects*
(a) Player 1 starts with four possible actions. After each action, player 2 has three possible actions. After that, player 1 has each time two possible actions.
(b) Player 1 has 4×2^{12} possible strategies. Player 2 has 3^4 strategies. [If strategies where player 1 makes moves excluded by own earlier actions are eliminated, then player 1 has only $4 \times 2^3 = 32$ different strategies.]
(c) In any subgame perfect equilibrium the game is played as follows: player 1 picks O_3, then player 2 picks O_2 or O_1, and finally player 1 picks O_4. These are the (two) subgame perfect equilibrium outcomes of the game. Due to ties (of player 2) there is more than one subgame perfect equilibrium, namely eight in total. All subgame perfect equilibria result in the same distribution of the objects.
(d) Consider the following strategies. Player 1 first picks O_4 and, at his second turn, the best of the remaining objects. Player 2 has the following strategy: pick O_3 if player 1 has picked O_4, and pick O_4 in the other three cases.

4.5 *A Bidding Game*
(a) The game starts with a decision node of player 1, at which this player has five possible actions: P, M, bid 1, bid 2, and bid 3. If player 1 plays P the game ends with payoffs $(0, 2)$. If player 1 player M then the game ends with payoffs $(1, 1)$. After the other three actions, player 2 continues.

If player 1 has bid 1, then player 2 has four possible actions: P, M, bid 2, and bid 3. (i) If player 2 plays P then the game ends with payoffs $(1, 0)$. (ii) If player 2 plays M then the game ends with payoffs $(1/2, 1/2)$. (iii) If player 2 bids 2, then player 1 continues with three possible actions: P, M, and bid 3. If player 1 plays P then the game ends with payoffs $(0, 0)$. If player 1 plays M then the game ends with payoffs $(0, 0)$. If player 1 bids 3, then player 2 continues with P or M. If player 2 plays P then the game ends with payoffs $(-1, 0)$. if player 2 plays M then the game ends with payoffs $(-1/2, -1/2)$. (iv) If player 2 bids 3, then player 1 continues with P or M. If player 1 plays P then the game ends with payoffs $(0, -1)$. If player 1 plays M then the game ends with payoffs $(-1/2, -1/2)$.

If player 1 has bid 2, then player 2 has three possible actions: P, M, and bid 3. (i) If player 2 plays P then the game ends with payoffs $(0, 0)$. (ii) If player 2 plays M, then the game ends with payoffs $(0, 0)$. (iii) If player 2 bids 3, then player 1 continues with P or M. If player 1 plays P then the game ends with payoffs $(0, -1)$. If player 1 plays M then the game ends with payoffs $(-1/2, -1/2)$.

If player 1 has bid 3, then player 2 can only play P, resulting in payoffs $(-1, 0)$, or M, resulting in $(-1/2, -1/2)$.

(b) Player 1 has $5\times3\times2\times2=60$ different strategies. Player 2 has $3\times3\times2\times2=36$ different strategies.
(c) Due to ties, there are four different subgame perfect equilibria. They all result in the same outcome, namely player 1 playing M at the first decision node.
(d) Any strategy combination where player 1 plays M at the first decision node is a Nash equilibrium and vice versa. Hence, the outcome of any Nash equilibrium is the same as in any subgame perfect Nash equilibrium.

4.6 *An extensive form game*
The strategic form is:

$$\begin{array}{c} \\ L \\ M \\ R \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 2,0 & 0,1 \\ 0,1 & 3,0 \\ 2,2 & 2,2 \end{pmatrix} \end{array} .$$

There is a unique pure strategy Nash equilibrium, namely (R,l). This is also subgame perfect, trivially, since the only subgame is the entire game. Let the belief of player 2 that player 1 has played L be equal to α. Then l is optimal for player 2 if $1-\alpha \geq \alpha$, i.e., if $\alpha \leq 1/2$. So for these beliefs, (R,l) is also perfect Bayesian.

4.7 *Another extensive form game*
The strategic form is:

$$\begin{array}{c} \\ L \\ M \\ R \end{array} \begin{array}{c} \begin{array}{ccc} l & m & r \end{array} \\ \begin{pmatrix} 1,3 & 1,2 & 4,0 \\ 4,2 & 0,2 & 3,3 \\ 2,4 & 2,4 & 2,4 \end{pmatrix} \end{array} .$$

The unique Nash equilibrium (in pure strategies) is (R,m). Since there is only one subgame, namely the entire game, this equilibrium is also subgame perfect. Denote the belief of player 2 that player 1 has played L by α. Then m is optimal if $2 \geq 3\alpha+2(1-\alpha)=2+\alpha$ and $2 \geq 3(1-\alpha)=3-3\alpha$. Clearly, these inequalities cannot both hold, so this equilibrium is not perfect Bayesian.

4.8 *Still Another Extensive Form Game*
(a) This is the following bimatrix game:

$$\begin{array}{c} \\ RA \\ RB \\ DA \\ DB \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 4,4 & 4,4 \\ 4,4 & 4,4 \\ 2,2 & 3,0 \\ 0,0 & 6,6 \end{pmatrix} \end{array} .$$

(b) (RA,l), (RB,l), (DB,r).
(c) The pure strategy Nash equilibria in the subgame starting with player 2's decision node are (A,l) and (B,r). Thus, the subgame perfect Nash equilibria are (RA,l) and (DB,r).

(d) (DB, r) with $\alpha = 0$ (by Bayesian consistency) is a perfect Bayesian equilibrium. (RA, l) with $\alpha = 1$ (by Bayesian consistency, since player 2 plays l) is also a perfect Bayesian equilibrium.

4.9 *A centipede game*
(a) The subgame perfect equilibrium tells each player to stop at any decision node. So the associated outcome is that player 1 stops immediately, resulting in the payoffs $(2, 0)$.
(b) Consider any other Nash equilibrium. If the play of the game proceeds to the last decision node (of player 2, in this case), then player 2 should stop (otherwise player 2 can improve). But then player 1 should have stopped at the before last decision node. Hence, the play of the game must stop earlier. But then the last player who has continued could have improved by stopping. Hence, the play of the game must have stopped immediately.
To exhibit a non-subgame perfect Nash equilibrium, assume that player 1 always stops, and that player 2 also always stops except at his second decision node. Check that this is a Nash equilibrium. In general, any pair of strategies where each player stops at his first decision node, is a Nash equilibrium. [One can also write down the strategic form, which is an 8×8 bimatrix game.]

4.10 *Finitely Repeated Prisoners' Dilemma*
(a) There are five subgames, including the entire game. Each player has $2 \times 2^4 = 32$ strategies. [One can also restrict attention to 8 strategies per player, by not considering the possibilities precluded by an own action at stage 1.]
(b) By working backward, it follows that the players play D at the second stage (independent of what has been played at the first stage), and also D at the first stage. Even if the game is played $k > 2$ times the players still always play D in a subgame perfect equilibrium.

4.11 *A Twice Repeated* 2×2 *Bimatrix Game*
(a) The unique subgame perfect Nash equilibrium is where player 1 always plays B and player 2 always R. This is true for any finite repetition of the game.
(b) Player 1: play B at the first stage; if (B, L) was played at the first stage play B at the second stage, otherwise play T at the second stage. Player 2: play L at the first stage and play R at the second stage.

4.12 *Twice Repeated* 3×3 *Bimatrix Games*
(a) There are ten subgames, including the entire game. Each player has 3^{10} strategies.
(b) Player 1: play T at the first stage. Player 2: play L at the first stage. Second stage play is given by the following diagram:

$$\begin{array}{c} \\ T \\ C \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} B,R & C,R & C,R \\ B,M & B,R & B,R \\ B,M & B,R & B,R \end{pmatrix} \end{array} .$$

For instance, if first stage play results in (C, L), then player 1 plays B and player 2 plays M at stage 2. Verify that this defines a subgame perfect equilibrium in which (T, L) is played at the second stage. (Other solutions are possible, as long as players 1 and 2 are punished for unilateral deviations at stage 1.)
(c) Player 1: play B at the first stage. Player 2: play R at the first stage. Second stage play is given by the following diagram:

$$\begin{array}{c} \\ T \\ C \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} T, L & T, L & C, M \\ T, L & T, L & T, L \\ T, L & T, L & T, L \end{pmatrix} \end{array} .$$

Problems of Chapter 5

5.1 *Battle-of-the-Sexes*
The extensive form of this game starts with a chance move, drawing the type combinations y_1y_2, y_1n_2, n_1y_2, n_1n_2 with probabilities 1/3, 1/3, 1/6, 1/6, respectively. Next we can model player 1's moves. Player 1 has two information sets, one following y_1y_2 and y_1n_2, and the other one following n_1y_2 and n_1n_2. Each information set of player 1 has, thus, two nodes. At each node player 1 has two moves, namely F and B. Next, also player 2 has two information sets, one following all moves following the type combinations y_1y_2 and n_1y_2, and the other one following all moves following the type combinations y_1n_2 and n_1n_2. Each information set has, thus, four nodes. At each of these nodes player 2 has two possible moves, namely B and F. Finally, there are sixteen end nodes, with payoffs according to Fig. 5.3.

The strategic form is a 4×4 bimatrix game. List the strategies of the players as in the text. We can then compute the expected payoffs. E.g., if the first row corresponds to strategy FF of player 1 and strategies FF, FB, BF, and BB of player 2, then the payoffs are, respectively, 1/6 times $(8, 3)$, $(6, 9)$, $(6, 0)$, and $(4, 6)$.

The (pure strategy) Nash equilibria are (FF, FB) and (BF, BB).

5.2 *A Static Game of Incomplete Information*
There are three pure strategy Nash equilibria: (TT, L), (TB, R), and (BB, R). (The first letter in a strategy of player 1 applies to Game 1, the second letter to Game 2.)

5.3 *Another Static Game of Incomplete Information*
(a) $p(t_1t_2) = 9/13$, $p(t_1t_2') = 0$, $p(t_1't_2) = 3/13$, $p(t_1't_2') = 1/13$. A pictorially easy way to find these is to write down a matrix:

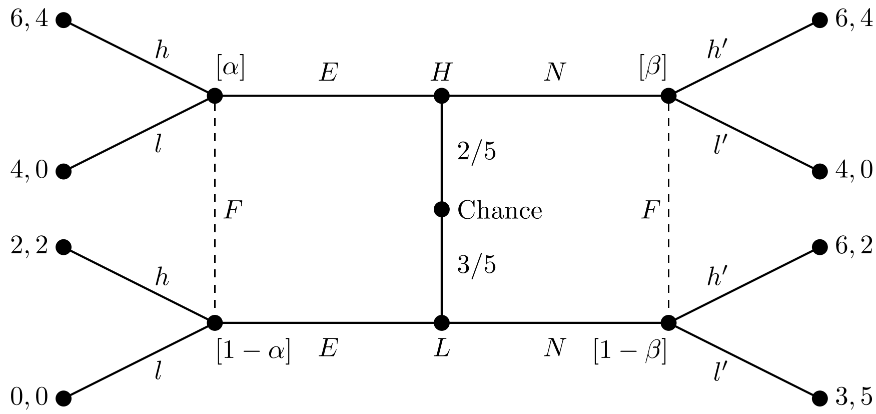
$$\begin{array}{c} \\ t_1 \\ t_2 \end{array} \begin{array}{c} \begin{array}{cc} t_2 & t_2' \end{array} \\ \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \end{array} .$$

The probabilities of the four type combinations are going to be the entries of this matrix. Clearly, we have a 0 at entry (t_1, t_2'), and we can put an x at entry (t_1', t_2'). Then by using the given conditional probabilities we obtain:

$$\begin{array}{c c} & \begin{array}{cc} t_2 & t_2' \end{array} \\ \begin{array}{c} t_1 \\ t_2 \end{array} & \begin{pmatrix} 9x & 0 \\ 3x & x \end{pmatrix} \end{array}.$$

Then use the equation $9x + 3x + x + 0 = 1$ to find $x = 1/13$.
(b) The unique pure strategy Nash equilibrium is: t_1 and t_1' play B, t_2 and t_2' play R. The analysis is simplified by starting with the observation that type t_1 will always play B, since T is strictly dominated.

5.4 *Job-Market Signaling*
(a,b) We provide the extensive form for the numerical specification of the game in part (b):



The strategic form is:

$$\begin{array}{c c} & \begin{array}{cccc} hh' & hl' & lh' & ll' \end{array} \\ \begin{array}{c} EE \\ EN \\ NE \\ NN \end{array} & \begin{pmatrix} 18,14 & 18,14 & 8,0 & 8,0 \\ 30,14 & 21,23 & 26,6 & 17,15 \\ 18,14 & 14,6 & 12,8 & 8,0 \\ 30,14 & 17,15 & 30,14 & 17,15 \end{pmatrix} \cdot \frac{1}{5} \end{array}.$$

The Nash equilibria in pure strategies are (EN, hl') and (NN, ll'). The first one is perfect Bayesian (separating) with $\alpha = 1$ and $\beta = 0$. The second one is not perfect Bayesian, since h strictly dominates l, i.e., there is no belief (no value of α) that makes l following E optimal. Thus, the IC does not apply.

5.5 *A Joint Venture*
(a) This is a game of incomplete information but not a signaling game.

(b) The strategic form is:

$$\begin{array}{c} \\ hh \\ hl \\ lh \\ ll \end{array} \begin{array}{c} \begin{array}{cc} \quad H \quad & \quad L \quad \end{array} \\ \begin{pmatrix} 45, 45 & 31\frac{1}{2}, 51\frac{1}{2} \\ 37\frac{1}{2}, 31\frac{1}{2} & 36, 50 \\ 59, 45 & 45\frac{1}{2}, 51\frac{1}{2} \\ 51\frac{1}{2}, 31\frac{1}{2} & 50, 50 \end{pmatrix} \end{array} .$$

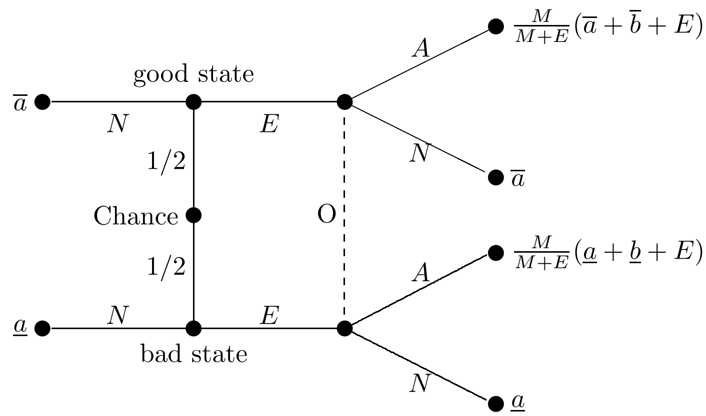
Hardware is the row player, the first letter corresponds to defective parts. Software is the column player. There is a unique Nash equilibrium (even in mixed strategies: apply iterated elimination of strictly dominated strategies), namely (ll, L). This is trivially subgame perfect and perfect Bayesian.

5.6 *Entry Deterrence*
For $x \leq 100$ the strategy combination where the entrant always enters and the incumbent colludes is a (pooling) perfect Bayesian equilibrium. For $x \geq 50$, the combination where the entrant always stays out and the incumbent fights is a (pooling) perfect Bayesian equilibrium if the incumbent believes that, if the entrant enters, then fighting yields 0 with probability at most $1 - \frac{50}{x}$. IC only applies to the second equilibrium where the entrant always stays out. Clearly both types, by entering, can get a payoff higher than their equilibrium payoff 0, so that IC puts no restrictions on the belief of the incumbent: the equilibrium survives IC.

5.7 *The Beer-Quiche Game*
(b) There are two pooling perfect Bayesian equilibria. In the first one, player 1 always eats quiche, and player 2 duels if and only if player 1 drinks beer; in that case, he believes that player 1 is weak with probability at least 1/2. This equilibrium does not survive the intuitive criterion since a weak player 1 could never benefit (compared to the equilibrium payoff) from drinking beer.

In the second one, player 1 always drinks beer, and player 2 duels if and only if player 1 eats quiche; in that case, he believes that player 1 is weak with probability at least 1/2. This equilibrium does survive the intuitive criterion for $\alpha = 1$: a weak player 1 could possibly benefit from eating quiche.

5.8 *Issuing Stock*
(a) The extensive form of this signaling game is as follows (we only write the payoffs for the manager, since the payoffs of the existing shareholder are identical):



(b) There is a pooling equilibrium in which the manager never proposes to issue new stock, and such a proposal would not be approved of by the existing shareholders since they believe that this proposal signals a good state with high enough probability. [The background of this is that a new stock issue would dilute the value of the stock of the existing shareholders in a good state of the world, see the original article Myers and Majluf (1984) for details.] This equilibrium (just about) survives the intuitive criterion.

There is also a separating equilibrium in which a stock issue is proposed in the bad state but not in the good state. If a stock issue is proposed, then it is approved of.

Finally, there is a separating equilibrium in which a stock issue is proposed in the good state but not in the bad state. If a stock issue is proposed, then it is not approved of.

(c) In this case, a stock issue proposal would always be approved of, so the 'bad news effect' of a stock issue vanishes. The reason is that the investment opportunity is now much more attractive.

5.9 *More Signaling Games*

(a) The perfect Bayesian equilibria are (LL, ud') for $\alpha = 1/2$ and $\beta \leq 2/3$ and (LL, uu') for $\alpha = 1/2$ and $\beta \geq 2/3$. In this case, type t obtains 2 in equilibrium and can get at most 1 by deviating to R. Type $\widetilde{t}$ obtains 4 in equilibrium and can get at most 2 by deviating to R. IC would require both $\beta = 0$ and $1 - \beta = 0$, which is clearly impossible, and therefore IC does not apply.

(b) There is a unique pooling perfect Bayesian equilibrium. Both types of player 1 play R; player 2 plays d after R and u after L, with belief $\alpha \geq \frac{1}{3}$. This equilibrium does not survive IC.

(c) Observe that after D player 2 always plays l, and that t_1 always plays D. There are two strategy combinations that are perfect Bayesian: i) t_2 plays D, t_3 plays U, after U player 2 plays r; ii) player 1 always plays D, after U player

2 plays l and believes that type t_3 has a probability of at most 1/2. The latter equilibrium does not survive IC.

Problems of Chapter 6

6.1 *Cournot with Asymmetric Costs*
To avoid corner solutions assume $0 \leq c_1, c_2 < a$ and $a \geq 2c_1 - c_2$, $a \geq 2c_2 - c_1$. Then the best reply functions are: $\beta_1(q_2) = (a - c_1 - q_2)/2$ if $q_2 \leq a - c_1$ and $\beta_1(q_2) = 0$ otherwise, and $\beta_2(q_1) = (a - c_2 - q_1)/2$ if $q_1 \leq a - c_2$ and $\beta_2(q_1) = 0$ otherwise. The point of intersection is $q_1 = (a - 2c_1 + c_2)/3$, $q_2 = (a - 2c_2 + c_1)/3$, which is the Nash equilibrium.

6.2 *Cournot Oligopoly*
(b) The reaction function of player i is: $\beta_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) = (a - c - \sum_{j \neq i} q_j)/2$ if $\sum_{j \neq i} q_j \leq a - c$, and $\beta_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) = 0$ otherwise.
(c) One should compute the point of intersection of the n reaction functions. This amounts to solving a system of n linear equations in n unknowns $q_1, \ldots, q_n$. Alternatively, one may guess that there is a solution $q_1 = q_2 = \ldots = q_n$. Then $q_1 = (a - c - (n-1)q_1)/2$, resulting in $q_1 = (a - c)/(n + 1)$. Hence, each firm producing $(a - c)/(n+1)$ is a Nash equilibrium. If the number of firms becomes large then this amount converges to 0, which is no surprise since demand is bounded by a.
(d) To show that this equilibrium is unique, it is sufficient to show that the determinant of the coefficient matrix associated with the system of n linear equations in n unknowns (the reaction functions) is unequal to zero.

6.3 *Quantity Competition with Heterogenous Goods*
(a) $\Pi_i(q_1, q_2) = q_i p_i(q_1, q_2) - c q_i$ for $i = 1, 2$.
(b) $\beta_1(q_2) = (5 - 2q_2 - c)/6$ (or 0 if this expression becomes negative), $\beta_2(q_1) = (4.5 - 1.5q_1 - c)/6$ (or 0 if this expression becomes negative). In equilibrium: $q_1 = (21 - 4c)/33$, $q_2 = (13 - 3c)/22$, $p_1 = (21 + 7c)/11$, $p_2 = (39 + 13c)/22$. The profits are $(21 - 4c)^2/363$ for firm 1 and $3(13 - 3c)^2/484$ for firm 2.
(c) $q_1 = (57 - 10c)/95$, $q_2 = (38 - 10c)/95$, $p_1 = (228 + 50c)/95$, $p_2 = (228 + 45c)/95$.
(d) $q_1 = \max\{1 - \frac{1}{2}p_1 + \frac{1}{3}p_2, 0\}$, $q_2 = \max\{1 - \frac{1}{2}p_2 + \frac{1}{4}p_1\}$. The profit functions are now $\Pi_1(p_1, p_2) = p_1 q_1 - c q_1$ and $\Pi_2(p_1, p_2) = p_2 q_2 - c q_2$, with q_1 and q_2 as given.
(e) $\beta_1(p_2) = 1 + \frac{1}{2}c + \frac{1}{3}p_2$ and $\beta_2(p_1) = 1 + \frac{1}{2}c + \frac{1}{4}p_1$. The equilibrium is $p_1 = (16 + 8c)/11$, $p_2 = (30 + 15c)/22$. Note that these prices are different from the ones in (c). The corresponding quantities are $q_1 = (16 - 3c)/22$ and $q_2 = (30 - 7c)/44$. The profit for firm 1 is $(16 - 3c)^2/242$ and for firm 2 it is $(30 - 7c)^2/968$. These profits are lower than under quantity competition, cf. (b) – price competition is more severe than quantity competition.
(f) These are the same prices and quantities as under (c).
(g) See the answers to (e) and (f).

6.4 *A Numerical Example of Cournot Competition with Incomplete Information*
$q_1 = 18/48$, $q_H = 9/48$, $q_L = 15/48$. In the complete information case with low cost we have $q_1 = q_2 = 16/48$, with high cost it is $q_1 = 20/48$ and $q_2 = 8/48$. Note that the low cost firm 'suffers' from incomplete information since firm 1 attaches some positive probability to firm 2 having high cost and therefore has higher supply. For the high cost firm the situation is reversed: it 'benefits' from incomplete information.

6.5 *Cournot Competition with Two-Sided Incomplete Information*
Similar to (6.3) we derive:

$$q_\ell = q_\ell(q_H, q_L) = \frac{a - c_\ell - \vartheta q_H - (1-\vartheta)q_L}{2} ,$$

$$q_h = q_h(q_H, q_L) = \frac{a - c_h - \vartheta q_H - (1-\vartheta)q_L}{2} ,$$

$$q_L = q_L(q_h, q_\ell) = \frac{a - c_L - \pi q_h - (1-\pi)q_\ell}{2} ,$$

$$q_H = q_H(q_h, q_\ell) = \frac{a - c_H - \pi q_h - (1-\pi)q_\ell}{2} .$$

Here, q_ℓ and q_h correspond to the low and high cost types of firm 1 and q_L, and q_H correspond to the low and high cost types of firm 2. The (Bayesian) Nash equilibrium follows by solving these four equations in the four unknown quantities.

6.6 *Incomplete Information about Demand*
The reaction functions are $q_1 = (1/2)(\vartheta a_H + (1-\vartheta)a_L - c - \vartheta q_H - (1-\vartheta)q_L)$, $q_H = (1/2)(a_H - c - q_1)$, $q_L = (1/2)(a_L - c - q_1)$. The equilibrium is: $q_1 = (\vartheta a_H + (1-\vartheta)a_L - c)/3$, $q_H = (a_H - c)/3 + ((1-\vartheta)/6)(a_H - a_L)$, $q_L = (a_L - c)/3 - (\vartheta/6)(a_H - a_L)$. (Assume that all these quantities are positive.)

6.7 *Variations on Two-Person Bertrand*
(a) If $c_1 < c_2$ then there is no Nash equilibrium. (Write down the reaction functions or – easier – consider different cases.)
(b) (i) There are two equilibria: $p_1 = p_2 = 2$ and $p_1 = p_2 = 3$. (ii) There are again two equilibria: $p_1 = p_2 = 2$ and $p_1 = 2$, $p_2 = 3$.

6.8 *Bertrand with More Than Two Firms*
A strategy combination is a Nash equilibrium if and only if at least two firms charge a price of c and the other firms charge prices higher than c.

6.9 *Variations on Stackelberg*
(a) The reaction function of firm 2 is $\beta_2(q_1) = (1/2)(a - c_2 - q_1)$ in the relevant range. Hence, firm 1 as a leader maximizes $q_1(a - q_1 - c_1 - (1/2)(a - c_2 - q_1))$.

This yields $q_1 = (1/2)(a - 2c_1 + c_2)$ and, thus, $q_2 = (1/4)(a + 2c_1 - 3c_2)$. With firm 2 as a leader we have $q_2 = (1/2)(a - 2c_2 + c_1)$ and $q_1 = (1/4)(a + 2c_2 - 3c_1)$.
(b) The leader in the Stackelberg game can always play the Cournot quantity: since the follower plays the best reply, this results in the Cournot outcome. Hence, the Stackelberg equilibrium – where the leader maximizes – can only give a higher payoff. (This argument holds for an arbitrary game where one player moves first and the other player moves next, having observed the move of the first player.)
(c) $q_i = (1/2^i)(a - c)$ for $i = 1, 2, \ldots, n$ is the subgame perfect equilibrium outcome, which can be found by backward induction.

6.10 *First-Price Sealed-Bid Auction*
(a) Player 1 wins and obtains $v_1 - v_2 \geq 0$. For player 1, bidding higher only reduces payoff, bidding lower and lose the auction reduces payoff to 0. The other players can only change their payoffs by bidding higher than v_2 and winning the auction, but this results in a negative payoff.
(b) Suppose that in some Nash equilibrium player i wins with valuation $v_i < v_1$. Then the winning bid b_i must be at most v_i otherwise player i makes a negative profit and therefore can improve by bidding (e.g.) v_i. But then player 1 can improve by bidding higher than b_i (and win) but lower than v_1 (and make positive profit).

Other Nash equilibria: $(v_1, v_1, 0, 0, \ldots, 0)$, $(b, b, b, \ldots, b)$ with $v_1 \geq b \geq v_2$, etc.
(c) Bidding $b \geq v_i$ is weakly dominated by (e.g.) bidding $v_i/2$, for every $i = 1, \ldots, n$. Bidding $0 < b < v_i$ is not weakly dominated, which can be seen as follows. Consider some other b'. If $b' > b$ then b gives higher payoff if all other players bid 0. If $b' < b$ then b gives higher payoff if all other players bid between b' and b.
(d) If not, then there would be a Nash equilibrium in which – in view of (c) – all players bid below their valuations. By (b) a player with the highest valuation wins the auction, so this must be player 1 if each player bids below his true valuation. But then player 1 can improve if $b_1 \geq v_2$ and player 2 can improve if $b_1 < v_2$.

6.11 *Second-Price Sealed-Bid Auction*
(d) Also $(v_1, 0 \ldots, 0)$ is a Nash equilibrium.
(e) The equilibria are: $\{(b_1, b_2) \mid b_2 \geq v_1,\ 0 \leq b_1 \leq v_2\} \cup \{(b_1, b_2) \mid b_1 \geq v_2,\ b_2 \leq b_1\}$.

6.12 *Third-Price Sealed-Bid Auction*
(a) Let $b_i < v_i$. If b_i is winning then v_i is also winning and the price to be paid is the same. If b_i is losing then the payoff from this bid is zero, whereas the payoff from bidding v_i is zero or positive. This shows that v_i weakly dominates any lower bid.

If $b_i > v_i$, then, if b_i is winning, v_i is losing, and the third highest bid is below v_i, then b_i gives positive payoff while v_i gives zero payoff. Hence, v_i does not weakly dominate any higher bid.
(b) Suppose $v_1 > v_2 > v_3 > \ldots$, then bidder 2 could improve by bidding higher than v_1.
(c) Everybody bidding the highest valuation v_1 is a Nash equilibrium. Also everybody bidding the second highest valuation v_2 is a Nash equilibrium. (There are many more!)

6.13 *n-Player First-Price Sealed-Bid with Incomplete Information*
Suppose every player $j \neq i$ plays s_j^*. If player i's type is v_i and he bids b_i (which can be assumed to be at most $1-1/n$ since no other bidder bids higher than this) then the probability of winning the auction is equal to the probability that very bid b_j, $j \neq i$, is at most b_i (including equality since this happens with zero probability). In turn, this is equal to the probability that $v_j \leq n/(n-1)b_i$ for every $j \neq i$. Since the players's valuations are independently drawn from the uniform distribution, the probability that player i wins the auction is equal to $((n/(n-1))b_i)^{n-1}$, hence player i should maximize the expression $(v_i - b_i)((n/(n-1))b_i)^{n-1}$, resulting in $b_i = (1-1/n)v_i$.

6.14 *Double Auction*
(a) Given the strategy of the seller (who asks x or 1) it can never be better to offer a price above x or below x if $v_b \geq x$. If $v_b < x$ then offering 0 is clearly a best reply. A similar argument holds for the seller.
(b) Observe that trade does not occur if $v_s > v_b$. Suppose $v_b \geq v_s$. Then trade takes place if and only if the buyer offers x and the seller asks x. This is the case when $v_b \geq x$ and $v_s \leq x$. Hence we compute the probability

$$\mathrm{Prob}[v_s \leq x \leq v_b | v_s \leq v_b]$$

which is equal to $2x(1-x)$. Note that this is maximal for $x = 1/2$, and then it is equal to 1/2.
(c) Suppose the seller plays $p_s(v_s) = a_s + c_s v_s$. We use (6.8) to determine the best reply of the buyer. Then $E[p_s|p_s \leq p_b] = a_s + c_s E[v_s|a_s + c_s v_s \leq p_b] = a_s + c_s[(1/2)(p_b - a_s)/c_s] = (a_s + p_b)/2$. Now (6.8) becomes

$$\max_{p_b \in [0,1]} \left[v_b - \frac{1}{2}\left\{ p_b + \frac{(a_s + p_b)}{2} \right\} \right] \frac{p_b - a_s}{c_s} \ .$$

Solving this problem yields $p_b = (2/3)v_b + (1/3)a_s$.

Similarly, assuming that the buyer plays a strategy $p_b(v_b) = a_b + c_b v_b$, we obtain that the seller solves the problem

$$\max_{p_s \in [0,1]} \left[\frac{1}{2}\left\{ p_s + \frac{(p_s + a_b + c_b)}{2} \right\} - v_s \right] \frac{a_b + c_b - p_s}{c_b} \ .$$

Solving this problem yields $p_s = (2/3)v_s + (1/3)(a_b + c_b)$.

Combining both results yields $p_b = (2/3)v_b + 1/12$ and $p_s = (2/3)v_s + 1/4$.
(d) Observe that no trade occurs if $v_s > v_b$. Suppose $v_s \leq v_b$. Then trade occurs when $p_b \geq p_s$, which is equivalent to $v_b \geq v_s + (1/4)$. Hence, the (conditional) probability of this is 9/16. Observe that this is larger than the maximal probability in (b).

6.15 *Mixed Strategies and Objective Uncertainty*
(a) $((1/2, 1/2), (2/5, 3/5))$.
(b) Consider the bimatrix game

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 4+\alpha, 1 & 1, 3 \\ 1, 2 & 3, \beta \end{pmatrix} \end{array},$$

where α and β are drawn from the uniform distribution over the interval $[0, x]$. We search for a Bayesian Nash equilibrium of the following form. Player 1 plays T if $\alpha \geq a$ and B otherwise; player 2 plays L if $\beta \leq b$ and R otherwise; $a, b \in [0, x]$. By a computation completely analogous to the one in the text we obtain $a = (1/2)(x - 5 + \sqrt{25 + x^2})$, $b = (2/5)(x + 5 - \sqrt{25 + x^2})$. By applying l'Hôpital's rule, we derive $\lim_{x \to 0} a/x = 1/2$ and $\lim_{x \to 0} b/x = 2/5$.

6.16 *Variations on Finite Horizon Bargaining*
(a) Adapt Table 6.1 for the various cases.
(b) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1 - \delta_2 + \delta_1\delta_2, \delta_2 - \delta_1\delta_2)$ at $t = 0$ and player 2 accepts.
(c) The subgame perfect equilibrium *outcome* in shares of the good is: player 1 proposes $(1 - \delta_2^2 + \delta_1\delta_2^2, \delta_2^2 - \delta_1\delta_2^2)$ at $t = 0$ and player 2 accepts.
(d) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1 - \delta + \delta^2 - \ldots + \delta^{T-1} - \delta^T s_1, \delta - \delta^2 + \ldots - \delta^{T-1} + \delta^T s_1)$ at $t = 0$ and player 2 accepts.
(e) The limits are $(1/(1+\delta), \delta(1+\delta))$, independent of $\mathbf{s}$.
(f) Consider the following strategies. Player 1 always proposes $\mathbf{s}$ and accepts a proposal (x_1, x_2) by player 2 if and only if $x_1 \geq s_1$. Player 2 always proposes $\mathbf{s}$ and accepts a proposal (x_1, x_2) by player 1 if and only if $x_2 \geq s_2$. These strategies are a Nash equilibrium and result in $\mathbf{s}$.

6.17 *Variations on Infinite Horizon Bargaining*
(a) Conditions (6.10) are replaced by $x_2^* = \delta_2 y_2^*$ and $y_1^* = \delta_1 x_1^*$. This implies $x_1^* = (1 - \delta_2)/(1 - \delta_1\delta_2)$ and $y_1^* = (\delta_1 - \delta_1\delta_2)/(1 - \delta_1\delta_2)$. In the strategies (σ_1^*) and (σ_2^*), replace δ by δ_1 and δ_2, respectively. The equilibrium outcome is that player 1's proposal x^* at $t = 0$ is accepted.
(b) Nothing essential changes. Player 2's proposal y^* is accepted at $t = 0$.
(c) Nothing changes compared to the situation in the text, since $\mathbf{s}$ is only obtained at $t = \infty$.
(d) In utilities, a Pareto optimal proposal (z_1, z_2) satisfies $z_2 = \sqrt{1 - z_1}$ (make picture). Hence, in a subgame perfect equilibrium as in the text we have $\sqrt{1 - x_1^*} = x_2^* = \delta y_2^* = \delta\sqrt{1 - y_1^*} = \delta\sqrt{1 - \delta x_1^*}$. Solving the equation yields $x_1^* = (1+\delta)/(1+\delta+\delta^2)$. In terms of distribution of the good, this means that

player 1 is going to receive this amount – since his utility is equal to his share – and player 2 is going to receive $1-(1+\delta)/(1+\delta+\delta^2)=\delta^2/(1+\delta+\delta^2)$, with utility $\sqrt{\delta^2/(1+\delta+\delta^2)}$.

(e) Let p denote the probability that the game ends. Then p is also the probability that the game ends given that it does not end at $t=0$. Hence, $p=(1-\delta)+\delta p$, so that $p=1$.

6.18 *A Principal-Agent Game*

(a) This is a game of complete information. The employer starts and has an infinite number of actions available, namely any pair (w_L, w_H) of nonnegative wages. After each of these actions, the worker has three possible actions: reject, resulting in 0 for the employer and 2 for the worker; accept and exert high effort, resulting in the (expected) payoffs of $10.8-0.8w_H-0.2w_L$ for the employer and $0.8w_H+0.2w_L-3$ for the worker; accept and exert low effort, resulting in the (expected) payoffs of $7.2-0.2w_H-0.8w_L$ for the employer and $0.2w_H+0.8w_L$ for the worker. The actions of the employer are also his strategies. A strategy for the worker is a function $(w_L, w_H) \mapsto \{$reject (r), accept and exert high effort (h), accept and exert low effort $(l)\}$.

(b) The subgame perfect equilibrium can be found by backward induction. Strategy h is optimal for the worker if it is at least as good as r, i.e., $8w_H+2w_L \geq 50$, and at least as good as l, i.e., $w_H-w_L \geq 5$. Subject to these to constraints, the employer maximizes the expected payoff $10.8-0.8w_H-0.2w_L$. Clearly, the maximum is obtained for any pair (w_H, w_L) with $8w_H+2w_L=50$ and $w_H-w_L \geq 5$. The associated profit is $10.8-5=5.8$.

Strategy l is optimal for the worker if it is at least as good as r, i.e., $2w_H+8w_L \geq 20$, and at least as good as h, i.e., $w_H-w_L \leq 5$. Subject to these to constraints, the employer maximizes the expected payoff $7.2-0.2w_H-0.8w_L$. Clearly, the maximum is obtained for any pair (w_H, w_L) with $2w_H+8w_L=20$ and $w_H-w_L \leq 5$. The associated profit is $7.2-2=5.2$.

Hence, it is optimal for the employer to induce high effort by a wage combination (w_H, w_L) with $8w_H+2w_L=50$ and $w_H-w_L \geq 5$. These are the equilibrium strategies of the employer; the worker chooses optimally, and in particular h following any equilibrium strategy of the employer.

6.19 *The Market for Lemons*

(b) There are many subgame perfect equilibria: the buyer offers $p \leq 5000$ and the seller accepts any price of at least 5000 if the car is bad and of at least 15000 if the car is good. All these equilibria result in expected payoff of zero for both. There are no other subgame perfect equilibria.

6.20 *Corporate Investment and Capital Structure*

(b) Suppose the investor's belief that $\pi=L$ after observing s is equal to q. Then the investor accepts s if and only if

$$s[qL+(1-q)H+R] \geq I(1+r)\ . \qquad (*)$$

The entrepreneur prefers to receive the investment if and only if

$$s \leq R/(\pi + R) \,, \qquad (**)$$

for $\pi \in \{L, H\}$.

In a pooling equilibrium, $q = p$. Note that $(**)$ is more difficult to satisfy for $\pi = H$ than for $\pi = L$. Thus, $(*)$ and $(**)$ imply that a pooling equilibrium exists only if

$$\frac{I(1+r)}{pL + (1-p)H + R} \leq \frac{R}{H+R} \,.$$

A separating equilibrium always exists. The low-profit type offers $s = I(1+r)/(L+R)$, which the investor accepts, and the high-profit type offers $s < I(1+r)/(H+R)$, which the investor rejects.

6.21 *A Poker Game*
(a) The strategic form of this game is as follows:

$$\begin{array}{c} \\ \text{believe} \\ \text{show} \end{array} \begin{array}{c} \begin{array}{cccc} \text{aa} & \text{aq} & \text{ka} & \text{kq} \end{array} \\ \begin{pmatrix} -1,1 & -1/3,1/3 & -2/3,2/3 & 0,0 \\ 2/3,-2/3 & 1/3,-1/3 & 0,0 & -1/3,1/3 \end{pmatrix} \end{array} .$$

Here, 'believe' and 'show' are the strategies of player I. The first letter in any strategy of player II is what player II says if the dealt card is a King, the second letter is what II says if the dealt card is a Queen – if the dealt card is an Ace player II has no choice.
(b) Player I has a unique optimal (maximin) strategy, namely $(1/3, 2/3)$. Also player 2 has a unique optimal (minimax) strategy, namely $(0, 0, 1/3, 2/3)$. The value of the game is $-2/9$.

6.22 *A Hotelling Location Problem*
(a) $x_1 = x_2 = \frac{1}{2}$.
(b) E.g. $x_1 = \frac{1}{3}$, $x_2 = x_3 = \frac{2}{3}$.
(c) $x_1 = x_2 = \frac{1}{2}$.
(d) For $n = 3$ there are no Nash equilibria (consider all possible cases). For $n = 4$ a Nash equilibrium is: $x_1 = x_2 = \frac{1}{4}$, $x_3 = x_4 = \frac{3}{4}$.

6.23 *Median Voting*
(a) The strategy set of each player is the interval $[0, 30]$. If each player i plays x_i, then the payoff to each player i is $-|((x_1 + \ldots + x_n)/n) - t_i|$.

Such a game typically may have a Nash equilibrium where players 1 up to k propose 0 and players $k+1$ up to n propose 30, for some $0 \leq k \leq n$. Such a strategy combination is a Nash equilibrium if the average is above t_k and below t_{k+1}. If such a configuration does not exist, then start with an arbitrary k and assume, without loss of generality, that the average is below t_k. If, by changing player k's proposal to 30, the average is still below t_k, then continue with this new configuration, consider player $k-1$, and repeat the argument. If, however, the average by k proposing 30 would be above t_k, then let player k propose a temperature that makes the average equal to t_k: then we have a Nash equilibrium again.

(b) The payoff to player i is now $-|\operatorname{med}(x_1,\ldots,x_n) - t_i|$, where $\operatorname{med}(\cdot)$ denotes the median. For each player, proposing a temperature different from his true ideal temperature either leaves the median unchanged or moves the median farther away from the ideal temperature, whatever the proposals of the other players. Hence, proposing one's ideal temperature is a weakly dominant strategy.

There are many other Nash equilibria. E.g., everyone proposing the same temperature is always a Nash equilibrium, since no player can change the median (i.e., the commonly proposed temperature) unilaterally.

6.24 *The Uniform Rule*

(a) In general, this game has no Nash equilibrium: if, in some strategy combination a player gets less [more] than his ideal amount, he can improve by reporting a higher [lower amount]. In specific cases an equilibrium may exist. E.g. if $\sum_{i=1}^n t_i = M$ then reporting truthfully is a Nash equilibrium.

(b) $M = 4:\ (1, 3/2, 3/2)$, $M = 5:\ (1, 2, 2)$, $M = 5.5:\ (1, 2, 5/2)$, $M = 6:\ (1, 2, 3)$, $M = 7:\ (2, 2, 3)$, $M = 8:\ (5/2, 5/2, 3)$, $M = 9:\ (3, 3, 3)$.

(c) If player i reports t_i and receives $s_i > t_i$ then, apparently the total reported quantity is above M and thus, player i can only further increase (hence, worsen) his share by reporting a different quantity. If player i reports t_i and receives $s_i < t_i$ then, apparently the total reported quantity is below M and thus, player i can only further decrease (hence, worsen) his share by reporting a different quantity.

There exist other Nash equilibria, but they do not give different outcomes (shares). E.g., if $M > \sum_{j=1}^n t_j$, then player 1 could just as well report 0 instead of t_1.

6.25 *Reporting a Crime*

(a) The Nash equilibria in pure strategies are those pure strategy combinations where exactly one person calls the police. There are n of these, and none is symmetric.

(b) If each person plays C with probability $0 < p < 1$, then each person must be indifferent between playing C and N, hence $v - c = (1-p)^{n-1} \cdot 0 + [1 - (1-p)^{n-1}] \cdot v$. This yields $p = 1 - (c/v)^{1/(n-1)}$.

(c) The probability of the crime being reported in this equilibrium is $1-(1-p)^n = 1 - (c/v)^{n/(n-1)}$. This converges to $1 - (c/v)$ for n going to infinity. Observe that both p and the the probability of the crime being reported decrease if n becomes larger.

6.26 *Firm Concentration*

Let, in equilibrium, n firms locate downtown and m firms in the suburbs, with $m+n = 10$. Then a downtown firm does not want to deviate, so $5n - n^2 + 50 \geq 48 - (m+1) = 37 + n$. This implies $n \in \{0, \ldots, 6\}$. Similarly, a suburb firm does not want to deviate, so $48 - m = 48 - (10-n) \geq 5(n+1) - (n+1)^2 + 50$. This implies $n \in \{6, \ldots, 10\}$. Hence $n = 6$ and $m = 4$.

6.27 *Tragedy of the Commons*
(d) Suppose, to the contrary, $G^* \leq G^{**}$. Then $v(G^*) \geq v(G^{**})$ since $v' < 0$, and $0 > v'(G^*) \geq v'(G^{**})$ since $v'' < 0$. Also, $G^*/n < G^{**}$. Hence

$$v(G^*) + (1/n)G^* v'(G^*) - c > v(G^{**}) + G^{**} v'(G^{**}) - c \,,$$

a contradiction since both sides should be zero.

Problems of Chapter 7

7.1 *Nash and Subgame Perfect Equilibrium in a Repeated Game (1)*
(a) The unique Nash equilibrium is (U, R); $v(A) = 1$ and the minimax strategy in A is R; $v(-B) = -1$ and the maximin strategy in $-B$ is D.
(b) Only $(1, 5)$, independent of δ.
(c) All payoff pairs in the convex hull of the points $(2, 3)$, $(1, 5)$, and $(0, 1)$ which have both coordinates strictly larger than $(1, 1)$.
(d) Player 1 plays always U but after a deviation switches to D forever. Player 2 always plays L but after a deviation switches to R forever. We need $\delta \geq \frac{1}{2}$, to keep player 2 from deviating to R.

7.2 *Nash and Subgame Perfect Equilibrium in a Repeated Game (2)*
(a) The limiting average payoffs $(2, 1)$, $(1, 2)$, and $(2/3, 2/3)$, resulting from playing, respectively, the Nash equilibria (U, L), (D, R), and $((2/3, 1/3), (1/3, 2/3))$ at every stage; and all payoffs (x_1, x_2) with $x_1, x_2 > 2/3$.
(b) $v(A) = 2/3$ and $-v(-B) = 2/3$. Hence, all payoffs (x_1, x_2) with $x_1, x_2 > 2/3$.
(c) The players play (U, L) at even times and (D, R) at odd times. Since at each time they play a Nash equilibrium of the stage game, no trigger strategies (describing punishment after a deviation) are needed.
(d) The players play (U, L) at $t = 0, 3, 6, \ldots$; (D, L) at $t = 1, 4, 7, \ldots$; and (D, R) at $t = 2, 5, 8, \ldots$. After a deviation player 1 plays $(2/3, 1/3)$ forever and player 2 plays $(1/3, 2/3)$ forever. To make this a subgame perfect equilibrium, we need $2 + (2/3)\delta/(1-\delta) \leq (0 + 1 \cdot \delta + 2 \cdot \delta^2)/(1 - \delta^3)$ to avoid deviation by player 1 and $2 + (2/3)\delta/(1-\delta) \leq (0 + 2 \cdot \delta + 1 \cdot \delta^2)/(1 - \delta^3)$ to avoid deviation by player 2. The first inequality gives the lower bound on δ.

7.3 *Nash and Subgame Perfect Equilibrium in a Repeated Game (3)*
(a) The stage game has a unique Nash equilibrium, namely $((1/2, 1/2), (2/3), (1/3))$ with payoffs $(14/3, 1)$. Hence these payoffs as well as all payoff pairs in the convex hull of $(3, 2)$, $(8, 0)$, $(4, 0)$, and $(6, 2)$ strictly larger than $(14/3, 1)$, can be obtained as limit average payoffs in a subgame perfect equilibrium of $G^\infty(\delta)$.
(b) $v(A) = 4$ since (D, L) is a saddlepoint in A. The minimax strategy of player 2 is L. The value of $-B$ is -1 and the maximin strategy of player 1 is $(1/2, 1/2)$. The associated payoffs in G are $(4, 1)$. Hence all payoff pairs in

the convex hull of $(3, 2)$, $(8, 0)$, $(4, 0)$, and $(6, 2)$ strictly larger for than $(4, 1)$, can be obtained as limit average payoffs in a Nash equilibrium of $G^\infty(\delta)$.
(c) A Nash equilibrium is obtained by letting the players play (U, L) at even times and (D, R) at odd times. After any deviation the players switch to playing (D, L) forever. Player 1 has an incentive to deviate at even times, say at $t = 0$. To avoid this we need $4 + 4\delta/(1 - \delta) \le 3/(1 - \delta^2) + 6\delta/(1 - \delta^2)$, which holds exactly if $\delta \ge 1/2$. Player 2 never has an incentive to deviate (so we do not need a trigger strategy for player 1 in this case, but we could let player 1 play $(1/2, 1/2)$ instead of D after a deviation). This equilibrium is subgame perfect for no value of δ: in a subgame after a deviation, (D, L) (or $((1/2, 1/2), L)$) is played forever, which is not a Nash equilibrium.

7.4 *Subgame Perfect Equilibrium in a Repeated Game*
(a) (M, C) and (B, R).
(b) $(1, 1)$ and all payoff pairs in the convex hull of the nine payoff pairs in G strictly larger than 1 for both players.
(c) Alternate between (T, L) and (M, C). After any deviation, player 1 switches to B and player 2 to R forever. At even moments, say at $t = 0$, player 1 has an incentive to deviate to M. This is avoided if $8 + \delta/(1 - \delta) \le 6/(1 - \delta^2) + 4\delta/(1 - \delta^2)$, which holds if δ is at least (approximately) 0.36. Also at $t = 0$, player 2 has an incentive to deviate to C. This is avoided if $7 + \delta/(1 - \delta) \le 4/(1 - \delta^2) + 6\delta/(1 - \delta^2)$, which holds if δ is at least (approximately) 0.40. Hence, 0.40 is approximately the lower bound on δ for which this is a subgame perfect equilibrium.

7.5 *The Strategies Tr_1^* and Tr_2^**
An optimal moment for player 1 to deviate would be $t = 1$. We then have the inequality

$$40 + \frac{40\delta}{1 - \delta} \le \frac{30 + 30\delta + 60\delta^2 + 40\delta^3 + 50\delta^4}{1 - \delta^5} .$$

An optimal moment for player 2 to deviate would be $t = 3$. The associated inequality is

$$40 + \frac{40\delta}{1 - \delta} \le \frac{30 + 40\delta + 50\delta^2 + 60\delta^3 + 60\delta^4}{1 - \delta^5} .$$

7.6 *Repeated Cournot and Bertrand*
(a) Each player offers half of the monopoly quantity (half of $(a - c)/2$) at each time, but if a deviation from this occurs, then each player offers the Cournot quantity $(a - c)/3$ forever. The relevant restriction on δ is given by $(1/8) \ge (9/16)(1 - \delta) + (1/9)\delta$, which yields $\delta \ge 9/17$.
(b) In this case, each player asks the monopoly price $(a + c)/2$ at each time; if a deviation from this occurs, each player switches to the Bertrand equilibrium price $p = c$ forever. The relevant restriction on δ is given by $1/4 \le (1/8)/(1 - \delta)$, which gives $\delta \ge 1/2$.

7.7 *Repeated Duopoly*

(a) $q_2 = \max\{10 - (1/3)p_1 - (2/3)p_2, 0\}$.
(b) For firm 1, maximize $p_1(10 - (2/3)p_1 - (1/3)p_2)$. This yields $p_1 = 15/2 - (1/4)p_2$. Similarly, the reaction function of firm 2 is $p_2 = 15/2 - (1/4)p_1$. This yields the Nash equilibrium $p_1 = p_2 = 6$ with profit 24 for each.
(c) Joint profit is maximized at $p_1 = p_2 = 5$. At these prices, each firm has profit 25.
(d) Ask prices $p_1 = p_2 = 5$, but after a deviation switch to the equilibrium prices $p_1 = p_2 = 6$. An optimal deviation, say for firm 1, would be to charge $p_1 = 15/2 - (1/4) \cdot 5 = 25/4$, yielding an instantaneous profit of $(25/4)(10 - (2/3)(25/4) - (1/3)5) = 25^2/24$. The relevant inequality is $25^2/24 + 24\delta/(1-\delta) \leq 25/(1-\delta)$, simplifying to $\delta \geq 25/49$.

7.8 *On Discounting*
See the solution to Problem 6.17(e).

7.9 *On Limit Average*
A sequence like $1, 3, 5, 7, \ldots$ has a limit average of infinity. More interestingly, one may construct a sequence containing only the numbers $+1$ and -1 of which the finite averages 'oscillate', e.g, below $-1/2$ and above $+1/2$, so that the limit does not exist.

Problems of Chapter 8

8.1 *Symmetric Games*
(a) $(0,1)$ is the only ESS.
(b) Both $(1,0)$ and $(0,1)$ are ESS: the unique best reply against $(1,0)$ is $(1,0)$, and the unique best reply against $(0,1)$ is $(0,1)$. The (Nash equilibrium) strategy $(1/3, 2/3)$ is not an ESS. For the latter, (8.1) yields $2/3 > 2y^2 + (1-y)^2$, which is equivalent to $(3y-1)^2 < 0$. This holds for no value of y.

8.2 *More Symmetric Games*
(a) The replicator dynamics is $\dot{p} = p(p-1)(p-1/2)$, with rest points $p = 0, 1, 1/2$, of which only $p = 1/2$ is stable. The game (A, A^T) has a unique symmetric Nash equilibrium, namely $((1/2, 1/2), (1/2, 1/2))$. The unique ESS is $(1/2, 1/2)$.
(b) The replicator dynamics is $\dot{p} = 3p^2(1-p)$, with rest points $p = 0, 1$, of which only $p = 1$ is stable. The game (A, A^T) has two symmetric Nash equilibria, namely $((1,0), (1,0))$ and $((0,1), (0,1))$, and a unique ESS, namely $(1,0)$.

8.3 *Asymmetric Games*
(a) The replicator dynamics is given by the equations $\dot{p} = p(1-p)(1-2q)$ and $\dot{q} = q(1-2p)(1-q)$. There are five rest points, namely $p = q = 0$, $p = q = 1$, $p = q = 1/2$, and $p = 0$, $q = 1$, and $p = 1$, $q = 0$. The last two rest points are stable. They correspond to the two strict Nash equilibria of the game, namely $((0,1), (1,0))$ and $((1,0), (0,1))$.

(b) The replicator dynamics is given by the equations $\dot{p} = pq(1-p)$ and $\dot{q} = pq(1-q)$. There is one stable rest point, namely $p = q = 1$, corresponding to the unique strict Nash equilibrium $((1,0),(1,0))$ of the game. The other rest points are all points in the set

$$\{(p,q) \mid \ p = 0 \text{ and } 0 \leq q \leq 1 \text{ or } q = 0 \text{ and } 0 \leq p \leq 1\} .$$

8.4 *More Asymmetric Games*
(a) Let $(x, 1-x)$ be the shares of the row population and $(y, 1-y)$ the shares of the column population. The replicator dynamics are $dx/dt = x(1-x)(2-3y)$ and $dy/dt = 2y(1-2x)(y-1)$. The rest points are $(x,y) = (0,0)$, $(x,y) = (1,0)$, $(x,y) = (1,1)$, $(x,y) = (0,1)$, $(x,y) = (1/2, 2/3)$. None of these is stable. The game has no pure Nash equilibria and therefore no strict Nash equilibria (Remark 8.9).
(b) The replicator dynamics are $dx/dt = x(x-1)(2y-1)$ and $dy/dt = y(y-1)(2x-1)$. The rest points are $(x,y) = (0,0)$, $(x,y) = (1,0)$, $(x,y) = (1,1)$, $(x,y) = (0,1)$, $(x,y) = (1/2, 1/2)$. The rest points $(0,1)$ and $(1,0)$ are stable. They correspond to the strict Nash equilibria $((0,1),(1,0))$ and $((1,0),(0,1))$.

8.5 *Frogs Call For Mates*
(a) If $P - z > 1 - m$ then Call is ESS. If $m - z < 0$ then Don't Call is ESS. If $P - z < 1 - m$ and $m - z > 0$ then there is a mixed ESS. Part (b) follows from the cases mentioned in (a): if $z < 0.4$ then Call is ESS; if $z > 0.6$ then Don't Call is ESS; if $0.4 < z < 0.6$ then there is a mixed ESS. Note that for (a) and (b) Prop. 8.5 can be used. Similarly, for (c) we can use Prop. 8.8, by stating the conditions under which each of the four pure strategy combinations is a strict Nash equilibrium: if $z_1 < P + m - 1$ and $z_2 < P + m - 1$ then (Call,Call) is a strict Nash equilibrium, etc.

8.6 *Video Market Game*
There are four rest points, namely: $x = y = 0$, $x = y = 1$, $(x = 0, y = 1)$, and $(x = 1, y = 0)$ [$(x, 1-x)$ is the row 'population', $(y, 1-y)$ corresponds to the columns]. The only stable rest point is $x = 0, y = 1$.

Problems of Chapter 9

9.1 *Number of Coalitions*
An arbitrary subset $S \subseteq N$ with $|N| = n$ can be represented by a vector $\mathbf{x} \in \{0,1\}^N$, where $i \in S \Leftrightarrow x_i = 1$. There are 2^n different vectors in $\{0,1\}^N$.

9.2 *Computing the Core*
(a) $\{(0,0,1)\}$; (b) polygon with vertices $(15,5,4)$, $(9,5,10)$, $(14,6,4)$, and $(8,6,10)$; (c) $\{(x_1, \ldots, x_5, 0, \ldots, 0) \in \mathbb{R}^{15} \mid x_1 \geq 0, \ldots, x_5 \geq 0, x_1 + \ldots + x_5 = 1\}$.

9.3 *The Core of a Two-Person Game*

$c \geq a+b$, $C(\{1,2\},v) = \{(x_1,x_2) \mid x_1 \geq a, x_2 \geq b, x_1+x_2 = c\}$.

9.4 *The Core of the General Glove Game*
Let $\ell = r$. Every two-person coalition consisting of a left-hand and a right-hand glove owner has worth 1, so needs to obtain at least 1 in the core. Hence, every such pair obtains exactly 1 in the core. This implies that all left-hand glove owners receive the same amount and all right-hand glove owners receive the same amount. Altogether, the core consists of all payoff vectors in which all right-hand glove owners receive an amount $0 \leq \alpha \leq 1$ and all left-hand glove owners an amount $1-\alpha$. For $\alpha = 1/2$, we obtain the Shapley value, so the Shapley value is in the core.

9.5 *A Condition For Nonemptiness of the Core of a Three-Person Game*
For a core element $\mathbf{x}$, we have $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq (x_1+x_2) + (x_1+x_3) + (x_2+x_3) = 2v(\{1,2,3\})$.

9.6 *Non-Monotonicity of the Core*
(b) The core of (N,v') is the set $\{(0,0,1,1)\}$ (use the fact that $C(N,v') \subseteq C(N,v)$). Hence, player 1 can only obtain less in the core although the worth of coalition $\{1,3,4\}$ has increased.

9.7 *Efficiency of the Shapley Value*
Consider an order $i_1, i_2, \ldots, i_n$ of the players. The sum of the coordinates of the associated marginal vector is

$$\begin{aligned}
&[v(\{i_1\}) - v(\emptyset)] \\
&+[v(\{i_1,i_2\}) - v(\{i_1\})] \\
&+[v(\{i_1,i_2,i_3\}) - v(\{i_1,i_2\})] \\
&+\ldots \\
&+[v(N) - v(N \setminus \{i_n\})] \\
&= v(N) - v(\emptyset) = v(N)\ .
\end{aligned}$$

Hence, every marginal vector is efficient, so the Shapley value is efficient since it is the average of the marginal vectors.

9.8 *Computing the Shapley Value*
(a) $\Phi(N,v) = (1/6, 1/6, 2/3) \notin C(N,v)$; (b) $(9\frac{1}{2}, 6\frac{1}{2}, 8)$, not in the core.
(c) The Shapley value assigns to each nonpermanent member the number $\binom{9}{3} \cdot (8! \cdot 6!)/15! \approx 0.002$. Hence, each permanent member is assigned approximately $\frac{1}{5} \cdot (1 - 0.02) = 0.196$. Clearly the Shapley value is not on the core.

9.9 *The Shapley Value and the Core*
(a) $a = 3$ (use Problem 9.5).
(b) $(2.5, 2, 1.5)$.
(c) The Shapley value is $(a/3 + 1/2, a/3, a/3 - 1/2)$. The minimal value of a for which this is in the core is $15/4$.

9.10 *Shapley Value in a Two-Player Game*
$\Phi(N,v) = (v(\{1\}) + (v(\{1,2\}) - v(\{1\}) - v(\{2\}))/2, v(\{2\}) + (v(\{1,2\}) - v(\{1\}) - v(\{2\}))/2)$.

9.11 *Computing the Nucleolus*
(a) $(0,0,1)$.
(b) $(11.5, 5.5, 7)$.
(c) $(1/5, 1/5, 1/5, 1/5, 1/5, 0, \ldots, 0) \in \mathbb{R}^{15}$.
(d) In (N,v): $(1/2, 1/2, 1/2, 1/2)$; in (N,v'): $(0,0,1,1)$.

9.12 *Nucleolus of Two-Player Games*
The nucleolus is $(v(\{1\})+(v(\{1,2\})-v(\{1\})-v(\{2\}))/2, v(\{2\})+(v(\{1,2\})-v(\{1\})-v(\{2\}))/2)$.

9.13 *Computing the Core, the Shapley Value, and the Nucleolus*
(a) The nucleolus and Shapley value coincide and are equal to $(1.5, 2, 2.5)$.
(c) The maximal value of $v(\{1\})$ is 2. For that value the core is the line segment with endpoints $(2,1,3)$ and $(2,3,1)$.

9.14 *Voting (1)*
(a) The winning coalitions, i.e., the coalitions with worth 1, are $\{1,2\}$, $\{1,3,4\}$, and all coalitions containing at least one of these two. To compute the Shapley value, note that players 3 and 4 only make a contribution of 1 to respectively the coalitions $\{1,4\}$ and $\{1,3\}$. Hence, $\Phi_3(N,v) = \Phi_4(N,v) = 2!1!/4! = 1/12$. Player 2 makes a contribution of 1 to the coalitions $\{1\}$, $\{1,3\}$, and $\{1,4\}$, and thus $\Phi_2(N,v) = 1!2!/4! + 2 \cdot 2!1!/4! = 3/12$. Hence, the Shapley value is $\Phi(N,v) = (7/12, 3/12, 1/12, 1/12)$.
(b) Let $\mathbf{x}$ be in the core. Since $v(\{1,2\}) = 1$, we must have that $x_3 = x_4 = 0$. Since $v(\{1,3,4\}) = 1$, we must have that $x_2 = 0$. Therefore, $C(N,v) = \{(1,0,0,0)\}$. By Prop. 9.7, the nucleolus is in the core, so the nucleolus is $(1,0,0,0)$.

9.15 *Voting (2)*
(a) The winning coalitions of minimal size are $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$.
(b) $v(S) = 1$ for every S containing one of the coalitions in (a), and $v(S) = 0$ for all other coalitions.
(c) Player 1 makes a contribution of 1 to the coalitions $\{3,4\}$, $\{3,5\}$, $\{4,5\}$, and $\{3,4,5\}$. Hence, $\Phi_1(N,v) = 3 \cdot 2!2!/5! + 1 \cdot 3!1!/5! = 9/60$. We obtain $\Phi(N,v) = (1/60)(9,9,14,14,14)$. Players 3,4, and 5 are most powerful.
(d) The nucleolus is of the form $(\alpha, \alpha, (1-2\alpha)/3, (1-2\alpha)/3, (1-2\alpha)/3)$, where $0 \le \alpha \le 1/2$ to make it an imputation. The maximal excess is reached for the coalitions in (a), and this excess is equal to $1 - \alpha - 2(1-\alpha)/3 = (\alpha+1)/3$, which is minimal for $\alpha = 0$. Hence the nucleolus is $(0,0,1/3,1/3,1/3)$. Players 3,4, and 5 are still most powerful.
(e) The nucleolus is not in the core (e.g., $v(\{1,3,4\} = 1 > 2/3)$, so the core must be empty. This can also be seen directly. Let $\mathbf{x}$ be in the core. Since

$v(\{1,3,4\}) = 1$, we must have $x_5 = 0$. Similarly, $x_3 = x_4 = 0$. Hence, $x_1 = 1$, but similarly $x_2 = 1$. Hence $x(N) = 2 > 1 = v(N)$, which means that $\mathbf{x}$ violates a core constraint.

9.16 *Two Buyers and a Seller*
(a) $v(\{1,3\}) = 1$, $v(\{2,3\}) = 2$, $v(N) = 2$, and $v(S) = 0$ in all other cases.
(b) $C(N,v) = \{(x_1,x_2,x_3) \mid x_1 = 0,\ x_2 + x_3 = 2,\ 1 \le x_3 \le 2\}$.
(c) $\Phi(N,v) = (1/6, 4/6, 7/6)$.
(d) The nucleolus is $(0, 1/2, 3/2)$.

9.17 *Properties of the Shapley Value*
(a) In $\Phi_i(N,v)$ the term $v(S \cup \{i\}) - v(S)$ occurs the same number of times as the term $v(S \cup \{j\}) - v(S)$ in $\Phi_j(N,v)$, for every coalition $S \subseteq N \setminus \{i,j\}$. Let S be a coalition with $i \in S$ and $j \notin S$. Then $v(S \setminus \{i\} \cup \{j\}) = v(S \setminus \{i\} \cup \{i\})$, so that

$$\begin{aligned} v(S \cup \{j\}) - v(S) &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{i\}) \\ &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{j\}) \,, \end{aligned}$$

and also these expressions occur the same umber of times. Similarly for coalitions S that contain j but not i.
(b) This is obvious from Def. 9.4.
(c) Observe that it is sufficient to show $\sum_{S: i \notin S} \frac{|S|!(n-|S|-1)!}{n!} = 1$. To show this, note that $\frac{|S|!(n-|S|-1)!}{n!} = \frac{1}{n} \binom{n-1}{|S|}^{-1}$, so that

$$\begin{aligned} \sum_{S: i \notin S} \frac{|S|!(n-|S|-1)!}{n!} &= \frac{1}{n} \sum_{s=0,1,\ldots,n-1} \binom{n-1}{s} \binom{n-1}{s}^{-1} \\ &= \frac{1}{n} \cdot n = 1 \,. \end{aligned}$$

Problems of Chapter 10

10.1 *A Division Problem (1)*
(b) In terms of utilities: $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, in terms of distribution: $(\frac{1}{3}\sqrt{3}, 1 - \frac{1}{3}\sqrt{3})$.
(c) The Rubinstein outcome is x^* where $x_1^* = \sqrt{\frac{1}{1+\delta+\delta^2}}$ and $x_2^* = 1 - \frac{1}{1+\delta+\delta^2}$.
(d) $\lim_{\delta \to 1} x_1^* = \frac{1}{3}\sqrt{3}$, consistent with what was found under (a).

10.2 *A Division Problem (2)*
By symmetry and Pareto optimality the Nash bargaining solution would assign equal distribution of the good if the utility function of player 2 were $u(\cdot)$. By covariance, the distribution does not change if the utility function of player 2 is $v(\cdot) = 2u(\cdot)$. The resulting utilities are $(u(\frac{1}{2}), 2u(\frac{1}{2}))$.

10.3 *A Division Problem (3)*
(a) The distribution of the good is $\left(2\frac{1-\delta^3}{1-\delta^4}, 2-2\frac{1-\delta^3}{1-\delta^4}\right)$. In utility terms this is $\left(\frac{1-\delta^3}{1-\delta^4}, \sqrt[3]{2-2\frac{1-\delta^3}{1-\delta^4}}\right)$.
(b) By taking the limit for $\delta \to 1$ in (b), we obtain $(1.5, 0.5)$ as the distribution assigned by the Nash bargaining solution. In utilities: $(0.75, \sqrt[3]{0.5})$.
(c) Same as in (b): by independence of irrelevant alternatives (or by the definition of the Nash bargaining solution) nothing changes.

10.4 *An Exchange Economy*
(a) $x_1^A(p_1, p_2) = (3p_2 + 2p_1)/2p_1$, $x_2^A = (4p_1 - p_2)/2p_2$, $x_1^B = (p_1 + 6p_2)/2p_1$, $x_2^B = p_1/2p_2$.
(b) $(p_1, p_2) = (9, 5)$ (or any positive multiple thereof); the equilibrium allocation is $((33/18, 31/10), (39/18, 9/10))$.
(c) The (non-boundary part of the) contract curve is given by the equation $x_2^A = (17x_1^A+5)/(2x_1^A+8)$. The core is the part of this contract curve such that $\ln(x_1^A + 1) + \ln(x_2^A + 2) \geq \ln 4 + \ln 3 = \ln 12$ (individual rationality constraint for A) and $3\ln(5-x_1^A)+\ln(5-x_2^A) \geq 3\ln 2+\ln 4 = \ln 12$ (individual rationality constraint for B).
(d) The point $\mathbf{x}^A = (33/18, 31/10)$ satisfies the equation $x_2^A = (17x_1^A + 5)/(2x_1^A + 8)$.
(e) For the disagreement point $\mathbf{d}$ one can take the point $(\ln 12, \ln 12)$. The set S contains all points $\mathbf{u} \in \mathbb{R}^2$ that can be obtained as utilities from any distribution of the goods that does not exceed total endowments $\mathbf{e} = (4, 4)$. Unlike the Walrasian equilibrium allocation, the allocation obtained by applying the Nash bargaining solution is not independent of arbitrary monotonic transformations of the utility functions. It is a 'cardinal' concept, in contrast to the Walrasian allocation, which is 'ordinal'.

10.5 *The Matching Problem of Table 10.1 Continued*
(a) The resulting matching is (w_1, m_1), (w_2, m_2), w_3 and m_3 remain single.
(b) If, in a stable matching, we have (m_1, w_1), then clearly also (m_2, w_2), m_3 stays single, and hence also w_3 stays single. This is the stable matching resulting from the deferred acceptance algorithm, both with the men and the women proposing.

If (m_1, w_2) in a stable matching, then also (m_2, w_1), but this would be blocked by m_3 and w_1, a contradiction. Obviously, (m_1, w_3) cannot occur in a stable matching. So there are not other stable matchings than the one in (a).

10.6 *Another Matching Problem*
(a) With the men proposing: (m_1, w_1), (m_2, w_2), (m_3, w_3). With the women proposing: (m_1, w_1), (m_2, w_3), (m_3, w_2).
(b) Since in any stable matching we must have (m_1, w_1), the matchings found in (a) are the only stable ones.
(c) Obvious: every man weakly or strongly prefers the men proposing matching in (a); and *vice versa* for the women.

10.7 *Yet Another Matching Problem: Strategic Behavior*
(a) With the men proposing: (m_1, w_2), (m_2, w_3), (m_3, w_1). With the women proposing: (m_1, w_1), (m_2, w_3), (m_3, w_2).
(b) There are no other stable matchings: each of the other four matchings is blocked by $\{m_3, w_2\}$ or by $\{m_1, w_1\}$.
(c) The resulting matching is (m_1, w_1), (m_2, w_3), (m_3, w_2). This is clearly better for w_1. It is a Nash equilibrium: w_1 and w_2 get their top men, and w_3 cannot change the outcome of the algorithm by herself.

10.8 *Core Property of Top Trading Cycle Procedure*
All players in a top trading cycle get their top houses, and thus none of these players can be a member of a blocking coalition, say S. Omitting these players and their houses from the problem, by the same argument none of the players in a top trading cycle in the second round can be a member of S: the only house that such a player may prefer is no longer available in S; etc.

10.9 *House Exchange with Identical Preferences*
Without loss of generality, assume that each player has the same preference $h_1 h_2 \dots h_n$. In a core allocation, obviously, player 1 gets h_1. Hence, player 2 gets h_2; hence, player 3 gets h_3, etc. So there is a unique core allocation: each player keeps his own house (this is independent of the preference).

10.10 *A House Exchange Problem*
There are three core allocations namely: (i) $1 : h_3$, $2 : h_4$, $3 : h_1$, $4 : h_2$; (ii) $1 : h_2$, $2 : h_4$, $3 : h_1$, $4 : h_3$; (iii) $1 : h_3$, $2 : h_1$, $3 : h_4$, $4 : h_2$. Allocation (i) is in the strong core.

10.11 *Cooperative Oligopoly*
(a)–(c) Analogous to Problems 6.1, 6.2. Parts (d) and (f) follow directly from (c). For parts (e) and (g) use the methods of Chap. 9.

Problems of Chapter 11

11.1 *Preferences*
(a) If aPb then aRB and not bRa, hence bPA does not hold.

If aPb and bPc then aRb, bRc, so aRc; cRa would imply cRb and hence not bPc, a contradiction; hence aPc.

Clearly, aPa is not possible since aRa.

Finally, if $a \neq b$ and aRb and bRa then neither aPb nor bPa, so P is not necessarily complete.

(b) Since aRa we have aIa.

If aIb and bIc then aRb and bRc hence aRc; and cRb and bRa hence cRa; hence aIc.

I is not complete unless aRb for all $a, b \in A$.

I is only antisymmetric if R is a linear order.

11.2 *Pairwise Comparison*
(a) $C(r)$ is reflexive and complete. It is not antisymmetric: if $|N(a,b,r)| = |N(b,a,r)|$ for distinct a, b, then $aC(r)b$ and $bC(r)a$.
(b) For the given profile, $aC(r)bC(r)cC(r)a$.
(c) There is no Condorcet winner in this example.

11.3 *Independence of the Conditions in Theorem 11.1*
The social welfare function based on the Borda scores is Pareto efficient but does not satisfy IIA and is not dictatorial (cf. Sect. 11.1). The social welfare function that assigns to each profile of preferences the reverse preference of agent 1 satisfies IIA and is not dicatorial but also not Pareto efficient.

11.4 *Independence of the Conditions in Theorem 11.2*
A constant social welfare function (i.e., always assigning the same fixed alternative) is strategy-proof and nondictatorial but not surjective. The social welfare function that always assigns the bottom element of agent 1 is surjective, nondictatorial, and not strategy-proof.

11.5 *Independence of the Conditions in Theorem 11.3*
A constant social welfare function (i.e., always assigning the same fixed alternative) is monotonic and nondictatorial but not unanimous. A social welfare function that assigns the common top alternative to any profile where all agents have the same top alternative, and a fixed constant alternative to any other profile, is unanimous and nondictatorial but not monotonic.

11.6 *Copeland Score and Kramer Score*
(a) The Copeland ranking is based on scores and therefore reflexive, complete, and transitive. Thus, it is a preference. The Copeland ranking is not antisymmetric, consider e.g. the profile in Problem 11.2(b). It is easy to see that the Copeland ranking is Pareto efficient. By Arrow's Theorem therefore, it does not satisfy IIA.
(b) The Kramer ranking is based on scores and therefore reflexive, complete, and transitive. Thus, it is a preference. The Kramer ranking is not antisymmetric, consider e.g. the profile in Problem 11.2(b). The Kramer ranking is not Pareto efficient: if all preferences are of the form $\ldots abc \ldots$ then b and c have equal Kramer scores 0, although every agent prefers b over c. If, in such a profile, we move bc to the top for agent 1 – so his preference becomes of the form $bc\ldots$ – then b gets a Kramer score of 1 whereas c still has Kramer score 0: hence, IIA is violated.

11.7 *Two Alternatives*
Consider the social welfare function based on majority rule, i.e., it assigns aPb if $|N(a,b,r)| > |N(b,a,r)|$; bPa if $|N(a,b,r)| < |N(b,a,r)|$; and aIb if $|N(a,b,r)| = |N(b,a,r)|$. This clearly satisfies IIA, PE, and nondictatoriality. The social choice function that assigns the top alternative of this ranking and a if aIb, satisfies surjectivity unanimity, monotonicity, strategy-proofness, nondictatoriality.

Review Problems of Part I

RP 1 *Matrix Games (1)*
(a) All rows are (pure) maximin strategies (with minimum 0) and all columns are pure minimax strategies (with maximum 2). The value of the game is between 0 and 2 (which is obvious anyway in this case).
(b) The third column is strictly dominated by the second column and the third row is strictly dominated by the second row. Entry $(1,2)$ is a saddlepoint, hence the value of the game is 2. The unique maximin strategy is $(1,0,0)$, and the minimax strategies are the strategies in the set $\{(q, 1-q, 0) \mid 0 \leq q \leq 1/2\}$.
(c) The second and third rows are the maximin rows. The second column is the unique minimax column. From this we can conclude that the value of the game is between 1 and 2. The first and fourth columns are strictly dominated by the second. Next, the first row is strictly dominated by the last row. The unique maximin strategy is $(0, 2/3, 1/3)$ and the unique minimax strategy is $(0, 2/3, 1/3, 0)$. The value of the game is $5/3$.

RP 2 *Matrix Games (2)*
(a) The first row is the unique maximin row (with minimum 2) and both columns are minimax columns (with maximum 5). So the value is between 2 and 5. The game has no saddlepoint.
(b) $v(A_1) = 5/2$, $v(A_2) = 20/7$, $v(A_3) = 2$ (saddlepoint), $v(A_4) = 1$ (saddlepoint), $v(A_5) = 7/3$, $v(A_6) = 25/9$. Since player 1 can pick rows, the value must be the maximum of these amounts, hence $20/7$, the value of A_2.
(c) The unique maximin strategy is $(5/7, 0, 2/7, 0)$ and the unique minimax strategy is $(3/7, 4/7)$.

RP 3 *Matrix Games (3)*
(a) The unique maximin row is the first row, with minimum 8. The unique minimax column is the first column, with maximum 12. So the value of the game is between 8 and 12. The game has no saddlepoint.
(b) The second row is strictly dominated by for instance putting probability $1/2$ on the first row and $1/2$ on the third row. After eliminating the second row, the third column is strictly dominated by the first column.
(c) The unique maximin strategy is $(1/2, 0, 1/2)$ and the unique minimax strategy is $(3/4, 1/4, 0)$. The value of the game is 10.

RP 4 *Bimatrix Games (1)*
(a) D is strictly dominated by $3/5 \cdot U + 2/5 \cdot M$. Next, C is strictly dominated by R.
(b) In the reduced (two by two) game, the best reply function of player 1 is: play U if player 2 puts less than probability $2/5$ on L, play M if player 2 puts more than probability $2/5$ on L, and play any combination of U and M if player 2 puts probability $2/5$ on L. The best reply function of player 2 is: play R if player 1 puts positive probability on U, and play any combination of L and R if player 1 plays M. The set of Nash equilibria is: $\{((1,0),(0,1))\} \cup \{((0,1),(q,1-q)) \mid 1 \geq q \geq 2/5\}$.

(c) The set of Nash equilibria in the original game is: $\{((1,0,0),(0,0,1))\} \cup \{((0,1,0),(q,0,1-q)) \mid 1 \geq q \geq 2/5\}$.

RP 5 *Bimatrix Games (2)*
(a) For $x > 2$: $\{((1,0),(1,0))\}$. For $x = 2$: $\{((1,0),(1,0))\} \cup \{((p,1-p),(0,1)) \mid 0 \leq p \leq 1/2\}$. For $0 < x < 2$: $\{((1/2,1/2),((2-x)/2,x/2))\}$. For $x = 0$: $\{((0,1),(0,1))\} \cup \{((p,1-p),(1,0)) \mid 1 \geq p \geq 1/2\}$. For $x < 0$: $\{((0,1),(0,1))\}$.
(b) f is strictly dominated by $1/3 \cdot e + 2/3 \cdot g$. Next: b is strictly dominated by c, e by g, a by d. The remaining two by two game has a unique Nash equilibrium. In the original game the unique Nash equilibrium is $((0,0,4/9,5/9),(0,0,1/2,1/2))$.

RP 6 *Voting*
(a)

$$\begin{array}{c c} & \begin{array}{ccc} (4,0) & (3,1) & (2,2) \end{array} \\ \begin{array}{c} (4,0) \\ (3,1) \\ (2,2) \end{array} & \begin{pmatrix} 3/2,3/2 & 1,2 & 1,2 \\ 2,1 & 3/2,3/2 & 1,2 \\ 2,1 & 2,1 & 3/2,3/2 \end{pmatrix} \end{array}$$

(b) By iterated elimination of strictly dominated strategies it follows that the unique Nash equilibrium in this game is $((2,2),(2,2))$. (This is a constant sum game: $(2,2)$ is the optimal strategy for each party.)

RP 7 *A Bimatrix Game*
(a) For $a \neq 0$ the unique Nash equilibrium is $((1/2,1/2),(1/2,1/2))$. For $a = 0$ the set of Nash equilibria is $\{((p,1-p),(0,1)) \mid 1 \geq p > 1/2\} \cup \{((p,1-p),(1,0)) \mid 0 \leq p < 1/2\} \cup \{((1/2,1/2),(q,1-q)) \mid 0 \leq q \leq 1\}$.
(b) The strategic form of this game is

$$\begin{array}{c c} & \begin{array}{cccc} LL & LR & RL & RR \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} a,0 & a,0 & 0,1 & 0,1 \\ 0,1 & a,0 & 0,1 & a,0 \end{pmatrix} \end{array}$$

There are two subgame perfect equilibria in pure strategies: player 1 plays T and player 2 plays RL (i.e., R after T and L after B); and player 1 plays B and player 2 plays RL.

RP 8 *An Ice-cream Vendor Game*
(a) There are four different situations: (i) all vendors in the same location: each gets 400; (ii) two in the same location and the third vendor in a neighboring location: the first two get 300 and the third gets 600; (iii) two in the same location and the third vendor in the opposite location: the first two get 300 and the third gets 600; and (iv) all vendors in different locations: the middle one gets 300 and the others get 450 each. From this it is easily seen that (iii) and (iv) are Nash equilibria but (i) and (ii) are not Nash equilibria.
(b) There are many subgame perfect Nash equilibria, but they can be reduced to three types: (i) player 1 chooses arbitrarily, player 2 chooses the opposite location of player 1, and player 3 chooses a remaining optimal open location;

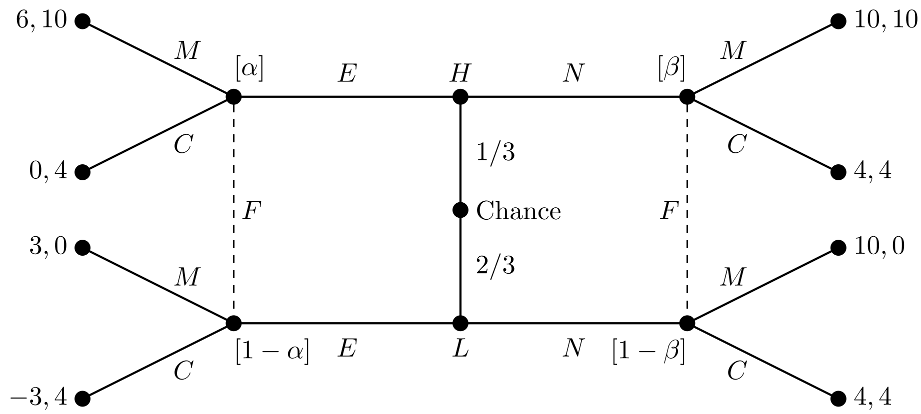
(ii) player 1 chooses arbitrarily, player 2 chooses one of the neighboring locations of player 1, and player 3 chooses the opposite location of player 2 if that is unoccupied, and otherwise the same location as player 2; (iii) player 1 chooses arbitrarily, player 2 chooses the same location as player 1, and player 3 chooses the opposite location of player 1.

RP 9 *A Repeated Game*
(a) (U, L, B) and (D, R, B).
(b) In the second period, after each action combination of the first period, one of the two equilibria in (a) has to be played.
(c) In the first period player 1 plays U, player 2 plays R, and player 3 plays A. In the second period, if the first period resulted in (U, R, A) then player 1 plays D, player 2 plays R, and player 3 plays B; in all other cases, player 1 plays U, player 2 plays L, and player 3 plays B.
(d) In the first period player 1 plays U, player 2 plays R, and player 3 plays B. In the second period, if the first period resulted in (U, R, B) then player 1 plays U, player 2 plays L, and player 3 plays B; in all other cases, player 1 plays D, player 2 plays R, and player 3 plays B.

RP 10 *Locating a Pub*
(a) Player 1 has 3 pure strategies and player 2 has 8 pure strategies.
(b) Player 1 chooses B. Player 2 chooses B, C, B, if player 1 chooses A, B, C respectively.
(c) Player 1 has 24 pure strategies and player 2 has 8 pure strategies.
(d) (i) Player 1 plays A; after A the subgame equilibrium (B, C) is played, after B the subgame equilibrium (A, C), and after C the subgame equilibrium (A, B). (ii) Player 1 plays B; after A the subgame equilibrium (B, C) is played, after B the subgame equilibrium (C, A), and after C the subgame equilibrium (A, B). (iii) Player 1 plays C; after A the subgame equilibrium (B, C) is played, after B the subgame equilibrium (C, A), and after C the subgame equilibrium (B, A).

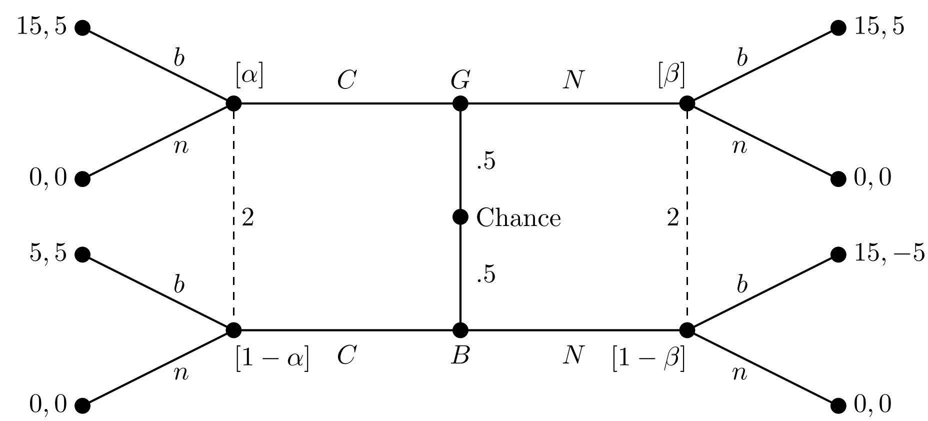
RP 11 *A Two-stage Game*
(a) In G_1: (D, R); in G_2: (T, X), (M, Y), and (B, Z).
(b) Each player has $2 \cdot 3^4 = 162$ pure strategies.
(c) In G_1 player 1 plays U and player 2 plays L. In G_2 the players play as follows. If (U, L) was played, then player 1 plays M and player 2 plays Y. If (D, L) was played, then player 1 plays B and player 2 plays Z. If (U, R) was played, then player 1 plays T and player 2 plays X. If (D, R) was played, then player 1 plays M and player 2 plays Y.
(d) In the second stage (in G_1) always (U, L) has to be played. Hence, there are three subgame perfect equilibria, corresponding to the three Nash equilibria of G_2.

RP 12 *Job Market Signaling*
(a)



(b) The Nash equilibria are: (i) type H plays E, type L plays N, F plays M after E and C after N; (ii) both types play N, F always plays C.
(c) The equilibrium in (i) is separating with (forced) beliefs $\alpha = 1$ and $\beta = 0$. The equilibrium in (ii) is pooling with $\beta = 1/3$ (forced) and $\alpha \leq 2/5$. According to the intuitive criterion we must have $\alpha = 1$, so that the intuitive criterion is not satisfied by the latter equilibrium. (It does not apply to the first equilibrium.)

RP 13 *Second-hand Cars (1)*
(a,b) The extensive form of this signaling game is as follows:



The strategic form is:

$$
\begin{array}{c}
\\ CC \\ CN \\ NC \\ NN
\end{array}
\begin{array}{c}
\begin{array}{cccc} bb & bn & nb & nn \end{array} \\
\begin{pmatrix}
10,\underline{5} & \underline{10},\underline{5} & 0,0 & \underline{0},0 \\
\underline{15},0 & 7.5,\underline{2.5} & 7.5,-2.5 & \underline{0},0 \\
10,\underline{5} & 2.5,2.5 & 7.5,2.5 & \underline{0},0 \\
\underline{15},\underline{0} & 0,\underline{0} & \underline{15},\underline{0} & \underline{0},\underline{0}
\end{pmatrix}
\end{array}
$$

The Nash equilibria are: (CC, bn), (NN, bb), (NN, nb), (NN, nn).
(c) (CC, bn) is pooling with $\beta \leq 1/2$, (NN, bb) is pooling for all α. The other two equilibria are not perfect Bayesian, since player 2 will play b after C.

RP 14 *Second-hand Cars (2)*
(a) This is a static game of incomplete information, represented by the pair G_1, G_2:

$$
G_1 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 3 & 5 \end{array} \\
\begin{pmatrix}
1,-1 & 0,0 & 0,0 \\
0,0 & -1,1 & 0,0 \\
-1,1 & -2,2 & -3,3
\end{pmatrix}
\end{array}
\qquad
G_2 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 3 & 5 \end{array} \\
\begin{pmatrix}
3,-3 & 0,0 & 0,0 \\
2,-2 & 1,-1 & 0,0 \\
1,-1 & 0,0 & -1,1
\end{pmatrix}
\end{array}
$$

where G_1 is played with probability 25% and G_2 with probability 75%. (The numbers should be multiplied by 1000, the buyer is the row and the seller the column player.)
(b) The buyer has one type and three pure strategies, the seller has two types and nine pure strategies.
(c) Strategy "5" is strictly dominated by strategy "3".
(d) Against strategy "3" of the buyer the best reply of the seller is the combination $(3, 5)$, but against this combination the best reply of the buyer is "1".
(e) Against strategy "1" of the buyer the seller has four best replies: $(3, 3)$, $(3, 5)$, $(5, 3)$, and $(5, 5)$. In turn, (only) against $(3, 5)$ and $(5, 5)$ is "1" a best reply. Hence there are two Nash equilibra in pure stategies: (i) $(1, (3, 5))$ and (ii) $(1, (5, 5))$. No trade is going to take place.

RP 15 *Signaling Games*
(a) The strategic form with best replies underlined is:

$$
\begin{array}{c}
\\ LL \\ LR \\ RL \\ RR
\end{array}
\begin{array}{c}
\begin{array}{cccc} uu & ud & du & dd \end{array} \\
\begin{pmatrix}
2,\underline{1} & \underline{2},\underline{1} & 1.5,0.5 & \underline{1.5},0.5 \\
\underline{2.5},\underline{1.5} & 1.5,1 & \underline{2},0.5 & 1,0 \\
1,0 & 0.5,0.5 & 1,0.5 & 0.5,\underline{1} \\
1.5,\underline{0.5} & 0,\underline{0.5} & 1.5,\underline{0.5} & 0,\underline{0.5}
\end{pmatrix}
\end{array}
$$

(LR, uu) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 1$ and $\beta = 0$. (LL, ud) is a pooling Bayesian equilibrium with beliefs $\alpha = 1/2$ and $\beta \geq 1/2$. For the latter, the intuitive criterion requires $\beta = 0$, so that this equilibrium does not satisfy it.
(b) The strategic form with best replies underlined is:

$$\begin{array}{c} \\ LL \\ LR \\ RL \\ RR \end{array} \begin{array}{c} \begin{array}{cccc} uu & ud & du & dd \end{array} \\ \begin{pmatrix} \underline{3}, \underline{1.5} & 3, \underline{1.5} & 0.5, 1 & 0.5, 1 \\ 2, 1 & 2.5, 0 & \underline{1}, \underline{1.5} & 1.5, 0.5 \\ 1.5, 1.5 & \underline{3.5}, \underline{2} & 0, 0.5 & 2, 1 \\ 0.5, \underline{1} & 3, 0.5 & 0.5, \underline{1} & \underline{3}, 0.5 \end{pmatrix} \end{array}$$

(LL, uu) is a pooling perfect Bayesian equilibrium with beliefs $\alpha = 1/2$ and $\beta \leq 2/3$. The intuitive criterion requires $\beta = 1$, so this pooling equilibrium does not satisfy it. (LR, du) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 1$ and $\beta = 0$, and (RL, ud) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 0$ and $\beta = 1$.

RP 16 *A Game of Incomplete Information*
(a) Start with the decision node of player 1. Player 1 has four actions/strategies: AA, AB, BA, BB. All these actions lead to one and the same information set of player 2, who has three actions/strategies: C, D, E.
(b) The strategic form is:

$$\begin{array}{c} \\ AA \\ AB \\ BA \\ BB \end{array} \begin{array}{c} \begin{array}{ccc} C & D & E \end{array} \\ \begin{pmatrix} 3, \underline{2} & 1.5, 1.5 & \underline{2.5}, 1.5 \\ \underline{4}, 2.5 & \underline{2.5}, \underline{3} & 1, 1.5 \\ 3, \underline{2} & 1.5, 1.5 & \underline{2.5}, 1.5 \\ \underline{4}, 2.5 & \underline{2.5}, \underline{3} & 1, 1.5 \end{pmatrix} \end{array}$$

The Nash equilibria in pure strategies are (AB, D) and (BB, D).
(c) Player 1 has now two pure strategies, namely A and B. If player 1 plays A then the best reply of player 2 is EC. Against EC, the payoff of A is 1.5 and the payoff of B is 2.5, so that A is not a best reply against EC. Against B, the best reply of player 2 is ED. In turn, B is player 1's best reply against ED (yields 2 whereas A only yields 1). So the unique Nash equilibrium in pure strategies is (B, ED).

RP 17 *A Bayesian Game*
(a) This is the game

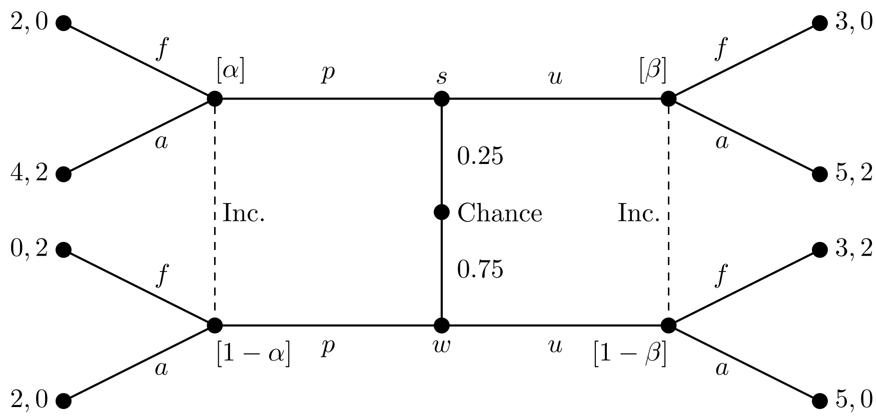
$$\begin{array}{c} \\ F \\ Y \end{array} \begin{array}{c} \begin{array}{cc} F & Y \end{array} \\ \begin{pmatrix} -1, 1 & 1, 0 \\ 0, 1 & 0, 0 \end{pmatrix} \end{array}$$

with (Y, F) as unique Nash equilibrium (also in mixed strategies).
(b) Start with the decision node for player 1, who has two actions/strategies: F and Y. Player 2 has a singe information set and four actions/strategies: FF, FY, YF, YY.
(c) The strategic form is:

$$\begin{array}{c} \\ F \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} FF & FY & YF & YY \end{array} \\ \begin{pmatrix} 1-2\alpha, 2\alpha-1 & 1-2\alpha, \alpha & 1, \alpha-1 & 1, 0 \\ 0, 1 & 0, \alpha & 0, 1-\alpha & 0, 0 \end{pmatrix} \end{array}$$

For $\alpha = 0$ the Nash equilibria in pure strategies are (F, FY) and (F, YY). For $0 < \alpha < 1/2$: (F, FY). For $\alpha = 1/2$: (F, FY) and (Y, FF). For $1/2 < \alpha < 1$: (Y, FF). For $\alpha = 1$: (Y, FF) and (Y, FY).

RP 18 *Entry as a Signaling Game*
(a) The extensive form of this signaling game is:



(b,c) The strategy combination (pu, af) (strong type p, incumbent a after p) is a Nash equilibrium. It is a separating perfect Bayesian equilibrium for $\alpha = 1$ and $\beta = 0$. Also (uu, ff) is a Nash equilibrium. It is pooling perfect Bayesian for $\beta = 1/2$ and $\alpha \leq 1/2$. It does not satisfy the intuitive criterion since that requires $\alpha = 1$.

RP 19 *Bargaining (1)*
(a) Player 1 has only one type. Player 2 has infinitely many types, namely each $v \in [0, 1]$ is a possible type of player 2. A typical strategy of player 1 consists of a price $p_1 \in [0, 1]$ and a yes/no decision depending on the price p_2 of player 2 if that player rejects p_1 – in principle, the yes/no decision may also depend on p_1.
(b) A typical strategy of player 2 consists, for every type $v \in [0, 1]$, of a yes/no decision depending on the price p_1 asked by player 1 and a price p_2 in case the decision was 'no'. In principle, p_2 may also depend on p_1 (not only via the yes/no decision).
(c) Player 2 accepts if $v - p_1 \geq \delta v$ (noting that he can offer $p_2 = 0$ if he does not accept the price p_1 of player 1); rejects and offers $p_2 = 0$ if $v - p_1 < \delta v$.
(d) Using (c) player 1 asks the price p_1 that maximizes $p_1 \cdot Pr[p_1 \leq (1 - \delta)v]$, i.e., his expected payoff – note that his payoff is 0 if player 2 rejects. Hence, player 1 solves $\max_{p_1 \in [0,1]} p_1 \cdot [1 - p_1/(1 - \delta)]$, which has solution $p_1 = (1 - \delta)/2$. So the equilibrium is, that player 1 asks this price and accepts any price of player 2; and player 2 accepts any price at most $(1 - \delta)/2$, and rejects any higher price and then offers $p_2 = 0$.

RP 20 *Bargaining (2)*
(a) The (Pareto) boundary of the feasible set consists of all pairs $(x, 1-x^2)$ for $x \in [0,1]$.
(b) The Nash bargaining solution outcome is found by maximizing the expression $x(1-x^2)$ over all $x \in [0,1]$. The solution is $((1/3)\sqrt{3}, 2/3)$. In distribution of the good: $((1/3)\sqrt{3}, 1-(1/3)\sqrt{3})$.
(c),(d) Let $(x, 1-x^2)$ be the proposal of player 1 and $(y, 1-y^2)$ that of player 2. Then the equations $1-x^2 = \delta(1-y^2)$ and $y = \delta x$ hold for the Rubinstein outcome. This results in $x = 1/\sqrt{1+\delta+\delta^2}$; taking the limit for $\delta \to 1$ gives $(1/3)\sqrt{3}$, which is indeed the Nash bargaining solution outcome for player 1.

RP 21 *Bargaining (3)*
(a) Player 1 proposes $(1-\delta+(1/2)\delta^2, \delta-(1/2)\delta^2)$ at $t=0$ and player 2 accepts. Note that $1-\delta+(1/2)\delta^2 > \delta-(1/2)\delta^2$, so the beginning player has an advantage.
(b) If the utility function of player 2 were the same as that of player 1, then the Nash bargaining solution would result in equal split. This is still the case if player 2's utility function is multiplied by 2, as is the case here: the maximum of $u(x) \cdot 2u(1-x)$ is attained at the same point as the maximum of $u(x) \cdot u(1-x)$. So the division of the good is $(1/2, 1/2)$. In terms of utilities, this gives $(u(1/2), 2u(1/2))$. (The Nash bargaining solution is symmetric, Pareto optimal, and scale covariant: see Chap. 10.)

RP 22 *Ultimatum Bargaining*
(a) Player 1 chooses an action/strategy $(1-m, m)$. Player 2 decides for each strategy of player 1 whether to accept or reject the offer. If he accepts, the payoffs are $(1-m, m+a(2m-1))$, otherwise the payoffs are $(0,0)$.
(b) Player 1 proposes $(1-a/(1+2a), a/(1+2a))$, and player 2 accepts $(1-m, m)$ if and only if $m \geq a/(1+2a)$. Hence, the outcome is $(1-a/(1+2a), a/(1+2a))$.
(c) If a becomes large, then this outcome converges to equal split: this is because then player 2 cares mainly about the division and not so much about what he gets.

RP 23 *An Auction (1)*
(a) The game has imperfect but complete information.
(b) The unique Nash equilibrium is each bidder bidding $v_1 = v_2$.
(c) There is no Nash equilibrium.
(d) The associated bimatrix game is:

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\begin{pmatrix}
1/2, 3/2 & 0, 2 & 0, 1 & 0, 0 \\
0, 0 & 0, 1 & 0, 1 & 0, 0 \\
-1, 0 & -1, 0 & -1/2, 1/2 & 0, 0 \\
-2, 0 & -2, 0 & -2, 0 & -1, 0
\end{pmatrix}
\end{array}
$$

The Nash equilibria are $(0,1)$, $(1,1)$, and $(1,2)$.

RP 24 *An Auction (2)*
(a) Let $b_i < v_i$. If b_i wins then v_i is equally good. If b_i loses and the winning bid is below v_i then v_i is a strict improvement. If b_i loses and the winning bid is at least v_i then v_i is at least as good. If, on the other hand, $b_i > v_i$, then, if b_i wins, the fourth-highest bid is below v_i and the second highest bid is above v_i, then bidding v_i results in zero instead of positive payoff.
(b) For instance, player 2 can improve by any bid above v_1.
(c) All bidders bid $\widetilde{v}$ where $\widetilde{v} \in [v_2, v_1]$.

RP 25 *An Auction (3)*
(a) The best reply function b_2 of player 2 is given by: $b_2(0) = \{1\}$, $b_2(1) = \{2\}$, $b_2(2) = \{3\}$, $b_2(3) = b_2(4) = \{0, \ldots, 4\}$, $b_2(5) = \{0, \ldots, 5\}$, $b_2(6) = \{0, \ldots, 6\}$. The best reply function b_1 of player 1 is given by: $b_1(0) = \{0\}$, $b_1(1) = \{1\}$, $b_1(2) = \{2\}$, $b_1(3) = \{3\}$, $b_1(4) = \{4\}$, $b_1(5) = \{5\}$, $b_1(6) = \{0, \ldots, 6\}$.
(b) The Nash equilibria are: $(3, 3)$, $(4, 4)$, $(5, 5)$, and $(6, 6)$.

RP 26 *Quantity Versus Price Competition*
(a) The profit functions are $q_1(4 - 2q_1 - q_2)$ and $q_2(4 - q_1 - 2q_2)$ respectively (or zero in case an expression is negative). The first-order conditions (best reply functions) are $q_1 = (4 - q_2)/4$ and $q_2 = (4 - q_1)/4$ (or zero) and the equilibrium is $q_1 = q_2 = 4/5$ with associated prices equal to $8/5$ and profits equal to $32/25$.
(b) Follows by substitution.
(c),(d) The profit functions are $(1/3)p_1(p_2 - 2p_1 + 4)$ and $(1/3)p_2(p_1 - 2p_2 + 4)$ (or zero) respectively. The first-order conditions (best reply functions) are $p_1 = (p_2 + 4)/4$ and $p_2 = (p_1 + 4)/4$. The equilibrium is $p_1 = p_2 = 4/3$ with associated quantities $q_1 = q_2 = 8/9$ and profits equal to $32/27$. Price competition is tougher.

RP 27 *An Oligopoly Game (1)*
(a),(b) Player 1 chooses $q_1 \geq 0$. Players 2 and 3 then choose q_2 and q_3 simultaneously, depending on q_1. The best reply functions of players 2 and 3 in the subgame following q_1 are $q_2 = (a - q_1 - q_3 - c)/2$ and $(a - q_1 - q_2 - c)/2$ (or zero), and the equilibrium in the subgame is $q_2 = q_3 = (a - q_1 - c)/3$. Player 1 takes this into account and maximizes $q_1(a - c - q_1 - 2(a - q_1 - c)/3)$, which gives $q_1 = (a - c)/2$. Hence, the subgame perfect equilibrium is: player 1 plays $q_1 = (a - c)/2$; players 2 and 3 play $q_2 = q_3 = (a - q_1 - c)/3$. The outcome is player 1 playing $(a - c)/6$ and players 2 and 3 playing $(a - c)/6$.

RP 28 *An Oligopoly Game (2)*
(a) The best-reply functions are $q_1 = (10 - q_2 - q_3)/2$, $q_2 = (10 - q_1 - q_3)/2$, $q_3 = (9 - q_1 - q_2)/2$.
(b) The equilibrium is $q_1 = q_2 = 11/4$, $q_3 = 7/4$.
(c) To maximize joint profit, $q_3 = 0$ and $q_1 + q_2 = 5$. (This follows by using intuition: firm 3 has higher cost, or by solving the problem as a maximization problem under nonnegativity constraints.)

RP 29 *A Duopoly Game with Price Competition*

(a) The monopoly price of firm 1 is $p_1 = 65$ and the monopoly price of player 2 is $p_2 = 75$.
(b)

$$p_1(p_2) = \begin{cases} \{x \mid x > p_2\} & \text{if } p_2 < 30 \\ \{x \mid x \geq 30\} & \text{if } p_2 = 30 \\ \{31\} & \text{if } p_2 = 31 \\ \{p_2 - 1\} & \text{if } p_2 \in [32, 65] \\ \{65\} & \text{if } p_2 \geq 66 \end{cases} \quad p_2(p_1) = \begin{cases} \{x \mid x > p_1\} & \text{if } p_1 < 50 \\ \{x \mid x \geq 50\} & \text{if } p_1 = 50 \\ \{51\} & \text{if } p_1 = 51 \\ \{p_1 - 1\} & \text{if } p_1 \in [52, 75] \\ \{75\} & \text{if } p_1 \geq 76 \end{cases}$$

(c) $(p_1, p_2) = (31, 32)$.
(d) $(p_1, p_2) = (50, 51)$.

RP 30 *Contributing to a Public Good*
(a) The Nash equilibria in pure strategies are all strategy combinations where exactly two persons contribute.
(b) The expected payoff of contributing is equal to $-3 + 8(1 - (1-p)^2)$, which in turn is equal to $16p - 8p^2 - 3$.
(c) A player should be indifferent between contributing or not if the other two players contribute, hence $16p - 8p^2 - 3 = 8p^2$. This holds for $p = 1/4$ and for $p = 3/4$.

RP 31 *A Demand Game*
(a) Not possible: each player can gain by raising his demand by 0.1. (b) Not possible: at least one player has $x_i > 0.2$ and can gain by decreasing his demand by 0.2. (c) The unique Nash equilibrium is $(0.5, 0.5, 0.5)$. (d) A Nash equilibrium is for instance $(0.6, 0.6, 0.6)$.
(e) All triples with sum equal to one, and all triples such that the sum of each pair is at least one.

RP 32 *A Repeated Game (1)*
(a) The unique Nash equilibrium in the stage game is $((2/3, 1/3), (1/2, 1/2))$, with payoffs $(8, 22)$. Therefore, all payoffs pairs in the quadrangle with vertices $(16, 24)$, $(0, 25)$, $(0, 18)$, and $(16, 16)$ which are strictly larger than $(8, 22)$, as well as $(8, 22)$, can be reached as long run average payoffs in a subgame perfect equilibrium in the repeated game, for suitable choices of δ.
(b) Write $G = (A, B)$, then $v(A) = 8$ and $-v(-B) = 18$. Therefore, all payoffs pairs in the quadrangle with vertices $(16, 24)$, $(0, 25)$, $(0, 18)$, and $(16, 16)$ which are strictly larger than $(8, 20)$, can be reached as long run average payoffs in a Nash equilibrium in the repeated game, for suitable choices of δ.
(c) The players alternate between (T, L) and (B, R). Player 1 has no incentive to deviate, but uses the eternal punishment strategy B to keep player 2 from deviating. Player 2 will not deviate provided

$$25 + 18\delta/(1-\delta) \leq 24/(1-\delta^2) + 16\delta/(1-\delta^2)$$

and

$$18 + 18\delta/(1-\delta) \leq 16/(1-\delta^2) + 24\delta/(1-\delta^2) \,.$$

The first inequality is satisfied if δ is at least (approximately) 0.55, and the second inequality if $\delta \geq 1/3$. Hence, this is a Nash equilibrium for $\delta \geq 0.55$. It is not subgame perfect since player 2 can obtain 22 by playing the stage game equilibrium strategy.

RP 33 *A Repeated Game (2)*
(a) (D, C), (D, R), and (M, R).
(b) Let $((p_1, p_2, p_3), (q_1, q_2, q_3))$ be a Nash equilibrium. First consider the case $q_3 < 1$. Then $p_1 = 0$ and therefore $q_1 = 0$. If $p_2 > 0$ then $q_2 = 0$ and $q_3 = 1$, a contradiction. Hence, $p_2 = 0$, and then $p_3 = 1$. We obtain the set of Nash equilibria $\{((0,0,1),(0,q_2,q_3)) \mid q_2, q_3 \geq 0, q_2 + q_3 = 1, q_3 < 1\}$.

Next, consider the case $q_3 = 1$. Then $9p_1 + p_2 + 4p_3 \leq p_1 + 2p_2 + 4p_3$, hence $8p_1 \leq p_2$. We obtain another set of Nash equilibria $\{((p_1, p_2, p_3), (0,0,1)) \mid p_1 \geq 0, 8p_1 \leq p_2, p_1 + p_2 = 1\}$.
(c) Each player has $3 \times 3^9 = 3^{10}$ pure strategies. In the first stage the players play (U, L) and in the second stage they play (for instance) according to the table

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} D,R & M,R & D,R \\ D,C & D,R & D,R \\ D,C & D,R & D,R \end{pmatrix} \end{array} .$$

(d) Always play (U, L) but after a deviation by player 1, player 2 reverts to C forever, to which player 1 replies by D, and after a deviation by player 2, player 1 reverts to M forever, to which player 2 replies by R. This is a subgame perfect equilibrium provided that

$$10 + 2\delta/(1-\delta) \leq 8/(1-\delta) \Leftrightarrow \delta \geq 1/4$$

and

$$9 + 2\delta/(1-\delta) \leq 8/(1-\delta) \Leftrightarrow \delta \geq 1/7$$

hence if $\delta \geq 1/4$.

RP 34 *A Repeated Game (3)*
(a) (D, L), (U, R), and (D, R).
(b) The second row and next the second column can be deleted by iterated elimination of strictly dominated strategies. This results in the sets of Nash equilibria $\{((0,0,1),(q_1,0,q_3)) \mid q_1, q_3 \geq 0, q_1 + q_3 = 1\}$ and $\{((p_1,0,p_3),(0,0,1)) \mid p_1, p_3 \geq 0, p_1 + p_3 = 1\}$.
(c) In the first stage the players play (M, C) and in the second stage they play (for instance) according to the table

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} D,R & D,L & D,R \\ U,R & D,R & D,R \\ D,R & D,L & D,R \end{pmatrix} \end{array} .$$

(d) Always play (M, C) but after a deviation by player 1 player 2 reverts to L forever, to which player 1 replies by D, and after a deviation by player 2 player 1 reverts to U, to which player 2 replies by R. This is a subgame perfect equilibrium provided that

$$12 + \delta/(1-\delta) \leq 10/(1-\delta)$$

which holds for $\delta \geq 2/11$.

RP 35 *A Repeated Game (4)*
(a) Player 1 plays B and player 2 plays L in both stages.
(b) They play (T, L) in the first stage. If player 1 would deviate to B, then player 2 plays R in the second stage, otherwise L. Player 1 plays B in the second stage.
(c) Since (B, L) is the unique Nash equilibrium in the stage game and there are no payoff pairs better for both players, the only possibility is that player 1 plays B and player 2 plays L forever. This is a subgame perfect equilibrium for any value of δ, with long run average payoffs $(5, 5)$.

RP 36 *A Repeated Game (5)*
(a) Only (T, L).
(b) The payoff pair $(2, 1)$, and all payoff pairs larger for both players in the triangle with vertices $(5, 0)$, $(0, 6)$, and $(1, 1)$.
(c) At even times play (B, L) and at odd times play (T, R). After a deviation revert to T (player 1) and L (player 2) forever. This is a subgame perfect Nash equilibrium provided that

$$2 + 2\delta/(1-\delta) \leq 5\delta/(1-\delta^2)$$

and

$$1 + \delta/(1-\delta) \leq 6\delta/(1-\delta^2)$$

which is equivalent to $\delta \geq \max\{2/3, 1/5\} = 2/3$.

RP 37 *An Evolutionary Game*
(a) The species consists of $100p\%$ animals of type C and $100(1-p)\%$ animals of type D.
(b) $\dot{p} = p(0p+2(1-p)-2p(1-p)-3(1-p)p-(1-p)^2)$ which after simplification yields $\dot{p} = 4p(p-1)(p-1/4)$. Hence the rest points are $p = 0, 1/4, 1$ and $p = 1/4$ is stable.
(c) The unique symmetric Nash equilibrium strategy is $(1/4, 3/4)$. One has to check that $(1/4, 3/4)A(q, 1-q) > (q, 1-q)A(q, 1-q)$ for all $q \neq 1/4$, which follows readily by computation.

RP 38 *An Apex Game*
(a) Suppose $(x_1, \ldots, x_5)$ is in the core. Since $x_1 + x_2 \geq 1$, and all x_i are nonnegative and sum to one, we must have $x_3 = x_4 = x_5 = 0$. Similarly, $x_2 = 0$, but this contradicts $x_2 + \ldots + x_5 \geq 1$. So the core is empty.

(b) $\Phi_2(v) = 1!3!/5!+3!1!/5! = 1/10$, hence $\Phi(v) = (6/10, 1/10, 1/10, 1/10, 1/10)$.
(c) Let $(1-4a, a, a, a, a)$ be the nucleolus of this game. The relevant (maximal) excesses to consider are $1-(1-4a)-a = 3a$ (e.g., $\{1, 2\}$) and $1-4a$ ($\{2, \ldots, 5\}$). Equating these yields $a = 1/7$.

RP 39 *A Three-person Cooperative Game (1)*
(a) For $a > 10$ the core is empty. For $a = 10$, a core element is for instance $(0, 5, 5)$. Hence, $a \leq 10$.
(b) The Shapley value is $((25-2a)/6, (19+a)/6, (16+a)/6)$. By writing down the core constraints, it follows that this is in the core for $-13 \leq a \leq 8.75$.
(c) At this vector, the excesses of the three two-player coalitions are equal, namely to $(a - 14)/3$. For this tobe the nucleolus we need that the excesses of the one-person coalitions are not larger than this, i.e.,

$$(2a - 16)/3 \leq (a - 14)/3, (-a - 4)/3 \leq (a - 14)/3, (-a - 7)/3 \leq (a - 14)/3$$

and it is straightforward to check that this is true for no value of a.

RP 40 *A Three-person Cooperative Game (2)*
(a) The core is nonempty for $a \leq 1$. In that case, the core is the quadrangle (or line segment if $a = 1$) with vertices $(1, 2, 2)$, $(a, 2, 3 - a)$, $(1, 1, 3)$, and $(a, 2 - a, 3)$.
(b) The Shapley value is $((2a + 7)/6, (10 - a)/6, (13 - a)/6)$, which is in the core for $-2 \leq a \leq -1/2$.
(c) By equating the excesses of the two-person coalitions we obtain the vector $(2/3, 5/3, 8/3)$ with excess $-1/3$. This is the nucleolus if $a - 2/3 \leq -1/3$, hence if $a \leq 1/3$.

RP 41 *Voting*
(a) The winning coalitions (omitting set braces) are AB, AC, ABC, ABD, ACD, $ABCD$, and BCD. Then $\Phi_A(v) = 1!2!/4! + 1!2!/4! + 2!1!/4! + 2!1!/4! + 2!1!/4! = 5/12$. Similarly, one computes the other values to obtain $\Phi(v) = (1/12)(5, 3, 3, 1)$. (In fact, it is sufficient to compute $\Phi_B(v)$ and $\Phi_C(v)$.)
(b) $p_A = 5$, $p_B = 3$, $p_C = 3$, $p_D = 1$; $\beta(A) = 5/12$, $\beta(B) = 3/12$, $\beta(C) = 3/12$, $\beta(D) = 1/12$.
(c) The winning coalitions are AB, AC, ABC. The Shapley value is $(2/3, 1/6, 1/6)$. Further, $p_A = 3$, $p_B = p_C = 1$; $\beta(A) = 3/5$, $\beta(B) = \beta(C) = 1/5$.

RP 42 *An Airport Game*
(a) $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = c_1$, $v(23) = c_2$, and $v(N) = c_1 + c_2$.
(b) The core is the quadrangle with vertices $(c_1, c_2, 0)$, $(0, c_2, c_1)$, $(0, c_1, c_2)$, and $(c_1, 0, c_2)$.
(c) $\Phi(v) = (1/6)(4c_1, 3c_2 + c_1, 3c_2 + c_1)$. This is a core element (check the constraints).
(d) The nucleolus is of the form $(a, (c_1+c_2-a)/2, (c_1+c_2-a)/2)$. By equating the excesses of the two-person coalitions it follows that $a = (3c_1-c_2)/3$, hence

the nucleolus would be $((3c_1 - c_2)/3, 2c_2/3, 2c_2/3)$ and the excess of the two-person coalitions is then $-c_2/3$. We need that the excesses of the one-person coalitions are not larger, that is, $-(3c_1 - c_2)/3 \le -c_2/3$ and $-(2/3)c_2 \le -c_2/3$. This results in the condition $c_1 \ge 2c_2/3$.

RP 43 *A Glove Game*
(a) By straightforward computation, $\Phi(v) = (1/60)(39, 39, 14, 14, 14)$: note that it is sufficient to compute one of these values.
(b) $C(v) = \{(1, 1, 0, 0, 0)\}$.
(c) By (b) and the fact that the nucleolus is in the core whenever the core is nonempty, the nucleolus is $(1, 1, 0, 0, 0,)$.

RP 44 *A Four-person Cooperative Game*
(a) $C(v) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_i \ge 0\,\forall i, x_1 + x_2 = x_3 + x_4 = 2, x_1 + x_3 \ge 3\}$. In the intended diagram, the core is a triangle with vertices $(2, 1)$, $(2, 2)$, and $(1, 2)$.
(b) $\Phi(v) = (1/4)(5, 3, 5, 3)$ (it is sufficient to compute one of these values).

RP 45 *A Matching Problem*
(a) The resulting matching is (x_1, y_4), (x_2, y_3), (x_3, y_2), (x_4, y_1).
(b) The resulting matching is (x_1, y_4), (x_2, y_3), (x_3, y_1), (x_4, y_2).
(c) x_1 prefers y_4 over y_1 and y_4 prefers x_1 over y_4.
(d) In any core matching, x_2 and y_3 have to be paired, since they are each other's top choices. Given this, x_1 and y_4 have to be paired. This leaves only the two matchings in (a) and (b).

RP 46 *House Exchange*
(a) There are two core allocations: $1 : h_1, 2 : h_3, 3 : h_2$ and $1 : h_2, 2 : h_3, 3 : h_1$.
(b) The unique top trading cycle is $1, 2, 3$, with allocation $1 : h_2, 2 : h_3, 3 : h_1$.
(c) Take preference h_1, h_2, h_3 with unique core allocation $1 : h_1, 2 : h_3, 3 : h_2$.

RP 47 *A Marriage Market*
(a) m_1 must be paired to his favorite woman in the core. Next, m_2 must be paired to his favorite of the remaining women, etc.
(b) (m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4).
(c) (m_1, w_4), (m_2, w_3), (m_3, w_2), (m_4, w_1).
(d) (m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4) (one can reason about this but also just try the six possibilities).

Problems of Chapter 12

12.1 *Solving a Matrix Game*
(a) Column 1 is strictly dominated by column 2. Next, row 4 is strictly dominated by row 3. Finally, column 3 is strictly dominated by α times column 2 and $1 - \alpha$ times column 4 for any $1/4 < \alpha < 1/3$.
(b) We obtain

$$B = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 0 & 4 \end{pmatrix}.$$

By solving the problem graphically we obtain $v_2(q, 1-q) = 4 - 4q$ for $0 \leq q \leq 2/5$, $v_2(q, 1-q) = q + 2$ for $2/5 \leq q \leq 1/2$, $v_2(q, 1-q) = 3q + 1$ for $1/2 \leq q \leq 1$. From this we obtain $v(B) = 12/5$ and the unique optimal strategy of player 2 is $(2/5, 3/5)$. The unique optimal strategy for player 1 is $(0, 4/5, 1/5)$ (graphically). Further, we have $v_1(p_1, p_2, p_3) = 4p_1 + 3p_2$ if $4p_1 + 3p_2 \leq p_1 + 2p_2 + 4p_3$, hence if $7p_1 + 5p_2 \leq 4$; and $v_1(p_1, p_2, p_3) = p_1 + 2p_2 + 4p_3$ if $7p_1 + 5p_2 \geq 4$.
(c) $v(A) = 12/5$ and the unique optimal strategies of players 1 and 2 are, respectively, $(0, 4/5, 1/5, 0)$ and $(0, 2/5, 0, 3/5)$.
(d) Independent of y, the strategies in (c) still guarantee $12/5$ to player 1 and $-12/5$ to player 2. Hence, the answer is independent of y and the same as in (c).

12.2 *Proof of Lemma 12.2*
For all $\mathbf{p} \in \Delta^m$ we have $\min_{\mathbf{q} \in \Delta^n} \mathbf{p}A\mathbf{q} \leq \min_{\mathbf{q} \in \Delta^n} (\max_{\mathbf{p}' \in \Delta^m} \mathbf{p}'A\mathbf{q}) = v_2(A)$. Hence, $v_1(A) = \max_{\mathbf{p} \in \Delta^m} \min_{\mathbf{q} \in \Delta^n} \mathbf{p}A\mathbf{q} \leq v_2(A)$.

12.3 2×2 *Games*
(a) To have no saddlepoints we need $a_{11} > a_{12}$ or $a_{11} < a_{12}$. By assuming the first, the other inequalities follow.
(b) For optimal strategies $\mathbf{p} = (p, 1-p)$ and $\mathbf{q} = (q, 1-q)$ we must have $0 < p < 1$ and $0 < q < 1$. Then it is easy to compute that $p = [a_{22} - a_{21}]/[(a_{22} - a_{21}) + (a_{11} - a_{12})]$ and $q = [a_{22} - a_{12}]/[(a_{22} - a_{21}) + (a_{11} - a_{12})]$. The value of the game is $v(A) = pa_{11} + (1-p)a_{12}$. It is then straightforward to check that these expressions yield the formulas as stated in the problem.

12.4 *Symmetric Games*
Let $\mathbf{x}$ be optimal for player 1. Then $\mathbf{x}A\mathbf{y} \geq v(A)$ for all $\mathbf{y}$; hence $\mathbf{y}A\mathbf{x} = -\mathbf{x}A\mathbf{y} \leq -v(A)$ for all $\mathbf{y}$; hence (take $\mathbf{y} = \mathbf{x}$) $v(A) \leq -v(A)$, so $v(A) \leq 0$.

Let $\mathbf{y}$ be optimal for player 2. Then $\mathbf{x}A\mathbf{y} \leq v(A)$ for all $\mathbf{x}$; hence $\mathbf{y}A\mathbf{x} = -\mathbf{x}A\mathbf{y} \geq -v(A)$ for all $\mathbf{x}$; hence (take $\mathbf{x} = \mathbf{y}$) $v(A) \geq -v(A)$, so $v(A) \geq 0$.

Thus, $v(A) = 0$.

Let $\mathbf{x}$ be optimal for player 1, then $\mathbf{x}A\mathbf{y} \geq 0$ for all $\mathbf{y}$; hence $-\mathbf{y}A\mathbf{x} \geq 0$ for all $\mathbf{y}$; hence $\mathbf{y}A\mathbf{x} \leq 0$ for all $\mathbf{y}$: this implies that $\mathbf{x}$ is also optimal for player 2. The converse is analogous.

12.5 *The Duality Theorem Implies the Minimax Theorem*
Let A be an $m \times n$ matrix game. Without loss of generality assume that all entries of A are positive. Consider the associated LP as in Sect. 12.2.

Consider the vector $\bar{\mathbf{x}} = (1/m, \ldots, 1/m, \eta) \in \mathbb{R}^{m+1}$ with $\eta > 0$. Since all entries of A are positive it is straightforward to check that $\bar{\mathbf{x}} \in V$ if $\eta \leq$

$\sum_{i=1}^{m} a_{ij}/m$ for all $j = 1, \ldots, n$. Since $\bar{\mathbf{x}} \cdot c = -\eta < 0$, it follows that the value of the LP must be negative.

Let $\mathbf{x} \in O_{\min}$ and $\mathbf{y} \in O_{\max}$ be optimal solutions of the LP. Then $-x_{m+1} = -y_{n+1} < 0$ is the value of the LP. We have $x_i \geq 0$ for every $i = 1, \ldots, m$, $\sum_{i=1}^{m} x_i \leq 1$, and $(x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ (> 0) for every $j = 1, \ldots, n$. Optimality in particular implies $\sum_{i=1}^{m} x_i = 1$, so that $v_1(A) \geq (x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ for all j, hence $v_1(A) \geq x_{m+1}$. Similarly, it follows that $v_2(A) \leq y_{n+1} = x_{m+1}$, so that $v_2(A) \leq v_1(A)$. The Minimax Theorem now follows.

12.6 *Infinite Matrix Games*

(a) A is an infinite matrix game with for all $i, j \in \mathbb{N}$: $a_{ij} = 1$ if $i > j$, $a_{ij} = 0$ if $i = j$, and $a_{ij} = -1$ if $i < j$.

(b) Fix a mixed strategy $\mathbf{p} = (p_1, p_2, \ldots)$ for player 1 with $p_i \geq 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} p_i = 1$. If player 2 plays pure strategy j, then the expected payoff for player 1 is equal to $-\sum_{i=1}^{j-1} p_i + \sum_{i=j+1}^{\infty} p_i$. Since $\sum_{i=1}^{\infty} p_i = 1$, this expected payoff converges to -1 as j approaches ∞. Hence, $\inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$, so $\sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$. Similarly, one shows $\inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathbf{p}A\mathbf{q} = 1$, hence the game has no 'value'.

12.7 *Equalizer Theorem*

Assume, without loss of generality, $v = 0$. It is sufficient to show that there exists $\mathbf{q} \in \mathbb{R}^n$ with $\mathbf{q} \geq \mathbf{0}$, $A\mathbf{q} \leq \mathbf{0}$, and $q_n = 1$. The required optimal strategy is then obtained by normalization.

This is equivalent to existence of a vector $(\mathbf{q}, \mathbf{w}) \in \mathbb{R}^{n+m}$ with $\mathbf{q} \geq \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$, such that

$$\begin{pmatrix} A & I \\ \mathbf{e}^n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} ,$$

where row vector $\mathbf{e}^n \in \mathbb{R}^n$, I is the $m \times m$ identity matrix, $\mathbf{0}$ is an $1 \times m$ vector on the left hand side and an $m \times 1$ vector on the right hand side. Thus, we have to show that the vector $\mathbf{x} := (\mathbf{0}, 1) \in \mathbb{R}^{m+1}$ is in the cone spanned by the columns of the $(m+1) \times (n+m)$ matrix on the left hand side. Call this matrix B and call this cone Z. We assume $\mathbf{x} \notin Z$ and derive a contradiction. By Theorem 22.1 there is a $\mathbf{p} \in \mathbb{R}^{m+1}$ such that $\mathbf{p} \cdot \mathbf{z} > \mathbf{p} \cdot \mathbf{x} = p_{m+1}$ for all $\mathbf{z} \in Z$. Since $\mathbf{0} \in Z$, it follows that $p_{m+1} < 0$. Let $i \in \{1, \ldots, m\}$. By considering α times column $n + i$ of B, it follows that $\alpha p_i > p_{m+1}$ for all positive α, but this implies $p_i \geq 0$. By a similar argument, $(p_1, \ldots, p_m)A\mathbf{e}^j \geq 0$ for all $j \in \{1, \ldots, n-1\}$. Also, $(p_1, \ldots, p_m)A\mathbf{e}^n + p_{m+1} > p_{m+1}$, so $(p_1, \ldots, p_m)A\mathbf{e}^n > 0$; so in particular $(p_1, \ldots, p_m) \neq \mathbf{0}$. It follows that we can normalize $(p_1, \ldots, p_m)$ to an optimal strategy of player 1 that gives positive payoff when played against column n. This is the desired contradiction.

Problems of Chapter 13

13.1 *Existence of Nash Equilibrium Using Brouwer*

(a) Clearly, $f_{i,s_i}(\sigma) \geq 0$ for every $i \in N$ and $s_i \in S_i$, and $\sum_{s_i \in S_i} f_{i,s_i}(\sigma) = 1$.
(b) The set $\prod_{i \in N} \Delta(S_i)$ is compact and convex, and f is continuous.
(c) Let $\sigma^* \in \prod_{i \in N} \Delta(S_i)$. If σ^* is a Nash equilibrium of G then

$$\sigma_i^*(s_i) = \frac{\sigma_i^*(s_i) + \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\}}{1 + \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}^*) - u_i(\sigma^*)\}} \qquad (*)$$

for all $i \in N$ and $s_i \in S_i$, so that σ^* is a fixed point of f. Conversely, let σ^* be a fixed point of f. Then $(*)$ holds for all $i \in N$ and $s_i \in S_i$. Hence

$$\sigma_i^*(s_i) \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}^*) - u_i(\sigma^*)\} = \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\} \ .$$

Multiply both sides of this equation by $u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)$ and next sum over all $s_i \in S_i$, to obtain

$$\begin{aligned} &\sum_{s_i \in S_i} \sigma_i^*(s_i)[u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)] \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}^*) - u_i(\sigma^*)\} \\ &\quad = \sum_{s_i \in S_i} [u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)] \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\} \ . \end{aligned}$$

Now for the first factor on the left-hand side we have

$$\begin{aligned} \sum_{s_i \in S_i} \sigma_i^*(s_i)[u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)] &= \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*) \\ &= u_i(\sigma^*) - u_i(\sigma^*) \\ &= 0 \ . \end{aligned}$$

Hence,

$$\sum_{s_i \in S_i} [u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)] \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\} = 0 \ .$$

But this implies $u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*) \leq 0$ for all s_i. Hence, player i cannot deviate profitably by any pure strategy, and therefore also not by any mixed strategy (cf. Sect. 13.2.1). We conclude that σ^* is a Nash equilibrium of G.

13.2 *Existence of Nash Equilibrium Using Kakutani*
For upper semi-continuity of β, take a sequence σ^k converging to σ, a sequence $\tau^k \in \beta(\sigma^k)$ converging to τ, and show $\tau \in \beta(\sigma)$. This is straightforward. Also convex-valuedness of β is straightforward.

13.3 *Lemma 13.2*
Consider player 1. The only-if direction is straightforward from the definition of best reply. For the if-direction, note that if $\mathbf{p}A\mathbf{q} \geq \mathbf{e}^i A\mathbf{q}$ for all $i = 1, \ldots, m$, then also $\mathbf{p}A\mathbf{q} \geq \sum_{i=1}^m p_i' \mathbf{e}^i A\mathbf{q} = \mathbf{p}' A\mathbf{q}$ for all $\mathbf{p}' = (p_1', \ldots, p_n') \in \Delta^m$. Similarly for player 2.

13.4 *Lemma 13.3*
Take i such that $\mathbf{e}^i A\mathbf{q} \geq \mathbf{e}^k A\mathbf{q}$ for all $k = 1, \ldots, m$. Then, clearly, $\mathbf{e}^i A\mathbf{q} \geq \mathbf{p}' A\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$, so $\mathbf{e}^i \in \beta_1(\mathbf{q})$. The second part is analogous.

13.5 *Dominated Strategies*
(a) Let $(\mathbf{p}^*, \mathbf{q}^*)$ be a Nash equilibrium and suppose $q_n^* > 0$. Define $\bar{\mathbf{q}}$ by $\bar{q}_j = q_j^* + q_n^* q_j$ for $j = 1, \ldots, n-1$ and $\bar{q}_n = 0$. Then $\bar{\mathbf{q}} \in \Delta^n$ and $\mathbf{p}^* B\bar{\mathbf{q}} = \sum_{j=1}^{n-1}(q_j^* + q_n^* q_j)\mathbf{p}^* B\mathbf{e}^j = q_n^*(\mathbf{p}^* B\mathbf{q}) + \sum_{j=1}^{n-1} q_j^*(\mathbf{p}^* B\mathbf{e}^j) > q_n^*(\mathbf{p}^* B\mathbf{e}^n) + \sum_{j=1}^{n-1} q_j^*(\mathbf{p}^* B\mathbf{e}^j) = \mathbf{p}^* B\mathbf{q}^*$, a contradiction.

(b) Denote by $NE(A,B)$ the set of Nash equilibria of (A,B). Then

$$
\begin{aligned}
(\mathbf{p}^*, \mathbf{q}^*) \in NE(A,B) &\Leftrightarrow (\mathbf{p}^*, (\mathbf{q}', 0)) \in NE(A,B) \text{ where } (\mathbf{q}', 0) = \mathbf{q}^* \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1}[\mathbf{p}^* A(\mathbf{q}', 0) \geq \mathbf{p}A(\mathbf{q}', 0), \\
&\qquad \mathbf{p}^* B(\mathbf{q}', 0) \geq \mathbf{p}^* B(\mathbf{q}, 0)] \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1}[\mathbf{p}^* A'\mathbf{q}' \geq \mathbf{p}A'\mathbf{q}', \\
&\qquad \mathbf{p}^* B'\mathbf{q}' \geq \mathbf{p}^* B'\mathbf{q}] \\
&\Leftrightarrow (\mathbf{p}^*, \mathbf{q}') \in NE(A', B') \,.
\end{aligned}
$$

Note that the first equivalence follows by part (a).

13.6 *A 3×3 Bimatrix Game*
(a) Suppose $\{1,2\} \subseteq C(\mathbf{p})$. Then $4q_2 + 5q_3 = 4q_1 + 5q_3$, so $q_1 = q_2$ and $\mathbf{q}$ has the form $(\alpha, \alpha, 1-2\alpha)$. But then the third row yields a payoff to player 1 of $6 - 6\alpha$ whereas the first and second rows yield only $5 - 6\alpha$, a contradiction.
(b) Suppose $\{2,3\} = C(\mathbf{p})$. Then $4q_2 + 5q_3 \leq 4q_1 + 5q_3$, so $q_2 \leq q_1$. However, since $p_1 = 0$ column 1 is strictly dominated for player 2, so $q_1 = 0$ and therefore $q_2 = 0$. Then, however, $C(\mathbf{p}) = \{3\}$, a contradiction.
(c) The only possibility left for a two-element carrier of $\mathbf{p}$ is $\{1,3\}$, but then the second column and next the first row is dominated, a contradiction. Hence $\mathbf{p}$ must have a one-element carrier, and that can only be $\{3\}$. So the unique Nash equilibrium is $((0,0,1),(0,0,1))$.

13.7 *A 3×2 Bimatrix Game*
The best reply function of player 1 is:

$$
\beta_1(\mathbf{q}) = \begin{cases} \{\mathbf{e}^1\} & \text{if } q_2 > q_1 \\ \Delta^3 & \text{if } q_1 = q_2 = \frac{1}{2} \\ \{\mathbf{p} \in \Delta^3 \mid p_1 = 0\} & \text{if } q_2 < q_1 \,. \end{cases}
$$

The best reply function of player 2 is:

$$
\beta_2(\mathbf{p}) = \begin{cases} \{\mathbf{e}^2\} & \text{if } p_1 > 0 \\ \Delta^2 & \text{if } p_1 = 0 \,. \end{cases}
$$

The set of Nash equilibria is $\{(\mathbf{p}, \mathbf{q}) \in \Delta^3 \times \Delta^2 \mid p_1 = 0,\ q_1 \geq \frac{1}{2}\} \cup \{((1,0,0),(0,1))\}$.

13.8 *The Nash Equilibria in Example 13.18*
(a) Let $\mathbf{p} = (p_1, p_2, p_3)$ be the strategy of player 1. We distinguish two cases: (i) $p_2 = 0$ (ii) $p_2 > 0$.

In case (i), reduce the game to

$$\begin{array}{c c} & \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{array}{c} p_1 \\ p_3 \end{array} & \begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 0,0 & 1,1 & 1,1 \end{pmatrix} \end{array}$$

where $\mathbf{q} = (q_1, q_2, q_3)$ is player 2's strategy. This game can be solved graphically and yields the following set of Nash equilibria: $\{((1,0),(1,0,0))\} \cup \{((1/2,1/2),(q,1/2,1/2-q)) \mid 0 \le q \le 1/2\} \cup \{((p,1-p),(0,1/2,1/2)) \mid 0 \le p \le 1/2\} \cup \{((0,1),(0,q,1-q)) \mid 1/2 \le q \le 1\}$. As long as player 1 gets at least 1 (the payoff from playing M) these equilibria are also equilibria of the original game G, so that we obtain: $\{((1,0,0),(1,0,0))\} \cup \{((p,0,1-p),(0,1/2,1/2)) \mid 0 \le p \le 1/2\} \cup \{((0,0,1),(0,q,1-q)) \mid 1/2 \le q \le 1\}$.

In case (ii), R gives a lower expected payoff to player 2 than C, so the game can be reduced to

$$\begin{array}{c c} & \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 1,2 & 1,2 \\ 0,0 & 1,1 \end{pmatrix} \end{array}.$$

Solving this game graphically and extending to G yields the collections (iii)–(v). Observe that the equilibrium $((1,0,0),(1,0,0))$ is also a member of (iii).
(b) Consider again the perturbed games $G(\varepsilon)$ as in Example 13.18. For $q = 0$ consider the strategy combination $((\varepsilon, 1-2\varepsilon, \varepsilon), (\varepsilon, 1-2\varepsilon, \varepsilon))$ in $G(\varepsilon)$: this is a Nash equilibrium in $G(\varepsilon)$, which for $\varepsilon \to 0$ converges to $((0,1,0),(0,1,0))$. For $q = 1$ consider, similarly, $((\varepsilon, 1-2\varepsilon, \varepsilon), (1-2\varepsilon, \varepsilon, \varepsilon))$ in $G(\varepsilon)$; for $0 < q < 1$ consider $((\varepsilon, 1-2\varepsilon, \varepsilon), (q-\varepsilon/2, 1-q-\varepsilon/2, \varepsilon))$.

13.9 *Proof of Theorem 13.8*
(ii) $\Rightarrow$ *(i)*: conditions (13.1) are satisfied and $f = 0$, which is optimal since $f \le 0$ always.
(i) $\Rightarrow$ *(ii)*: clearly we must have $a = \mathbf{p}A\mathbf{q}$ and $b = \mathbf{p}B\mathbf{q}$ (otherwise $f < 0$ which cannot be optimal). From the conditions (13.1) we have $\mathbf{p}'A\mathbf{q} \le a = \mathbf{p}A\mathbf{q}$ and $\mathbf{p}B\mathbf{q}' \le b = \mathbf{p}B\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$ and $\mathbf{q}' \in \Delta^n$, which implies that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

13.10 *Matrix Games*
This is a repetition of the proof of Theorem 12.5. Note that the solutions of program (13.3) give exactly the value of the game a and the optimal (minimax) strategies of player 2. The solutions of program (13.4) give exactly the value of the game $-b$ and the optimal (maximin) strategies of player 1.

13.11 *Tic-Tac-Toe*
(a) Start by putting a cross in the center square. Then player 2 has essentially two possibilities for the second move, and it is easy to see that in each of the

two cases player 1 has a forcing third move. After this, it is equally easy to see that player 1 can always enforce a draw.
(b) If player 1 does not start at the center, then player 2 can put his first circle at the center and then can place his second circle in such a way that it becomes forcing. If player 1 starts at the center then either a pattern as in (a) is followed, leading to a draw, or player 2's second circle becomes forcing, also resulting in a draw.
(c) Since the game has a pure strategy Nash equilibrium (it is a finite extensive form game of perfect information), the value must be -1, 0, or 1 (by Theorem 13.9). Here, -1 indicates that player 1 looses, 0 a draw, and 1 a win for player 1. By (a) and (b) its value must be 0.

13.12 *Iterated Elimination in a Three-Player Game*
R is strictly dominated by L, then U by D, then r by l. This results in the strategy combination (D, l, L).

13.13 *Never a Best Reply and Domination*
Strategy Y is not strictly dominated: this would require putting probability larger than $\frac{2}{3}$ on V and also probability larger than $\frac{2}{3}$ on X, which is impossible.
For Y to be a best reply, there must be strategies $(p, 1-p)$ of player 1 and $(q, 1-q)$ of player 2 such that:

$$\begin{aligned}
&6pq + 6(1-p)(1-q) \geq 9pq \\
&6pq + 6(1-p)(1-q) \geq 9(1-p)q + 9p(1-q) \\
&6pq + 6(1-p)(1-q) \geq 9(1-p)(1-q)\ .
\end{aligned}$$

This is not possible, as can be seen, for instance by making a diagram in p, q-space in which the sets of solutions of these three inequalities are depicted: there is no point where all three intersect.

13.14 *Completely Mixed Nash Equilibria are Perfect*
Let σ be a completely mixed Nash equilibrium in G. Let $\varepsilon := \min\{\sigma_i(h_i) \mid i \in N,\ h_i \in S_i\}$. Then $\varepsilon > 0$. Take any sequence of perturbed games $G(\mu^t)$ with (without loss of generality) $\mu^t_{ih} < \varepsilon$ for all $i \in N$, $h \in S_i$, $t = 1, 2, \ldots$ and with $\mu^t \to \mathbf{0}$ for $t \to \infty$. Then σ is a Nash equilibrium in $G(\mu^t)$ for every t, so $\sigma^t \to \sigma$ where $\sigma^t := \sigma$ for every t.

13.15 *A 3-Player Game with an Undominated but not Perfect Equilibrium*
(a) Note that r and R are strictly dominated. Therefore, the set of Nash equilibria of the game is $\{((p, 1-p), l, L) \mid 0 \leq p \leq 1\}$, where p is the probability with which player 1 plays U. In the Nash equilibrium of any perturbed game $G(\mu)$ with μ small, the three players would put maximal probability on the strategies U, l, and L, respectively. This implies that (U, l, L) is the only limit of Nash equilibria of perturbed games, and therefore the only perfect equilibrium.

(b) Clearly, l and L are undominated, and so is D. So the equilibrium (D, l, L) is undominated.

13.16 *Existence of Proper Equilibrium*
Tedious but straightforward.

13.17 *Strictly Dominated Strategies and Proper Equilibrium*
(a) The only Nash equilibria are $(U, l(, L))$ and $(D, r(, L))$. Obviously, only the first one is perfect and proper.
(b) Let $1 > \varepsilon > 0$ and consider the strategy combination σ^ε such that: σ_1^ε puts probability ε on U and $1-\varepsilon$ on D; σ_2^ε puts probability ε on l and $1-\varepsilon$ on r; and σ_3^ε puts probability ε on R and $1-\varepsilon$ on L. For ε sufficiently small, this is an ε-proper equilibrium, as is easily seen by checking the definition. Hence the limit for $\varepsilon \to 0$, (D, r, L), is a proper Nash equilibrium. (One can also let σ_3^ε put probability 2ε on R and $1-2\varepsilon$ on L in order to obtain that player 1 strictly prefers D over U and player 2 strictly prefers r over l.)

13.18 *Strictly Perfect Equilibrium*
(a) Identical to the proof of Lemma 13.16, see Problem 13.14: note that any sequence of perturbed games converging to the given game must eventually contain any given completely mixed Nash equilibrium σ.
(b) Let σ be a strict Nash equilibrium in the game G. Note that σ must be pure, hence $\sigma = (s_1, \ldots, s_n)$ for some $s_1 \in S_1, \ldots, s_n \in S_n$. Let $(G(\mu^t))_{t\in\mathbb{N}}$ be any sequence of perturbed games converging to G. For each t, consider the strategy combination σ^t in which player i puts probability μ_{ih}^t on any pure strategy $h \in S_i \setminus \{s_i\}$ and $1 - \sum_{h\in S_i\setminus\{s_i\}}$ on s_i. Since σ is strict, for large enough t the combination σ^t is a Nash equilibrium in $G(\mu^t)$, and $\sigma^t \to \sigma$ for $t \to \infty$.
(c) Note that M and R are strictly dominated. The set of Nash equilibria is $\{((p, 1-p), L) \mid 0 \le p \le 1\}$, where p is the probability on U. Consider a sequence of perturbed games $G(\mu^t)$ with $\mu_{2M}^t = \mu_{2R}^t$: in a Nash equilibrium of such a perturbed game player 2 plays M and R both with the same probability $\mu_{2M}^t = \mu_{2R}^t$, and thus player 1 is indifferent between the two rows. Hence, any $((p, 1-p), L)$ can be obtained as the limit of Nash equilibria of perturbed games, so every Nash equilibrium of the game (A, B) is perfect. By the same argument, all Nash equilibria are also proper. But none of these is strictly perfect: for $\mu_{2M}^t > \mu_{2R}^t$ any sequence of Nash equilibria of perturbed games converges to (U, L) whereas for $\mu_{2M}^t < \mu_{2R}^t$ any sequence of Nash equilibria of perturbed games converges to (D, L).

13.19 *Correlated Equilibria in the Two-Driver Example (1)*
Inequalities (13.5) and (13.6) result in: $-10p_{11} + 6p_{12} \ge 0$, $10p_{21} - 6p_{22} \ge 0$, $-10p_{11} + 6p_{21} \ge 0$, $10p_{12} - 6p_{22} \ge 0$. Altogether, we obtain the conditions: $p_{11} + p_{12} + p_{21} + p_{22} = 1$, $p_{ij} \ge 0$ for all $i, j \in \{1, 2\}$, $p_{11} \le \frac{3}{5}\min\{p_{12}, p_{21}\}$, $p_{22} \le \frac{5}{3}\min\{p_{12}, p_{21}\}$.

13.20 *Nash Equilibria are Correlated*

Let $i, k \in \{1, \ldots, m\}$. Then

$$\begin{aligned}\sum_{j=1}^{n}(a_{ij} - a_{kj})p_{ij} &= \sum_{j=1}^{n}(a_{ij} - a_{kj})p_i q_j \\ &= p_i \sum_{j=1}^{n} q_j(a_{ij} - a_{kj}) \\ &= p_i(\mathbf{e}^i A\mathbf{q} - \mathbf{e}^k A\mathbf{q}) .\end{aligned}$$

If $p_i > 0$, then row i is a pure best reply, hence $\mathbf{e}^i A\mathbf{q} \geq \mathbf{e}^k A\mathbf{q}$, so that the last expression above is always nonnegative. This proves (13.5). The proof of (13.6) is analogous..

13.21 *The Set of Correlated Equilibria is Convex*
Let P and Q be correlated equilibria and $0 \leq t \leq 1$. Check that (13.5) and (13.6) are satisfied for $tP + (1-t)Q$.

13.22 *Correlated vs. Nash Equilibrium*
(a) The Nash equilibria are: $((1,0),(0,1))$, $((0,1),(1,0))$, and $((2/3,1/3),(2/3,1/3))$.
(b) To verify that P is a correlated equilibrium, check conditions (13.5) and (13.6). These are, for both player 1 and player 2, $(1/2)\cdot 6 + (1/2)\cdot 2 \geq (1/2)\cdot 7 + (1/2)\cdot 0$ and $1\cdot 7 + 0\cdot 0 \geq 1\cdot 6 + 0\cdot 2$.

The associated payoffs are 5 for each. The payoff pair $(5,5)$ lies 'above' the triangle of payoffs with vertices $(7,2)$, $(2,7)$, and $(4\frac{2}{3}, 4\frac{2}{3})$, which are the payoffs of the Nash equilibria of the game.

13.23 *Correlated Equilibria in the Two-Driver Example (2)*
The payoff matrices are:

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} \quad 1' \quad & \quad 2' \quad \end{array} \\ \begin{pmatrix} -10,-10 & 5,0 \\ 0,5 & -1,-1 \end{pmatrix} \end{array} .$$

The matrix C is:

$$\begin{array}{c} \\ (1,1') \\ (1,2') \\ (2,1') \\ (2,2') \end{array}\begin{array}{c} \begin{array}{cccc} (1,2) & (2,1) & (1',2') & (2',1') \end{array} \\ \begin{pmatrix} -10 & 0 & -10 & 0 \\ 6 & 0 & 0 & 10 \\ 0 & 10 & 6 & 0 \\ 0 & -6 & 0 & -6 \end{pmatrix} \end{array} .$$

The optimal (maximin) strategy of player 1 in C is $(0, \frac{1}{2}, \frac{1}{2}, 0)$: this guarantees a payoff of at least 3, and by playing $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ player 2 guarantees to pay at most 3, so 3 is the value of the game. Clearly, no other strategy of player 1 guarantees 3. (Alternatively, this matrix game can be analyzed by first deleting strategies $(1,1')$ and $(2,2')$ of player 1: these are strictly dominated by any mixed strategy that puts positive weights on $(1,2')$ and $(2,1')$. The resulting 2×4 matrix game can be analyzed by using the graphical method of Chap. 2.)

The associated correlated equilibrium is:

$$\begin{array}{cc} & \begin{array}{cc} c & s \end{array} \\ \begin{array}{c} c \\ s \end{array} & \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \end{array},$$

resulting in each of the two pure Nash equilibria played with probability $\frac{1}{2}$.

13.24 *Finding Correlated Equilibria*
Let P be a correlated equilibrium. Then

$$\begin{aligned} 5p_{11}+p_{12} \geq 2p_{11}+4p_{12} &\Leftrightarrow p_{11} \geq p_{12} \\ 2p_{21}+4p_{22} \geq 5p_{21}+p_{22} &\Leftrightarrow p_{22} \geq p_{21} \\ 2p_{11}+3p_{21} \geq 3p_{11}+p_{21} &\Leftrightarrow 2p_{21} \geq p_{11} \\ 3p_{12}+p_{22} \geq 2p_{12}+3p_{22} &\Leftrightarrow p_{12} \geq 2p_{22} \end{aligned}$$

which implies

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

So this is the unique correlated equilibrium.
The matrix C is:

$$\begin{array}{cc} & \begin{array}{cccc} (1,2) & (2,1) & (1',2') & (2',1') \end{array} \\ \begin{array}{c} (1,1') \\ (1,2') \\ (2,1') \\ (2,2') \end{array} & \begin{pmatrix} 3 & 0 & -1 & 0 \\ -3 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 \\ 0 & 3 & 0 & -2 \end{pmatrix} \end{array}.$$

Then the (unique) maximin strategy is $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, and the value of the game is $v(C) = 0$ – hence, maximin strategies correspond one-to-one with correlated equilibria. (The (unique) minimax strategy is $(\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{1}{3})$.)

13.25 *Nash, Perfect, Proper, Strictly Perfect, and Correlated Equilibria*
(a) Suppose $p_3 = 0$. If $p_1 > 0$ then $q_1 = 1$, hence $p_3 = 1$. Contradiction. Hence $\mathbf{p} = (0,1,0)$.
(b) Suppose $p_1, p_3 > 0$. Then $6q_1 = 6q_3$, so $q_1 = q_3$. Also, $4q_1 + 4q_3 \leq 6q_1$, so $8q_1 \leq 6q_1$, which implies $q_1 = q_3 = 0$. Hence $\mathbf{q} = (0,1,0)$.
(c) If $p_3 = 0$ then $\mathbf{p} = (0,1,0)$, so player 2 gets 0. If $p_1, p_3 > 0$, then $\mathbf{q} = (0,1,0)$, so player 1 gets 0. If $p_1 = 0$ and $p_3 > 0$, then $q_3 = 1$, but then $p_1 = 1$, a contradiction.
(d) If $p_3 = 0$ then $\mathbf{p} = (0,1,0)$ and $6q_3 \leq 4q_1 + 4q_3$ and $6q_1 \leq 4q_1 + 4q_3$, so we obtain a set of Nash equilibria $\{((0,1,0),(q_1,q_2,q_3)) \mid (q_1,q_2,q_3) \in \Delta^3,\ q_3 \leq$

$2q_1 \le 4q_3\}$. The case $p_3 > 0$ and $p_1 = 0$ does not result in a Nash equilibrium. If $p_3 > 0$ and $p_1 > 0$, then $\mathbf{q} = (0,1,0)$ and $6p_1 \le 4p_1 + 4p_3$ and $6p_3 \le 4p_1 + 4p_3$, so we obtain a second set of Nash equilibria $\{((p_1,p_2,p_3),(0,1,0)) \mid 0 < p_1 \le 2p_3 \le 4p_1\}$.
(e) Any strategy (p_1,p_2,p_3) with $p_1,p_3 > 0$ is weakly dominated by a strategy $(p_1-\varepsilon, p_2+2\varepsilon, p_3-\varepsilon)$ for small $\varepsilon > 0$. Similarly, any strategy (q_1,q_2,q_3) with $q_1,q_3 > 0$ is weakly dominated. Hence, the only perfect equilibrium is $((0,1,0),(0,1,0))$.
(f) By (e) and the fact that a proper equilibrium always exists, $((0,1,0),(0,1,0))$ is the only proper equilibrium. Again by (e), since $((0,1,0),(0,1,0))$ is the only perfect equilibrium, it is also strictly perfect: since any converging sequence of Nash equilibria of perturbed games leads to a perfect equilibrium, this must be $((0,1,0),(0,1,0))$, which is therefore strictly perfect.
(g) $4\beta+4\gamma \ge 6\gamma$, $4\beta+4\gamma \ge 6\beta$, hence $\beta \le 2\gamma \le 4\beta$; $4\alpha+4\delta \ge 6\alpha$, $4\alpha+4\delta \ge 6\delta$, hence $\delta \le 2\alpha \le 4\delta$.
(h) $4\beta + 4\gamma = 3$ (payoff to player 1) and $4\alpha + 4\delta = 1$ (payoff to player 2), for instance $\beta = \gamma = 3/8$, $\alpha = \delta = 1/8$.

13.26 *Independence of the Axioms in Corollary 13.40*
Not OPR: take the set of all strategy combinations in every game. Not CONS: in games with maximal player set take all strategy combinations, in other games take the set of Nash equilibria. Not COCONS: drop a Nash equilibrium in some game with maximal player set, but otherwise always take the set of all Nash equilibria.

13.27 *Inconsistency of Perfect Equilibria*
Observe that the perfect equilibria in G_0 are all strategy combinations where player 2 plays L, player 3 plays D, and player 1 plays any mixture between T and B – this follows easily by first applying Theorem 13.21, noting that R and U are (even strictly) dominated strategies; and next consider perturbed games where player 2 plays L and R with probabilities $1-\varepsilon$ and ε, respectively, and player 3 plays D and U with probabilities $1-\varepsilon$ and ε, respectively, so that player 1 is indifferent between T and B. Consider now the reduced game by fixing player 3's strategy at D. Then B is (weakly) dominated for player 1, so the only remaining perfect equilibrium is the pair (T,L). This shows that the perfect Nash equilibrium correspondence is not consistent.

Problems of Chapter 14

14.1 *Mixed and Behavioral Strategies*
LL' should be played with probability $1/8$, LR' with probability $3/8$, and RL' and RR' with any probabilities adding up to $1/2$.

14.2 *An Extensive Form Structure without Perfect Recall*
(a) The paths $\{(x_0,x_1)\}$ and $\{(x_0,x_2)\}$ contain different player 1 actions.

(b) Any behavioral strategy generating the same probability distribution over the end nodes as σ_1, should assign positive probabilities to L, R, l, and r. Therefore it generates positive probabilities on x_4 and x_5 as well, a contradiction.

14.3 *Consistency Implies Bayesian Consistency*
With notations as in Def. 14.13, for $h \in H$ with $\mathbb{P}_b(h) > 0$ and $x \in h$ we have: $\beta_h(x) = \lim_{m\to\infty} \beta_h^m(x) = \lim_{m\to\infty} \mathbb{P}_{b^m}(x)/\mathbb{P}_{b^m}(h) = \mathbb{P}_b(x)/\mathbb{P}_b(h)$. Here, the second equality follows from Bayesian consistency of the (b^m, β^m).

14.4 *(Bayesian) Consistency in Signaling Games*
The idea of the proof is as follows. Let (b, β) be a Bayesian consistent assessment. This means that β is determined on every information set of player 2 that is reached with positive probability, given b_1. Take $m \in \mathbb{N}$. Assign the number $1/m^2$ to action a of a type i of player 1 if that type does not play a but some other type of player 1 plays a with positive probability. Assign the number $1/m^2$ also to action a of type i if no type of player 1 plays a and player 2 attaches zero belief probability to type i conditional on player 1 having played a. To every other action of a of player 1, assign the number $\beta(i, a)/m$, where $\beta(i, a)$ is the (positive) belief that player 2 attaches to player 1 being of type i conditional on having played a. Next, normalize all these numbers to behavioral strategies b_1^m of player 1. For player 2, just take completely mixed behavioral strategies b_2^m converging to b_2. Then $(b^m, \beta^m) \to (b, \beta)$, where the β^m are determined by Bayesian consistency.

14.5 *Sequential Equilibria in a Signaling Game*
Player 2 plays u if $\alpha > 1/3$, d if $\alpha < 1/3$, and is indifferent if $\alpha = 1/3$. Player 2 plays u' if $\beta > 2/3$, d' if $\beta < 2/3$, and is indifferent if $\beta = 2/3$. This results in nine different combinations of values of α and β. Only two of those lead to sequential equilibria:

- $\alpha > 1/3$, $\beta < 2/3$. Then $b_2(u) = b_2(d') = 1$, $b_1(R) = b_1(R') = 1$, hence $\beta = 1/2$ and $\alpha > 1/3$.
- $\alpha = 1/3$, $\beta = 2/3$. Then $b_1(L) = b_1(R') = 1/3$, $b_2(u) = b_2(u') = 1/2$.

14.6 *Computation of Sequential Equilibrium (1)*
The unique sequential equilibrium consists of the behavioral strategies where player 1 plays B with probability 1 and C with probability $1/2$, and player 2 plays L with probability $1/2$; and player 1 believes that x_3 and x_4 are equally likely.

14.7 *Computation of Sequential Equilibrium (2)*
(b) The Nash equilibria are (L, l), and $(R, (\alpha, 1-\alpha))$ for all $\alpha \leq 1/2$, where α is the probability with which player 2 plays l. All these equilibria are subgame perfect, since the only subgame is the whole game.
(c) Let π be the belief player 2 attaches to node y_1. Then the sequential equilibria are: (L, l) with belief $\pi = 1$; (R, r) with belief $\pi \leq 1/2$; and $(R, (\alpha, 1-\alpha))$ for any $\alpha \leq 1/2$ with belief $\pi = 1/2$.

14.8 *Computation of Sequential Equilibrium (3)*
(b) First observe that there is no Nash equilibrium in which player 2 plays pure. Let player 2's strategy be $(q, 1-q)$ with q the probability on l. Since player 2 has to be indifferent, player 1 has to put equal probability on L and M, but then it is better for player 1 to play R. Hence, the Nash equilibria are $(R,(q,1-q))$ with $1/3 \le q \le 2/3$. (The conditions on q keep player 1 from deviating to L or M.)
(c) With player 2 attaching equal belief to y_1 and y_2 (since player 2 should be indifferent between l and r), the equilibria in (b) are sequential.

14.9 *Computation of Sequential Equilibrium (4)*
The Nash equilibria in this game are: $(R,(q_1,q_2,q_3))$ with $q_3 \le 1/3$ and $q_1 \le 1/2 - (3/4)q_3$, where q_1, q_2, q_3 are the probabilities put on l, m, r, respectively; and $((1/4, 3/4, 0), (1/4, 0, 3/4)$ (probabilities on L, M, R and l, m, r, respectively).

Let π be the belief attached by player 2 to y_1. Then with $\pi = 1/4$ the equilibrium $((1/4, 3/4, 0), (1/4, 0, 3/4)$ becomes sequential. There is no π that makes m optimal for player 2; therefore, the first set of equilibria contains no equilibrium that can be extended to a sequential equilibrium, since $q_2 > 0$ there.

14.10 *Computation of Sequential Equilibrium (5)*
The Nash equilibria are: (DB, r); $((R,(s,1-s)),(q,1-q))$ with $0 \le s \le 1$ and $q \ge 1/3$, where s is the probability on A and q is the probability on l. The subgame perfect equilibria are: (DB, r); (RA, l); $((R,(3/4,1/4)),(3/5,2/5))$. The first one becomes sequential with $\beta = 0$; the second one with $\beta = 1$; and the third one with $\beta = 3/5$.

Problems of Chapter 15

15.1 *Computing ESS in* 2×2 *Games (1)*
$ESS(A)$ can be computed using Proposition 15.4.
(a) $ESS(A) = \{\mathbf{e}^2\}$. (b) $ESS(A) = \{\mathbf{e}^1, \mathbf{e}^2\}$. (c) $ESS(A) = \{(2/3, 1/3)\}$.

15.2 *Computing ESS in* 2×2 *Games (2)*
Case (1): $ESS(A') = \{\mathbf{e}^2\}$; case (2): $ESS(A') = \{\mathbf{e}^1, \mathbf{e}^2\}$; case (3): $ESS(A') = \{\hat{\mathbf{x}}\} = \{(a_2/(a_1+a_2), a_1/(a_1+a_2))\}$.

15.3 *Rock-Paper-Scissors (1)*
The unique Nash equilibrium is $((1/3,1/3,1/3),(1/3,1/3,1/3))$, which is symmetric. But $(1/3,1/3,1/3)$ is not an ESS: e.g., $(1/3,1/3,1/3)A(1,0,0) = 1 = (1,0,0)A(1,0,0)$.

15.4 *Uniform Invasion Barriers*
Case (1), $\mathbf{e}^2$: maximal uniform invasion barrier is 1.
Case (2), $\mathbf{e}^1$: maximal uniform invasion barrier is $a_1/(a_1+a_2)$.

Case (2), $\mathbf{e}^2$: maximal uniform invasion barrier is $a_2/(a_1+a_2)$.
Case (3), $\hat{\mathbf{x}}$: maximal uniform invasion barrier is 1.

15.5 *Replicator Dynamics in Normalized Game (1)*
Straightforward computation.

15.6 *Replicator Dynamics in Normalized Game (2)*
The replicator dynamics can be written as $\dot{x} = [x(a_1+a_2) - a_2]x(1-x)$, where $\dot{x} = \dot{x}_1$. So $x = 0$ and $x = 1$ are always stationary points. In case (1) the graph of $\dot{x}$ on $(0,1)$ is below the horizontal axis. In case (2) there is another stationary point, namely at $x = a_2/(a_1+a_2)$; on $(0, a_2/(a_1+a_2))$ the function $\dot{x}$ is negative, on $(a_2/(a_1+a_2), 1)$ it is positive. In case (3) the situation of case (2) is reversed: the function $\dot{x}$ is positive on $(0, a_2/(a_1+a_2))$ and negative on $((a_2/(a_1+a_2), 1)$.

15.7 *Weakly Dominated Strategies and Replicator Dynamics*
(a) For population shares $(x, 1-x)$ the replicator dynamics is $\dot{x} = x(1-x)^2$. The only (Lyapunov and asymptotically) stable stationary point is $x = 1$. The strategy $(1,0)$ is the unique ESS. The strategy $(0,1)$ is weakly dominated.
(b) In this case the replicator dynamics are given by

$$\begin{aligned}
\dot{x}_1 &= x_1[1 - (x_1+x_2)^2 - x_1x_3] \\
\dot{x}_2 &= x_2[x_1 + x_2 - (x_1+x_2)^2 - x_1x_3] \\
\dot{x}_3 &= x_3[-(x_1+x_2)^2 - x_1x_3]\ .
\end{aligned}$$

The stationary points are $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and all points with $x_3 = 0$. Except $\mathbf{e}^3$, all stationary points are Lyapunov stable. None of these points is asymptotically stable. Note that $\mathbf{e}^3$ is strictly dominated (by $\mathbf{e}^1$)). One can also derive $d(x_1/x_2)/dt = x_1x_3/x_2 > 0$ at completely mixed strategies, i.e., at the interior of Δ^3. Hence, the share of subpopulation 1 grows faster than that of 2 but this difference goes to zero if x_3 goes to zero ($\mathbf{e}^2$ is weakly dominated by $\mathbf{e}^1$).

15.8 *Stationary Points and Nash Equilibria (1)*
(a) $NE(A) = \{(\alpha, \alpha, 1-2\alpha) \mid 0 \le \alpha \le 1/2\}$.
(b) By Proposition 15.18 and (a) it follows that $\{(\alpha, \alpha, 1-2\alpha) \mid 0 \le \alpha \le 1/2\} \cup \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\} \subseteq ST(A)$, and that possibly other stationary points must be boundary points of Δ^3. The replicator dynamics are given by

$$\begin{aligned}
\dot{x}_1 &= x_1(x_1 - x_2)(x_1 - x_2 - 1) \\
\dot{x}_2 &= x_2(x_1 - x_2)(1 + x_1 - x_2) \\
\dot{x}_3 &= x_3(x_1 - x_2)^2\ .
\end{aligned}$$

Inspection of this system yields no additional stationary points. All stationary points except $\mathbf{e}^1$ and $\mathbf{e}^2$ are Lyapunov stable, but no point is asymptotically stable.

15.9 *Stationary Points and Nash Equilibria (2)*

(a) Let $(x_1, x_2, x_3) \in NE(A)$. If $x_3 = 0$ then $x_2 = 1$, resulting in $(0, 1, 0)$. If $x_3 > 0$, then we must have $2x_2 + 4x_3 \geq 3x_1 + 3x_2 + x_3$, i.e., $4x_3 \geq 2x_1 + 1$, from the first row; and $2x_2 + 4x_3 \geq 4x_1 + 4x_2$, i.e., $3x_3 \geq x_1 + 1$ from the second row. If $x_1, x_2 > 0$ then both inequalities have to be equalities, but this is impossible. If $x_1 > 0$ and $x_2 = 0$ then we have $4x_3 = 2x_1 + 1$, resulting in $(1/2, 0, 1/2)$. Similarly, the case $x_1 = 0$ and $x_2 > 0$ results in $(0, 2/3, 1/3)$. Finally, the case $x_1 = x_2 = 0$ results in $(0, 0, 1)$.
(b) Use Proposition 15.18. This implies that $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1/2, 0, 1/2)$, and $(0, 2/3, 1/3)$ all are stationary states. Any other stationary state must be on the boundary of Δ^3 and have exactly one zero coordinate. If $x_1 = 0$ and $x_2, x_3 > 0$, then the replicator dynamics $\dot{x}_2 = x_2(4x_2 - 4x_2^2 - 4x_3^2 - 2x_2x_3)$ implies after simplification that $2x_3(2x_3 + x_2) = 0$, which is impossible. If $x_2 = 0$ and $x_1, x_3 > 0$, then the replicator dynamics $\dot{x}_1 = -x_1[(x_1 + x_2)(6x_1 + 6x_2 - 8) - (x_1 - 3)]$ implies after simplification that $6x_1^2 - 9x_1 + 3 = 0$, yielding $x_1 = 1$ or $x_1 = 1/2$, and thus no new stationary state. Finally, the case $x_3 = 0$ and $x_1, x_2 > 0$ yields via the replicator dynamics $\dot{x}_1 = -x_1[(1 - x_3)(-6x_3 - 2) - (x_1 - 3)]$ that $x_1 = 1$.
(c) By Proposition 15.19, $(1, 0, 0)$ is not Lyapunov stable. Since $\dot{x}_2 = -2x_2x_3 \cdot (2x_3 + x_2)$, which is negative for positive values of x_2 and x_3, also the states $(0, 2/3, 1/3)$ and $(0, 1, 0)$ are not Lyapunov stable. Since $\dot{x}_2 = -2x_2x_3(2x_3 + x_2)$ and $\dot{x}_1 = -x_1[3 + (x_1 + x_2)(6x_1 + 6x_2 - 8) - x_1]$, the state $(0, 0, 1)$ is asymptotically stable. Finally, at $(1/2 + \varepsilon, 0, 1/2 - \varepsilon)$ for $\varepsilon > 0$ we have $\dot{x}_1 > 0$, so that $(1/2, 0, 1/2)$ is not Lyapunov stable.
(d) $(1/2, 0, 1/2)A(1/2, \lambda, 1/2 - \lambda) = 2$, whereas $(1/2, \lambda, 1/2 - \lambda)A(1/2, \lambda, 1/2 - \lambda) = 2 + 6\lambda^2$. Hence $(1/2, 0, 1/2)$ is not locally superior.

15.10 *Lyapunov Stable States in* 2×2 *Games*
Case (1): $\mathbf{e}^2$; case (2): $\mathbf{e}^1$ and $\mathbf{e}^2$; case (3): $\hat{\mathbf{x}}$. (Cf. Problem 15.6.)

15.11 *Nash Equilibrium and Lyapunov Stability*
$NE(A) = \{\mathbf{e}^1\}$. If we start at a completely mixed strategy close to $\mathbf{e}^1$, then first x_3 increases, and we can make the solution trajectory pass $\mathbf{e}^3$ as closely as desired. This shows that $\mathbf{e}^1$ is not Lyapunov stable.

15.12 *Rock-Paper-Scissors (2)*
(a) Replicator dynamics:

$$\begin{aligned}
\dot{x}_1 &= x_1[x_1 + (2 + a)x_2 - \mathbf{x}A\mathbf{x}] \\
\dot{x}_2 &= x_2[x_2 + (2 + a)x_3 - \mathbf{x}A\mathbf{x}] \\
\dot{x}_3 &= x_3[x_3 + (2 + a)x_1 - \mathbf{x}A\mathbf{x}]\ .
\end{aligned}$$

(b)

$$\begin{aligned}
\dot{h}(\mathbf{x}) &= \frac{1}{x_1x_2x_3}(\dot{x}_1x_2x_3 + x_1\dot{x}_2x_3 + x_1x_2\dot{x}_3) \\
&= \frac{\dot{x}_1}{x_1} + \frac{\dot{x}_2}{x_2} + \frac{\dot{x}_3}{x_3}
\end{aligned}$$

$$= (x_1 + x_2 + x_3) + (2 + a)(x_1 + x_2 + x_3) - 3\mathbf{x}A\mathbf{x}$$
$$= 3 + a - 3\mathbf{x}A\mathbf{x} \ .$$

(c) Since $1 = (x_1 + x_2 + x_3)^2 = ||\mathbf{x}||^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$, it follows that $\mathbf{x}A\mathbf{x} = 1 + a(x_1x_2 + x_1x_3 + x_2x_3) = 1 + \frac{a}{2}(1 - ||\mathbf{x}||^2)$, and hence $\dot{h}(\mathbf{x}) = \frac{a}{2}(3||\mathbf{x}||^2 - 1)$.
(d) Directly from (c).
(e) Follows from (d). If $a > 0$ then any trajectory converges to the maximum point of $x_1x_2x_3$, i.e. to $(1/3, 1/3, 1/3)$. If $a = 0$ then the trajectories are orbits ($x_1x_2x_3$ constant) around $(1/3, 1/3, 1/3)$. If $a < 0$ then the trajectories move outward, away from $(1/3, 1/3, 1/3)$.

Problems of Chapter 16

16.1 *Imputation Set of an Essential Game*
Note that $I(v)$ is a convex set and $\mathbf{f}^i \in I(v)$ for every $i = 1, \ldots, n$. Thus, $I(v)$ contains the convex hull of $\{\mathbf{f}^i \mid i \in N\}$. Now let $\mathbf{x} \in I(v)$, and write $\mathbf{x} = (v(1), \ldots, v(n)) + (\alpha_1, \ldots, \alpha_n)$, where $\sum_{i \in N} \alpha_i = v(N) - \sum_{i \in N} v(i) =: \alpha$. Then $\mathbf{x} = \sum_{i \in N} (\alpha_i/a)\mathbf{f}^i$, so that $\mathbf{x}$ is an element of the convex hull of $\{\mathbf{f}^i \mid i \in N\}$.

16.2 *Convexity of the Domination Core*
Claim: For each $\mathbf{x} \in I(v)$ and $\emptyset \neq S \subseteq N$ we have

$$\exists \mathbf{z} \in I(v) : \mathbf{z} \operatorname{dom}_S \mathbf{x} \Leftrightarrow x(S) < v(S) \text{ and } x(S) < v(N) - \sum_{i \notin S} v(i) \ .$$

Proof. $\Rightarrow$: Let $\mathbf{z}$ satisfy the left hand side of the equivalence. Then $v(N) = z(N) = z(S) + \sum_{i \notin S} z_i \geq z(S) + \sum_{i \notin S} v(i)$ and $x(S) < z(S) \leq v(S)$, which imply the right hand side.
$\Leftarrow$: Assume the right hand side is true. Since $x(N \setminus S) > \sum_{i \notin S} v(i)$, we can take $j \notin S$ with $x_j > v(j)$. Let $\alpha := \min\{x_j - v(j), v(S) - x(S)\}$, and define $\mathbf{z}$ by $z_i = x_i + \alpha/|S|$ if $i \in S$, $z_j = x_j - \alpha$, and $z_i = x_i$ otherwise. Then $\mathbf{z} \operatorname{dom}_S \mathbf{x}$. This completes the proof of the claim.

Because of the claim, we have for $S \neq \emptyset$:

$$I(v) \setminus D(S) = \{\mathbf{x} \in I(v) \mid x(S) \geq v(S) \text{ or } x(S) \geq v(N) - \sum_{i \notin S} v(i)\}$$
$$= \{\mathbf{x} \in I(v) \mid x(S) \geq \min\{v(S), v(N) - \sum_{i \notin S} v(i)\}\}$$

and therefore $I(v) \setminus D(S)$ is a convex set. Hence

$$DC(v) = I(v) \setminus \bigcup_{S \neq \emptyset} D(S) = \bigcap_{S \neq \emptyset} (I(v) \setminus D(S))$$

is convex.

16.3 *Dominated Sets of Imputations*
(a) For any $x \in I(v)$ and any $i \in N$, $x_i \geq v(i)$. So there is no $z \in I(v)$ with $x_i < z_i \leq v(i)$. Hence, $D(i) = \emptyset$. Also, $x(N) = v(N)$ so it is not possible that $z_j > x_j$ for all $j \in N$, hence $D(N) = \emptyset$.
(b) In both games, $D(ij) = \{\mathbf{x} \in I(v) \mid x_i + x_j < v(ij)\}$, $i, j \in \{1, 2, 3\}$, $i \neq j$.

16.4 *The Domination Relation*
(a) Clearly, $\mathbf{x}\,\mathrm{dom}\,\mathbf{x}$ and $\mathbf{x}\,\mathrm{dom}_S\mathbf{x}$ are not possible, hence dom and dom_S are irreflexive.
Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in I(v)$ with $\mathbf{x}\,\mathrm{dom}_S\mathbf{y}\,\mathrm{dom}_S\mathbf{z}$. Then $x_i > y_i > z_i$ for all $i \in S$, and $x(S) \leq v(S)$. Hence $\mathbf{x}\,\mathrm{dom}_S\mathbf{z}$, so dom_S is transitive.
If $\mathbf{x}\,\mathrm{dom}_S\mathbf{y}$, then $x_i > y_i$ for all $i \in S$, so $\mathbf{y}\,\mathrm{dom}_S\mathbf{x}$ is not possible. Hence, dom_S is antisymmetric.
(b) $N = \{1, 2, 3, 4\}$, $v(N) = 8$, $v(12) = 6$, $v(34) = 6$, $v(S) = 0$ otherwise. Then $(3, 3, 1, 1)\,\mathrm{dom}_{\{1,2\}}(1, 1, 3, 3)$ whereas $(1, 1, 3, 3)\,\mathrm{dom}_{\{3,4\}}(3, 3, 1, 1)$.
(c) A trivial example is the one in (ii) with $\mathbf{z} = (3, 3, 1, 1)$. Other example: $N = \{1, \ldots, 6\}$, $v(N) = 10$, $v(12) = 6$, $v(34) = 6$, $v(56) = 5$, $v(S) = 0$ otherwise. Then

$$(3, 3, 1, 1, 1, 1)\,\mathrm{dom}_{\{1,2\}}(1, 1, 3, 3, 1, 1)\,\mathrm{dom}_{\{3,4\}}(1, 1, 1, 1, 3, 3)$$

but not $(1, 1, 1, 1, 3, 3)\,\mathrm{dom}(3, 3, 1, 1, 1, 1)$.

16.5 *Stable Sets in a Three-Person Game*
(a) Let $\mathbf{x} \in I(v)$. There are $i, j \in N$, $i \neq j$, with $x_i + x_j < 1$. Take $\mathbf{z} \in I(v)$ with $z_i > x_i$, $z_j > x_j$, and $z_i + z_j = 1$. Then $\mathbf{z}\,\mathrm{dom}_{\{i,j\}}\mathbf{x}$.
(b) Let $\mathbf{x} \in I(v) \setminus A$. Then there are $i \neq j$ with $x_i, x_j < 1/2$.
(c) Let $\mathbf{x} \in I(v) \setminus B$.

Case 1: $x_3 < c$. Note that $x_1 < 1 - c$ or $x_2 < 1 - c$ otherwise $x_1 + x_2 \geq 2 - 2c > 1$ which is impossible. If $x_1 < 1 - c$ then $(1 - c, 0, c)\,\mathrm{dom}_{\{1,3\}}\mathbf{x}$, and if $x_2 < 1 - c$ then $(0, 1 - c, c)\,\mathrm{dom}_{\{2,3\}}\mathbf{x}$.

Case 2: $x_3 > c$. Then $x_1 + x_2 = 1 - x_3 < 1 - c$. Take α, β with $\alpha + \beta = 1 - c$ and $\alpha > x_1$, $\beta > x_2$. Then $(\alpha, \beta, c)\,\mathrm{dom}_{\{1,2\}}\mathbf{x}$.

16.6 *Singleton Stable Set*
Let $\{\mathbf{x}\}$ be a one-element stable set and assume that $v(N) > \sum_{i \in N} v(i)$. Then there is a player j with $x_j > v(j)$. Take some other player k and an imputation $\mathbf{y}$ with $x_j > y_j$, $x_k < y_k$, and all other coordinates of $\mathbf{x}$ and $\mathbf{y}$ equal. Then $\mathbf{x}$ should dominate $\mathbf{y}$ but this is only possible through the coalition $\{j\}$. Hence, $x_j \leq v(j)$, a contradiction.

16.7 *A Glove Game*
(a) Let $\mathbf{x} = (x_1, x_2, x_3) \neq (0, 1, 0)$ be any imputation. Without loss of generality assume $x_1 + x_2 < 1$. Define $\varepsilon = 1 - x_1 - x_2$ (> 0). Then the imputation $(x_1 + \varepsilon/2, x_2 + \varepsilon/2, 0)$ dominates $\mathbf{x}$ via $\{1, 2\}$.

(b) The core and the domination core are both equal to $\{(0,1,0)\}$, cf. Theorem 16.12.
(c) The imputation $(1/3,1/3,1/3)$ (for instance) is not dominated by $(0,1,0)$, hence external stability is not satisfied.
(d) Consider any $0 \le \lambda, \lambda' \le 1/2$. Then, clearly, $(\lambda, 1-2\lambda, \lambda)$ does not dominate $(\lambda', 1-2\lambda', \lambda')$, since $\lambda > \lambda'$ implies $1-2\lambda < 1-2\lambda'$. (Note that, since $v(\{1,3\}) = 0$, domination is only possible via $\{1,2\}$ or $\{2,3\}$.) Hence B is internally stable.

Let $(x_1,x_2,x_3) \notin B$, then without loss of generality $x_1 < x_3$ and in particular also $x_1 < \frac{1}{2}$. Take $0 \le \lambda \le \frac{1}{2}$ with $x_1 < \lambda < \frac{1}{2}(x_1+x_3)$. Then $x_2 = 1 - x_1 - x_3 = 1 - 2\left(\frac{x_1+x_3}{2}\right) < 1-2\lambda$. Hence, $(\lambda, 1-2\lambda, \lambda)$ dominates (x_1,x_2,x_3) via $\{1,2\}$. So B is externally stable.

16.8 *Proof of Theorem 16.15*
Take $\mathbf{x}, \mathbf{y} \in \Delta^S$. Suppose that $\mathbf{x} \,\mathrm{dom}_T \mathbf{y}$ for some coalition T. Then $T \subsetneq S$, and hence $v(T) = 0$. This is a contradiction, hence internal stability is satisfied. If $\mathbf{x} \in I(v) \setminus \Delta^S$, then there is a player $j \notin S$ with $x_j > 0$. Define $\mathbf{y} \in \Delta^S$ by $y_i = x_i + y(N \setminus S)/|S|$ for all $i \in S$ and $y_i = 0$ for all $i \notin S$. Then $\mathbf{y} \,\mathrm{dom}_S \mathbf{x}$. This proves external stability.

16.9 *Example 16.16*
Let A denote the set in (16.3). We first show that A is internally stable for any $0 \le \alpha \le 1$. Consider $(x, x, 1-2x) \in A$, hence $\frac{\alpha}{2} \le x \le \frac{1}{2}$. It is sufficient to show that this imputation is not dominated by any other element of A.

First suppose it would be dominated by an element in A of the form $(y, y, 1-2y)$. If it is dominated via $\{1,2\}$, then $2y \le \alpha \le 2x$, a contradiction. If it is dominated via $\{1,3\}$, then $y > x$ and $1-2y > 1-2x$, again a contradiction. Likewise, it can not be dominated via $\{2,3\}$.

Next, suppose it would be dominated by an element of A of the form $(y, 1-2y, y)$. If it is dominated via $\{1,2\}$ then $y > x \ge \frac{\alpha}{2}$ and $1-2y > x \ge \frac{\alpha}{2}$, hence $y + (1-2y) > \alpha$, a contradiction. If it is dominated via $\{1,3\}$, then $2y \le \alpha$ and so $x < y \le \frac{\alpha}{2}$, a contradiction. If it is dominated via $\{2,3\}$, then $1-2y > x \ge \frac{\alpha}{2}$ and $y \ge \frac{\alpha}{2}$, so $y + (1-2y) > \alpha$, a contradiction.

The case where $(x, x, 1-2x)$ would be dominated by an element of the form $(1-2y, y, y)$ leads to a contradiction in the same way. So we have proved that A is internally stable for any $0 \le \alpha \le 1$.

We next show that A is externally stable whenever $\alpha \ge \frac{2}{3}$. Let $(x_1,x_2,x_3) \in I(v) \setminus A$. If there are at least two coordinates, say x_1, x_2, smaller than $\frac{\alpha}{2}$, then (x_1,x_2,x_3) is dominated by $(\frac{\alpha}{2}, \frac{\alpha}{2}, 1-\alpha)$ via $\{1,2\}$. Otherwise, w.l.o.g., $x_1, x_2 \ge \frac{\alpha}{2}$ and (say) $x_2 < x_1$. Then $x_2 < \frac{1-x_3}{2}$. Choose $x_2 < y < \frac{1-x_3}{2}$. Then $1-2y > x_3$. Moreover, $y + (1-2y) = 1-y < 1-x_2 \le 1-\frac{\alpha}{2} \le \alpha$, where the last inequality follows from $\alpha \ge \frac{2}{3}$. So $(y, y, 1-2y)$ dominates (x_1,x_2,x_3) via $\{2,3\}$, and $(y,y,1-2y) \in A$.

Let now $\alpha < \frac{2}{3}$. We show that A is not externally stable. Note that in this case $C(v) = \{(x_1, x_2, x_3) \in I(v) \mid x_1, x_2, x_3 \leq 1 - \alpha\}$. Take any $\mathbf{x} \in C(v) \setminus A$, then $\mathbf{x}$ is undominated since $C(v) \subseteq DC(v)$. So A is not externally stable.

We prove that $A \cup C(v)$ is a stable set whenever $\alpha < \frac{2}{3}$. For internal stability, it suffices to show that an $\mathbf{x} \in A \setminus C(v)$ is not dominated by anything in the core. W.l.o.g. let $\mathbf{x} = (x, x, 1-2x)$, hence, since $1 - 2x \leq 1 - \alpha$, we have $x > 1 - \alpha$. Then $(x, x, 1-2x)$ is not dominated by any $(y_1, y_2, y_3) \in C(v)$, since $y_1, y_2, y_3 \leq 1 - \alpha$ whereas $x > 1 - \alpha$. This shows internal stability of A.

To show external stability of $A \cup C(v)$, observe that the complement of this set in $I(v)$ consists of three pairwise disjoint subsets of $I(v)$ each of the form $\{(x_1, x_2, x_3) \in I(v) \mid x_i > \max\{1 - \alpha, x_j, x_k\}\}$ for different i, j, k. W.l.o.g. let $i = 1$, $j = 2$, $k = 3$ and take (x_1, x_2, x_3) in the associated subset. Then either there is an $\varepsilon > 0$ such that $x_1 - 2\varepsilon = 1 - \alpha$ and $x_2, x_3 \leq 1 - \alpha$; or there is an $\varepsilon > 0$ with $x_1 - 2\varepsilon = x_3 + \varepsilon > 1 - \alpha$; or there is an $\varepsilon > 0$ with $x_1 - 2\varepsilon = x_2 + \varepsilon > 1 - \alpha$. In each case, $(x_1 - 2\varepsilon, x_2 + \varepsilon, x_3 + \varepsilon)$ dominates (x_1, x_2, x_3) via $\{2, 3\}$; in the first case $(x_1 - 2\varepsilon, x_2 + \varepsilon, x_3 + \varepsilon) \in C(v)$ and in the last two cases $(x_1 - 2\varepsilon, x_2 + \varepsilon, x_3 + \varepsilon) \in A$. This shows external stability of $A \cup C(v)$.

Finally, for $\alpha \leq \frac{1}{2}$ we have $A \subseteq C(v)$. Clearly, $C(v)$ is internally stable, and external stability follows in the same way as in the preceding paragraph. Hence, the core is a stable set, and therefore the unique stable set, cf. Theorem 16.17, which also holds for the core instead of the D-core (cf. Problem 16.10).

16.10 *Proof of Theorem 16.17*

(a) Let A be a stable set and suppose $\mathbf{x} \in DC(v) \setminus A$. Then $\mathbf{x}$ is dominated by some element of A, contradiction.

(b) Let $A \subsetneq B$ and let A be a stable set. Take $x \in B \setminus A$. Then $y \operatorname{dom} x$ for some $y \in A$, hence $y \in B$. So B is not internally stable and, therefore, not a stable set.

(c) Follows from (a) and (b).

16.11 *Core and D-Core*

Condition (16.1) is not a necessary condition for equality of the core and the D-core. To find a counterexample, first note that if $C(v) \neq \emptyset$ then (16.1) must hold. Therefore, a counterexample involves a game with empty core and D-core. Take the following game for $n \geq 3$. If $|S| = 2$ then let $v(S) = 2$, let $v(N) = 1$, and let $v(T) = 0$ for all other coalitions T. Then $\emptyset = C(v) = DC(v)$ but (16.1) is not fulfilled.

16.12 *Strategic Equivalence*

Straightforward using the definitions.

16.13 *Proof of Theorem 16.20*

Write $B = \begin{pmatrix} A \\ -A \end{pmatrix}$. Then

$$\begin{aligned}\max\{\mathbf{b}\cdot\mathbf{y} \mid A\mathbf{y}=\mathbf{c},\mathbf{y}\geq\mathbf{0}\} &= \max\{\mathbf{b}\cdot\mathbf{y} \mid B\mathbf{y}\leq(\mathbf{c},-\mathbf{c}),\mathbf{y}\geq\mathbf{0}\}\\ &= \min\{(\mathbf{c},-\mathbf{c})\cdot(\mathbf{x},\mathbf{z}) \mid (\mathbf{x},\mathbf{z})B\geq\mathbf{b},(\mathbf{x},\mathbf{z})\geq\mathbf{0}\}\\ &= \min\{\mathbf{c}\cdot(\mathbf{x}-\mathbf{z}) \mid (\mathbf{x}-\mathbf{z})A\geq\mathbf{b},(\mathbf{x},\mathbf{z})\geq\mathbf{0}\}\\ &= \min\{\mathbf{c}\cdot\mathbf{x}' \mid \mathbf{x}'A\geq\mathbf{b}\}\ .\end{aligned}$$

The second equality follows from Theorem 22.6.

16.14 *Infeasible Programs in Theorem 16.20*
Follow the hint.

16.15 *Proof of Theorem 16.22 Using Lemma 22.5*
List the non-empty coalitions $S\subseteq N$ as $S_1,\ldots,S_p$ ($p=2^n-1$) with $S_p=N$. Define the $(n+n+p)\times p$ matrix A as follows. Column $k<p$ is $(\mathbf{e}^{S_k},-\mathbf{e}^{S_k},-\mathbf{e}^k)$ where: $\mathbf{e}^{S_k}\in\mathbb{R}^n$, $e_i^{S_k}=1$ if $i\in S_k$, $e_i^{S_k}=0$ if $i\notin S_k$. Column p is $(\mathbf{e}^N,-\mathbf{e}^N,\mathbf{0})$. Then $C(N,v)\neq\emptyset$ iff there exists $(\mathbf{z},\mathbf{z}',\mathbf{w})\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^p$ with $(\mathbf{z},\mathbf{z}',\mathbf{w})\geq\mathbf{0}$ and $(\mathbf{z},\mathbf{z}',\mathbf{w})A=\mathbf{b}$, where $\mathbf{b}=(v(S_k))_{k=1}^p$. This has the form as in (a) of Lemma 22.5. The associated system in (b) of Lemma 22.5 is as follows: there is a $\mathbf{y}\in\mathbb{R}^p$ with $A\mathbf{y}\geq\mathbf{0}$ and $\mathbf{b}\cdot\mathbf{y}<0$. Hence, for such a $\mathbf{y}$ we have $\sum_{S:i\in S}y_S\geq 0$ and $-\sum_{S:i\in S}y_S\geq 0$ for all $i\in N$. Thus, $\sum_{S:i\in S}y_S=0$ for all $i\in N$ or, equivalently, $y_N+\sum_{S:i\in S,S\neq N}y_S=0$ for all $i\in N$. Further, $-y_S\geq 0$ for all $S\neq N$, hence $y_S\leq 0$ for all $S\neq N$. Also, $\mathbf{b}\cdot\mathbf{y}<0$ implies $y_Nv(N)+\sum_{S\subset N,S\neq N}y_Sv(S)<0$.

Now suppose that λ is a balanced map. Define $\mathbf{y}\in\mathbb{R}^p$ by $y_N=1-\lambda(N)$ and $y_S=-\lambda(S)$ for all $S\neq N$. Then for every $i\in N$ we have $\sum_{S:i\in S}y_S=\sum_{S:i\in S,S\neq N}-\lambda(S)+1-\lambda(N)=0$ since λ is balanced. Also, $y(S)=-\lambda(S)\leq 0$ for all $S\neq N$. Thus, this vector $\mathbf{y}$ satisfies the conditions established above.

Now we have:

$C(N,v)\neq\emptyset\Leftrightarrow$ (a) in Farkas' Lemma is true
$\Leftrightarrow$ (b) in Farkas' Lemma is not true
$\Leftrightarrow$ for every balanced map λ we have $y_Nv(N)+\sum_{S\neq N}y_Sv(S)\geq 0$ for $\mathbf{y}$ associated with λ as above
$\Leftrightarrow\sum_{S\neq N}-\lambda(S)v(S)+(1-\lambda(N))v(N)\geq 0$ for every balanced map λ
$\Leftrightarrow\sum_{S\subset N}\lambda(S)v(S)\leq v(N)$ for every balanced map λ.

16.16 *Balanced Maps and Collections*
(a) Let λ be a balanced map with associated balanced collection B. Since, for all $i\in N$, we have $\sum_{S:i\in S}\lambda(S)=1$, it follows that $\sum_{S\in 2^N:\ S\neq\emptyset}\lambda(S)\geq 1$. In particular, $\sum_{S\in 2^N:\ S\neq\emptyset}\lambda(S)=1\Leftrightarrow\forall i\in N,S\in B\ [i\in S]\Leftrightarrow B=\{N\}$.
(b) Define the map λ^c by

$$\lambda^c(T)=\frac{\lambda(N\setminus T)}{\sum_{S\in B}\lambda(S)-1}$$

for all $T\in B^c$, and $\lambda^c(T)=0$ otherwise. Then, for every $i\in N$,

$$\sum_{T:\ i\in T}\lambda^c(T)=\frac{\sum_{S:\ i\notin S}\lambda(S)}{\sum_{S\in B}\lambda(S)-1}=1\ .$$

(c) Follows directly from (b), since $\{N \setminus S, \{i\} \mid i \in S\}$ is a partition and therefore a balanced collection. The weights for $\{S, N \setminus \{i\} \mid i \in S\}$ are equal to $1/|S|$.
(d) If λ and λ' are two balanced maps and $0 \leq \alpha \leq 1$ then for each $i \in N$:

$$\sum_{S:\ i \in S} (\alpha\lambda + (1-\alpha)\lambda')(S) = \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1 .$$

16.17 *Minimum of Balanced Games*
Follows by using the definition of balancedness or by Theorem 16.22.

16.18 *Balanced Simple Games*
Let (N, v) be a simple game.

Suppose i is a veto player. Let B be a balanced collection with balanced map λ. Then

$$\sum_{S \in B} \lambda(S)v(S) = \sum_{S \in B: i \in S} \lambda(S)v(S) \leq 1 = v(N) ,$$

since i is a veto player. Hence, v is balanced.

For the converse, suppose v is balanced. We distinguish two cases.
Case 1: There is an i with $v(\{i\}) = 1$. In this case, take $S \subset N \setminus \{i\}$, and define λ by $\lambda(\{i\}) = 1$, $\lambda(S) = 1$, and $\lambda(N \setminus (S \cup \{i\})) = 1$, and $\lambda(T) = 0$ for every other nonempty coalition T. Then λ is balanced, and thus $\lambda(\{i\})v(\{i\}) + \lambda(S)v(S) + \lambda(N \setminus (S \cup \{i\}))v(N \setminus (S \cup \{i\})) \leq 1$. This implies $v(S) = 0$, hence i is a veto player.
Case 2: $v(\{i\}) = 0$ for every $i \in N$. In this case, suppose there are no veto players. Then there are nonempty coalitions $S_1, \ldots, S_m$ such that $v(S_j) = 1$ for each $j = 1, \ldots, m$ and for every player $i \in N$ there is an S_j with $i \notin S_j$. Let, for every $i \in N$, $n_i := |\{j \in \{1, \ldots, m\} \mid i \in S_j\}|$, then $n_i \leq m - 1$. Define λ by $\lambda(S_j) = \frac{1}{m-1}$ for each $j = 1, \ldots, m$, $\lambda(\{i\}) = 1 - \frac{n_i}{m-1}$ for every $i \in N$, and $\lambda(T) = 0$ for every other coalition T. Then for each $i \in N$ we have

$$\sum_{S: i \in S} \lambda(S) = \left(1 - \frac{n_i}{m-1}\right) + \frac{n_i}{m-1} = 1 ,$$

so that λ is a balanced map. Hence,

$$1 = v(N) \geq \sum_{S \neq \emptyset} \lambda(S)v(S) \geq \sum_{j=1}^{n} \lambda(S_j)v(S_j) = \frac{m}{m-1} > 1 ,$$

a contradiction. Thus, v has a veto player.

Problems of Chapter 17

17.1 *The Games 1_T*

(a) The system is complete (for each v we can write $v = \sum_{S \neq \emptyset} v(S)1_S$) and linearly independent ($\sum_{S \neq \emptyset} \alpha_S 1_S = 0$ implies $\alpha_S = 0$ for all S).
(b) Let $T \neq \emptyset$ and $i \in N$. If $i \in T$ then $1_T((T \setminus \{i\}) \cup \{i\}) - 1_T(T \setminus \{i\}) = 1 - 0 \neq 0$ and if $i \notin T$ then $1_T(T \cup \{i\}) - 1_T(T) = 0 - 1 \neq 0$. So 1_T has no null-player.
(c) For $i \notin T$:

$$\Phi_i(1_T) = \frac{|T|!(n-|T|-1)!}{n!}[1_T(T \cup \{i\}) - 1_T(T)] = -\frac{|T|!(n-|T|-1)!}{n!}$$

and for $i \in T$:

$$\begin{aligned}\Phi_i(1_T) &= \frac{(|T|-1)!(n-|T|)!}{n!}[1_T((T \setminus \{i\}) \cup \{i\}) - 1_T(T \setminus \{i\})] \\ &= \frac{(|T|-1)!(n-|T|)!}{n!} .\end{aligned}$$

17.2 *Unanimity Games*
(a) Suppose $\sum_{T \neq \emptyset} \alpha_T u_T = 0$ (where 0 means the zero-game) for some $\alpha_T \in \mathbb{R}$. Then $\sum_{T \neq \emptyset} \alpha_T u_T(\{i\}) = 0$ implies $\alpha_{\{i\}} = 0$, for each i. Next, $\sum_{T \neq \emptyset,\ |T| \geq 2} \alpha_T u_T(\{i,j\}) = 0$ implies $\alpha_{\{i,j\}} = 0$, for all i, j. Etc. Hence, all α_T are zero, so that $\{u_T \mid T \neq \emptyset\}$ is a linearly independent system.
(b) Let $W \in 2^N$ then

$$\begin{aligned}\sum_{T \neq \emptyset} c_T u_T(W) &= \sum_{T \neq \emptyset} \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) u_T(W) \\ &= \sum_{T \neq \emptyset, T \subseteq W} \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) \\ &= \sum_{S \subseteq W} v(S) \sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|} \\ &= v(W) + \sum_{S:\ S \subsetneq W} v(S) \sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|} .\end{aligned}$$

It is sufficient to show that the second term of the last expression is equal to 0, hence that $\sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|} = 0$. We can write

$$\sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|} = \sum_{k=0}^{|W|-|S|} (-1)^k \binom{|W|-|S|}{k} .$$

This last expression is of the form $\beta = \sum_{k=0}^{m} (-1)^k \binom{m}{k}$. By the binomial formula $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$, it follows that $\beta = 0$ by taking $a = 1$ and $b = -1$.

17.3 *If-Part of Theorem 17.4*
EFF: every marginal vector m^σ is efficient, so the average of these vectors is also efficient. NP: for a null-player i, $v(S \cup i) - v(S) = 0$ for every S, hence $\Phi_i(v) = 0$. ADD: follows from the fact that $(v + w)(S \cup i) - (v + w)(S) = v(S \cup i) - v(S) + w(S \cup i) - w(S)$ for all v, w, S, i.
SYM: Let i, j be symmetric in v. Note that for $S \subseteq N$ with $i \notin S$ and $j \in S$ we have $v((S \cup i) \setminus j) = v(S)$ by symmetry of i and j, since $v((S \cup i) \setminus j) = v((S \setminus j) \cup i)$ and $v(S) = v((S \setminus j) \cup j$. Write $\gamma(|S|) = \frac{|S|!(n-|S|-1)!}{n!}$, then

$$\begin{aligned}
\Phi_i(v) &= \sum_{S:i,j\notin S} \gamma(|S|)[v(S \cup i) - v(S)] + \sum_{S:i\notin S, j\in S} \gamma(|S|)[v(S \cup i) - v(S)] \\
&= \sum_{S:i,j\notin S} \gamma(|S|)[v(S \cup j) - v(S)] + \sum_{S:i\notin S, j\in S} \gamma(|S|)[v(S \cup i) - v((S \cup i) \setminus j)] \\
&= \sum_{S:i,j\notin S} \gamma(|S|)[v(S \cup j) - v(S)] + \sum_{T:i\in T, j\notin T} \gamma(|T|)[v(T \cup j) - v(T)] \\
&= \Phi_j(v)\ .
\end{aligned}$$

17.4 *Dummy Player Property and Anonymity*
That DUM implies NP and the Shapley value satisfies DUM is straightforward.
AN implies SYM: Let i and j be symmetric players, and let the value ψ satisfy AN. Consider the permutation σ with $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ otherwise. Since i and j are symmetric players it follows easily that $v = v^\sigma$. Then, by AN, $\psi_i(v) = \psi_{\sigma(i)}(v^\sigma) = \psi_j(v)$, proving SYM of ψ.

17.5 *Shapley Value, Core, and Imputation Set*
In the case of two players the core and the imputation set coincide. If the core is not empty then the Shapley value is in the core, cf. Example 17.2. In general, consider any game with $v(1) = 2$, $v(N) = 3$, and $v(S) = 0$ otherwise. Then $\Phi_1(v) = 5/n$, hence the Shapley value is not even in the imputation set as soon as $n \geq 3$.

17.6 *Shapley Value as a Projection*
If a is an additive game then $\Phi(a) = (a(1), a(2), \ldots, a(n))$. For a general game v let a^v be the additive game generated by $\Phi(v)$. Then $\Phi(a^v) = (a^v(1), \ldots, a^v(n)) = \Phi(v)$.

17.7 *Shapley Value of Dual Game*
Let $v = \sum_{T\neq\emptyset} \alpha_T u_T$. Then for any coalition S, $\sum_{T\neq\emptyset} \alpha_T u_T^*(S) = \sum_{T\neq\emptyset} \alpha_T[u_T(N) - u_T(N \setminus S)] = \sum_{T\neq\emptyset} \alpha_T u_T(N) - \sum_{T\neq\emptyset} \alpha_T u_T(N \setminus S) = v(N) - v(N \setminus S) = v^*(S)$. Hence, $v^* = \sum_{T\neq\emptyset} \alpha_T u_T^*$.

Take T arbitrary. For all players $i, j \in T$, $i \neq j$, and $S \subseteq N \setminus \{i, j\}$, we have $u_T^*(S \cup \{i\}) = 1 - u_T(S \cup \{i\}) = 1$ and $u_T^*(S \cup \{j\}) = 1 - u_T(S \cup \{j\}) = 1$, hence i and j are symmetric. Also, every player $i \notin T$ is a null player. Hence, since $u_T^*(N) = 1$, efficiency, symmetry and null player of the Shapley value imply $\Phi(\alpha_T u_T) = \Phi(\alpha_T u_T^*)$. By additivity, $\Phi(v) = \Phi(v^*)$.

This can also be proved directly, by using (17.4), as follows. For $i \in N$,

$$\begin{aligned}\Phi_i(v^*) &= \sum_{S:\ i\notin S} \gamma_S[v(N) - v(N \setminus (S \cup i)) - v(N) + v(N \setminus S)] \\ &= \sum_{S:\ i\notin S} \gamma_S[v(N \setminus S) - v(N \setminus (S \cup i))] \\ &= (*)\end{aligned}$$

where $\gamma_S = (|S|!(n-|S|-1)!/n!)$. Write $N \setminus S = T \cup i$, hence $N \setminus (S \cup i) = T$, and observe that $\gamma_T = \gamma_S$. Then

$$(*) = \sum_{T:\ i\notin T} \gamma_T[v(T \cup i) - v(T)] = \Phi_i(v)\ .$$

17.8 *Multilinear Extension*
(a) Let $c_T := \sum_{S: S\subseteq T}(-1)^{|T|-|S|}v(S)$. Let $T \in 2^N$, $T \neq \emptyset$. The product $\prod_{i\in T} x_i$ only occurs in the expression $\prod_{i\in S} x_i \prod_{i\in N\setminus S}(1 - x_i)$ if $S \subseteq T$, since each term in this expression contains the factor $\prod_{i\in S} x_i$. So let $S \subseteq T$, then the factor $\prod_{i\in T\setminus S} x_i$ occurs in $\prod_{i\in N\setminus S}(1 - x_i)$ with sign $(-1)^{|T\setminus S|} = (-1)^{|T|-|S|}$. Altogether, the factor $\prod_{i\in T} x_i$ occurs in the right hand side of formula (17.14) with coefficients $(-1)^{|T|-|S|}v(S)$ for every $S \subseteq T$. So $f(\mathbf{x}) = \sum_{T\subseteq N}\sum_{S\subseteq T}(-1)^{|T|-|S|}v(S)\left(\prod_{i\in T} x_i\right) = \sum_{T\subseteq N}\left(\prod_{i\in T} x_i\right)$.
(b) Let g be another multilinear extension of $\tilde{v}$ to $[0,1]^n$, say $g(\mathbf{x}) = \sum_{T\subseteq N} b_T\left(\prod_{i\in T} x_i\right)$. Since $g(\mathbf{e}^S) = f(\mathbf{e}^S) = v(S)$ for every $S \in 2^N$, we obtain:
$S = \{i\} \Rightarrow g(\mathbf{e}^S) = b_{\{i\}} = c_{\{i\}} = f(\mathbf{e}^S)$ for all $i \in N$,
$S = \{i,j\} \Rightarrow g(\mathbf{e}^S) = b_{\{i\}} + b_{\{j\}} + b_{\{i,j\}} = c_{\{i\}} + c_{\{j\}} + c_{\{i,j\}} \Rightarrow b_{\{i,j\}} = c_{\{i,j\}}$ for all $i \neq j$, etc.
So $b_T = c_T$ for all T, whence the uniqueness of the multilinear extension.

17.9 *The Beta-Integral Formula*

$$\begin{aligned}\int_0^1 t^{|S|}(1-t)^{n-|S|-1}dt &= -\frac{t^{|S|}(1-t)^{n-|S|}}{n-|S|}\Big|_0^1 \\ &\quad + \frac{|S|}{n-|S|}\int_0^1 t^{|S|-1}(1-t)^{n-|S|}dt \\ &= \frac{|S|(|S|-1)}{(n-|S|)(n-|S|+1)}\int_0^1 t^{|S|-2}(1-t)^{n-|S|+1}dt \\ &\ \ \vdots \\ &= \frac{|S|!}{(n-|S|)\cdot(n-|S|+1)\cdot\ldots\cdot(n-1)} \\ &\quad \cdot \int_0^1 t^0(1-t)^{n-1}dt\end{aligned}$$

$$= \frac{|S|!}{(n-|S|)\cdot(n-|S|+1)\cdot\ldots\cdot(n-1)\cdot n}$$
$$= \frac{|S|!(n-|S|-1)!}{n!} .$$

Here, the first term after the first equality sign is equal to zero, and similar terms are omitted in the following lines. Partial integration has been applied repeatedly.

17.10 *Path Independence of* Φ
Let $\tau : N \to N$ be a permutation. Then, using Theorem 17.12(c),

$$\begin{aligned}
\Phi_{\tau(1)}(\{\tau(1)\},v) &= P(\{\tau(1)\},v) - P(\emptyset,v) , \\
\Phi_{\tau(2)}(\{\tau(1),\tau(2)\},v) &= P(\{\tau(1),\tau(2)\},v) - P(\{\tau(1)\},v) , \\
\Phi_{\tau(3)}(\{\tau(1),\tau(2),\tau(3)\},v) &= P(\{\tau(1),\tau(2),\tau(3)\},v) - P(\{\tau(1),\tau(2)\},v) , \\
&\vdots \\
\Phi_{\tau(n)}(N,v) &= P(N,v) - P(\{\tau(1),\ldots,\tau(n-1)\},v) .
\end{aligned}$$

So $\sum_{k=1}^{n} \Phi_{\tau(k)}(\{\tau(1),\ldots,\tau(k)\},v) = P(N,v) - P(\emptyset,v) = P(N,v)$, which is independent of the permutation τ.

17.11 *An Alternative Characterization of the Shapley Value*
The Shapley value satisfies all these conditions. Conversely, (b)–(d) imply standardness for two-person games, so the result follows from Theorem 17.18.

Problems of Chapter 18

18.1 *Marginal Vectors and Dividends*
(a) This is straightforward from the definition of a dividend.
(b) For each $i \in N$, $m_i^\pi = v(P_\pi(i)\cup i) - v(P_\pi(i)) = \sum_{T\subseteq P_\pi(i)\cup i} \Delta_v(T) - \sum_{T\subseteq P_\pi(i)} \Delta_v(T) = \sum_{T\subseteq P_\pi(i)\cup i,\ i\in T} \Delta_v(T)$.

18.2 *Convexity and Marginal Vectors*
For the if-direction, note that the equalities imply that every marginal vector is in the core of the game: for $\pi \in \Pi(N)$ and non-empty coalition T we have $\sum_{i\in T} m_i^\pi(v) \geq v(T)$, hence $m^\pi(v) \in C(v)$. Hence, the Weber set is a subset of the core, and by Theorems 18.3 and 18.6 it follows that the game is convex.

For the only-if direction, if the game is convex then by Theorem 18.6 every marginal vector is in the core, so that for all $T \in 2^N \setminus \{\emptyset\}$:

$$v(T) \leq \sum_{i\in T} m_i^\pi(v) .$$

Since this inequality is an equality for any permutation π where the players of T come first, we are done.

18.3 *Strictly Convex Games*
Let π and σ be two different permutations and suppose that $k \geq 1$ is the minimal number such that $\pi(k) \neq \sigma(k)$. Then $m^{\pi}_{\pi(k)}(v) = v(\pi(1), \ldots, \pi(k-1), \pi(k)) - v(\pi(1), \ldots, \pi(k-1)) < v(P_\sigma(\pi(k)) \cup \pi(k)) - v(P_\sigma(\pi(k))) = m^{\sigma}_{\pi(k)}(v)$, where the inequality follows from strict convexity since $\{\pi(1), \ldots, \pi(k-1)\} \subsetneqq P_\sigma(\pi(k))$. Hence, $m^\pi \neq m^\sigma$.

18.4 *Sharing Profits*
(a) Let π be a permutation with $\pi(k) = 0$ then

$$m^{\pi}_{\pi(\ell)} = \begin{cases} 0 & \text{if } \ell < k \\ f(k-1) & \text{if } \ell = k \\ f(\ell-1) - f(\ell-2) & \text{if } \ell > k. \end{cases}$$

For the landlord: $\Phi_0(v) = \frac{1}{(n+1)!}\left[n!f(0) + n!f(1) + n!f(2) + \ldots + n!f(n)\right] = \frac{1}{n+1}\left[\sum_{s=0}^{n} f(s)\right]$.
For worker i: $\Phi_i(v) = \frac{1}{n}\left[v(N) - \Phi_0(v)\right] = \frac{nf(n)-f(0)-f(1)-\ldots-f(n-1)}{n(n+1)}$.
(b) $\mathbf{x} \in C(v)$ if and only if $\mathbf{x} \geq \mathbf{0}$ and $\sum_{i \in S} x_i \geq f(|S|-1)$ for every coalition S with $0 \in S$.
(c) Extend f to a piecewise linear function on $[0, n]$. Then v is convex if and only if f is convex.

18.5 *Sharing Costs*
(a) For every nonempty coalition S, $v(S) = \sum_{i \in S} c_i - \max\{c_i \mid i \in S\}$. If we regard $c = (c_1, \ldots, c_2)$ as an additive game we can write $v = c - c_{\max}$, where $c_{\max}(S) = \max\{c_i \mid i \in S\}$.
(b) For a coalition S and a player $i \notin S$, $v(S \cup i) - v(S) = \min\{c_i, c_{\max}(S)\}$. If T is another coalition with $S \subseteq T$ and $i \notin T$, then $v(S \cup i) - v(S) \leq v((T \cup i) - v(T)$ since $c_{\max}(S) \leq c_{\max}(T)$. Hence v is a convex game.
For each permutation π and each player $i \in N$, $m^\pi_i(v) = \min\{c_i, c_{\max}(P_\pi(i))\}$. Since the game is convex, both the Weber set and the core are equal to the convex hull of these marginal vectors, while the Shapley value is its barycenter. The Shapley value can also be computed as follows. First we compute the Shapley value of $c_{\max}$. Clearly, $\Phi_1(c_{\max}) = [(n-1)!/n!] \cdot c_1 = \frac{c_1}{n}$. Further,

$$\begin{aligned}
\Phi_2(c_{\max}) &= \frac{(n-1)!}{n!} c_2 + \frac{(n-2)!}{n!}(c_2 - c_1) \\
&= \frac{c_2}{n-1} - \frac{c_1}{n(n-1)} \\
\Phi_3(c_{\max}) &= \frac{(n-1)!}{n!} c_3 + \frac{(n-2)!}{n!}(c_3 - c_1) \\
&\quad + \frac{(n-2)!}{n!}(c_3 - c_2) + \frac{2!(n-3)!}{n!}(c_3 - c_2) \\
&= \frac{c_3}{n-2} - \frac{c_2}{(n-1)(n-2)} - \frac{c_1}{n(n-1)}
\end{aligned}$$

and in general

$$\Phi_i(c_{\max}) = \frac{c_i}{n-i+1} - \sum_{j=1}^{i-1} \frac{c_j}{(n-j+1)(n-j)} \ .$$

Hence,

$$\Phi_i(v) = c_i - \Phi_i(c_{\max}) = \frac{n-i}{n-i+1}\, c_i + \sum_{j=1}^{i-1} \frac{c_j}{(n-j+1)(n-j)} \ .$$

18.6 *Independence of the Axioms in Theorem 18.8*
(a) Consider the value which, for every game v, gives each dummy player his individual worth and distributes the rest, $v(N) - \sum_{i \in D} v(i)$ where D is the subset of dummy players, evenly among the players. This value satisfies all axioms except LIN.
(b) Consider the value which, for every game v, distributes $v(N)$ evenly among all players. This value satisfies all axioms except DUM.
(c) The value which gives each player his individual worth satisfies all axioms except EFF.
(d) Consider any set of weights $\{\alpha_\pi \mid \pi \in \Pi(N)\}$ with $\alpha_\pi \in \mathbb{R}$ for all π and $\sum_{\pi \in \Pi(N)} \alpha_\pi = 1$. The value $\sum_{\pi \in \Pi(N)} \alpha_\pi m^\pi$ satisfies LIN, DUM and EFF, but not MON unless all weights are nonnegative.

18.7 *Null-Player in Theorem 18.8*
Check that the dummy axiom in the proof of this theorem is only used for unanimity games. In those games, dummy players are also null-players, so it is sufficient to require NP. Alternatively, one can show that DUM is implied by ADD (and, thus, LIN), EFF and NP.

18.8 *Characterization of Weighted Shapley Values*
Check that every weighted Shapley value satisfies the Partnership axiom. Conversely, let ψ be a value satisfying the Partnership axiom and the four other axioms. Let $S^1 := \{i \in N \mid \psi_i(u_N) > 0\}$ and for every $i \in S^1$ let $\omega_i := \psi_i(u_N)$. Define, recursively, $S^k := \{i \in N \setminus (S^1 \cup \ldots \cup S^{k-1}) \mid \psi_i(u_{N \setminus (S^1 \cup \ldots \cup S^{k-1})}) > 0\}$ and for every $i \in S^k$ let $\omega_i := \psi_i(u_{N \setminus (S^1 \cup \ldots \cup S^{k-1})})$. This results in a partition $(S^1, \ldots, S^m)$ of N. Now define the weight system w by the partition $(S_1, \ldots, S_m)$ with $S_1 := S^m$, $S_2 := S^{m-1}$, …, $S_m := S^1$, and the weights $\boldsymbol{\omega}$. Then it is sufficient to prove that for each coalition S and each player $i \in S$ we have $\psi_i(u_S) = \Phi_i^w(u_S)$. Let $h := \max\{j \mid S \cap S_j \neq \emptyset\}$, then with $T = N \setminus (S_{h+1} \cup \ldots \cup S_m)$ we have by the Partnership axiom: $\psi_i(u_S) = \frac{1}{\psi(u_T)(S)} \psi_i(u_T)$. If $i \notin S_h$ then $\psi_i(u_T) = 0$, hence $\psi_i(u_S) = 0 = \Phi_i^w(u_S)$. If $i \in S_h$ then $\psi_i(u_S) = \frac{\omega_i}{\sum_{j \in S \cap S_h} \omega_j} = \Phi_i^w(u_S)$.

18.9 *Core and Weighted Shapley Values in Example 18.2*
First write the game as a sum of unanimity games:

$$v = u_{\{1,2\}} + u_{\{1,3\}} - u_{\{2,3\}} + 2u_N \;.$$

Then consider all possible ordered partitions of N, there are 13 different ones, and associated weight vectors. This results in a description of all payoff vectors associated with weighted Shapley values, including those in the core of the game.

Problems of Chapter 19

19.1 *Binary Relations*
Not (4): $\succeq$ on $\mathbb{R}$ defined by $x \succeq y \Leftrightarrow x^2 \geq y^2$.
Not (3): $\geq$ on $\mathbb{R}^2$.
Not (2): $\succeq$ on $\mathbb{R}$ defined by: for all x, y, $x \geq y$, let $x \succeq y$ if $x - y \geq 1$, and let $y \succeq x$ if $x - y < 1$.
Not (1): $>$ on $\mathbb{R}$.
The ordering on $\mathbb{R}$, defined by $[x \succeq y] \Leftrightarrow [x = y \text{ or } 0 \leq x, y \leq 1]$ is reflexive and transitive but not complete and not anti-symmetric.

19.2 *Linear Orders*
If $\mathbf{x} \succ \mathbf{y}$ then by definition $\mathbf{x} \succeq \mathbf{y}$ and not $\mathbf{y} \succeq \mathbf{x}$: hence $\mathbf{x} \neq \mathbf{y}$ since otherwise $\mathbf{y} \succeq \mathbf{x}$ by reflexivity.
If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then not $\mathbf{y} \succeq \mathbf{x}$ since otherwise $\mathbf{x} = \mathbf{y}$ by anti-symmetry. Hence $\mathbf{x} \succ \mathbf{y}$.

19.3 *The Lexicographic Order (1)*
Check (1)–(4) in Sect. 19.2 for $\succeq_{\text{lex}}$. Straightforward.

19.4 *The Lexicographic Order (2)*
This is the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid [x_1 = 3,\ x_2 \geq 1] \text{ or } [x_1 > 3]\}$. This set is not closed.

19.5 *Representability of Lexicographic Order (1)*
Consider Problem 19.4. Since $(\alpha, 0) \succeq_{\text{lex}} (3, 1)$ for all $\alpha > 3$, we have $u(\alpha, 0) \geq u(3, 1)$ for all $\alpha > 3$ and hence, by continuity of u, $\lim_{\alpha \downarrow 3} u(\alpha, 0) \geq u(3, 1)$. Therefore $(3, 0) \succeq_{\text{lex}} (3, 1)$, a contradiction.

19.6 *Representability of Lexicographic Order (2)*
Suppose that u represents $\succeq_{\text{lex}}$ on $\mathbb{R}^2$, that is, $\mathbf{x} \succeq_{\text{lex}} \mathbf{y}$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then for any $t \in \mathbb{R}$ let $q(t)$ be a rational number in the interval $[u(t, 0), u(t, 1)]$. Since $(t, \alpha) \succ_{\text{lex}} (s, \beta)$ and hence $u(t, \alpha) > u(s, \beta)$ for all $t > s$ and all $\alpha, \beta \in [0, 1]$, we have $[u(t, 0), u(t, 1)] \cap [u(s, 0), u(s, 1)] = \emptyset$ for all $t \neq s$. Hence, $q(t) \neq q(s)$ for all $t \neq s$. Therefore, we have found uncountably many different rational numbers, a contradiction.

19.7 *Single-Valuedness of the Pre-nucleolus*
Although the pre-nucleolus is defined with respect to the unbounded set $I^*(N, v)$, clearly the pre-nucleolus is equal to the nucleolus with respect to

some subset X of $I^*(N,v)$ of the form $X = \{\mathbf{x} \in I^*(N,v) \mid x_i \le M\}$ for some real number M large enough. Since X is compact and convex Theorem 19.3 applies.

19.8 *(Pre-)Nucleolus and Core*
For any core element all excesses are non-positive by definition of the core. This implies that for the (pre-)nucleolus all excesses must be non-positive as well, since otherwise any core element would have a smaller maximal excess. But then the (pre-)nucleolus is in the core itself. (Of course, nucleolus and pre-nucleolus must be equal since they are both equal to the nucleolus with respect to the core.)

19.9 *Kohlberg Criterion for the Nucleolus*
First observe that the following modification of Theorem 19.4 holds:

Theorem 19.4′ Let (N,v) be a game and $\mathbf{x} \in I(N,v)$. Then the following two statements are equivalent.

(1) $\mathbf{x} = \nu(N,v)$.
(2) *For every α such that $\mathcal{D}(\alpha,\mathbf{x},v) \neq \emptyset$ and for every side-payment $\mathbf{y}$ with $y(S) \ge 0$ for every $S \in \mathcal{D}(\alpha,\mathbf{x},v)$ and with $y_i \ge 0$ for all $i \in N$ with $x_i = v(i)$ we have $y(S) = 0$ for every $S \in \mathcal{D}(\alpha,\mathbf{x},v)$.*

The proof of this theorem is almost identical to the proof of Theorem 19.4. In the second sentence of the proof, note that $\mathbf{z}_\varepsilon \in I(N,v)$ for ε small enough. In the second part of the proof, (2)⇒(1), note that $y_i = z_i - x_i \ge 0$ whenever $x_i = v(i)$.

For the 'if'-part of the statement in this problem, let $\mathbf{x} \in I(N,v)$, $\mathcal{D}(\alpha,\mathbf{x},v) \neq \emptyset$, and $\mathcal{E}(\alpha,\mathbf{x},v)$ such that $\mathcal{D}(\alpha,\mathbf{x},v) \cup \mathcal{E}(\alpha,\mathbf{x},v)$ is balanced. Consider a side-payment $\mathbf{y}$ with $y(S) \ge 0$ for every $S \in \mathcal{D}(\alpha,\mathbf{x},v)$ and $y_i \ge 0$ for every i with $x_i = v(i)$ (hence in particular for every i with $\{i\} \in \mathcal{E}(\alpha,\mathbf{x},v)$). The argument in the first part of the proof of Theorem 19.5 now applies to $\mathcal{D}(\alpha,\mathbf{x},v) \cup \mathcal{E}(\alpha,\mathbf{x},v)$, and Theorem 19.4′ implies $\mathbf{x} = \nu(N,v)$.
For the 'only-if' part, consider the program (19.4) in the second part of the proof of Theorem 19.5 but add the constraints $-y_i \le 0$ for every $i \in N$ with $x_i = v(i)$. Theorem 19.4′ implies that the dual of this program is feasible, that is, there are $\lambda(S) \ge 0$, $S \in \mathcal{D}(\alpha,\mathbf{x},v)$, $\lambda(\{i\}) \ge 0$, i such that $x_i = v(i)$, and $\lambda(N) \in \mathbb{R}$ such that

$$-\sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}} - \sum_{S \in \mathcal{D}(\alpha,\mathbf{x},v)} \lambda(S)\mathbf{e}^S + \lambda(N)\mathbf{e}^N = \sum_{S \in \mathcal{D}(\alpha,\mathbf{x},v)} \mathbf{e}^S \, .$$

Hence $\lambda(N)\mathbf{e}^N = \sum_{S \in \mathcal{D}(\alpha,\mathbf{x},v)} (1 + \lambda(S))\mathbf{e}^S + \sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}}$. Let $\mathcal{E}(\alpha,\mathbf{x},v)$ consist of those one-person coalitions $\{i\}$ with $x_i = v(i)$ and $\lambda(\{i\}) > 0$, then $\mathcal{D}(\alpha,\mathbf{x},v) \cup \mathcal{E}(\alpha,\mathbf{x},v)$ is balanced.

19.10 *Proof of Theorem 19.5*

To formulate the dual program, use for instance the formulation in Theorem 16.20. For instance, the primal (19.4) can be converted to the minimization problem in Theorem 16.20; then the dual corresponds to the maximization problem in Theorem 16.20. Feasibility of the dual follows from Problem 16.14.

19.11 *Nucleolus of a Three-Person Game (1)*
At the imputation $(5, 4, 3)$ the excesses of the one-person coalitions are equal to -1 and the excesses of the two-person coalitions are equal to -5, so this is clearly the nucleolus.

19.12 *Nucleolus of a Three-Person Game (2)*
The core of this game is non-empty, so the nucleolus and pre-nucleolus coincide. First, we solve the minimization problem: $\min \alpha$ subject to: $x_1 + \alpha \geq 0$, $x_2 + \alpha \geq 0$, $x_3 + \alpha \geq 1$, $x_1 + x_2 + \alpha \geq 7$, $x_1 + x_3 + \alpha \geq 5$, $x_2 + x_3 + \alpha \geq 3$, $x_1 + x_2 + x_3 = 10$. The solution is: $\alpha = -1$, $\mathcal{B}_1 = \{3, 12\}$, $X_1 = \{\mathbf{x} \in I(v) \mid x_3 = 2, x_1 + x_2 = 8, x_1 \geq 4, x_2 \geq 2\}$.

Next, we solve the problem: $\min \alpha$ subject to: $x_1 + \alpha \geq 0$, $x_2 + \alpha \geq 0$, $x_1 + x_2 + \alpha \geq 7$, $x_1 + \alpha \geq 3$, $x_2 + \alpha \geq 1$, $\mathbf{x} \in X_1$. The solution is: $\alpha = -2$, $\mathcal{B}_2 = \{13, 23\}$, $X_2 = \{(5, 3, 2)\}$. So $\nu(v) = \nu^*(v) = (5, 3, 2)$.

19.13 *Nucleolus of a Two-Person Game*
$(v(1) + (v(12) - v(1) - v(2))/2, v(2) + (v(12) - v(1) - v(2))/2)$.

19.14 *Individual Rationality Restrictions for the Nucleolus*
The nucleolus is $(1, 0, 0)$: this is easily checked using the Kohlberg criterion in Problem 19.9 by adding coalitions $\{2\}$ and $\{3\}$ to the coalitions $\{1, 2\}$ and $\{1, 3\}$, which have the highest excess.
The pre-nucleolus (found by trying a payoff vector of the form $(1 - 2\alpha, \alpha, \alpha)$ with $\alpha < 0$) is $(5/3, -1/3, -1/3)$. Again, this is easily verified by the Kohlberg criterion, Theorem 19.4.

19.15 *Example 19.7*
The set $\mathcal{B}_1 = \{123, 124, 34\}$ is balanced with weights all equal to $1/2$. The set $\mathcal{B}_1 \cup \mathcal{B}_2 = \{123, 124, 34, 134, 234\}$ is balanced with weights, respectively, equal to $5/12$, $5/12$, $3/12$, $2/12$, $2/12$.

19.16 *(Pre-)Nucleolus of a Symmetric Game*
(a) Let $\mathbf{z} = (v(N)/n)\mathbf{e}^N$. Let $\alpha_1, \ldots, \alpha_p$ be the excesses at $\mathbf{z}$ such that $\alpha_1 > \ldots > \alpha_p$. Then for every $j = 1, \ldots, p$ there is an $M \subseteq N$ such that $=$ $\mathcal{D}(\alpha_j, \mathbf{z}, v) = \cup_{k \in M} \{S \subseteq N \mid |S| = k\}$. Hence $\mathcal{D}(\alpha_j, \mathbf{z}, v)$ is a union of balanced sets and therefore itself balanced. ($\{S \mid |S| = k\}$ is balanced with weights $\binom{n-1}{k-1}^{-1}$.) Thus, $\mathbf{z} = \nu^*(v)$ by the Kohlberg criterion, and since $\mathbf{z} \in I(v)$ it is also the nucleolus.
(b) The maximal excess is reached (at least) for all coalitions of some same size, say $s < n$. The equations $v(S) - (s/n)v(N) = v(S) - \sum_{i \in S} x_i$ for all S with $|S| = s$ determine $\mathbf{x}$ uniquely: the (pre-)nucleolus is a solution, and there are n independent equations. Hence X_1 consists of a unique element.

19.17 *COV and AN of the Pre-nucleolus*
Covariance of the pre-nucleolus follows since applying a transformation as in the definition of this property changes all excesses (only) by the same positive (multiplicative) factor.
Anonymity of the pre-nucleolus follows since a permutation of the players does not change the ordered vectors $\theta(\mathbf{x})$, but only permutes the coalitions to which the excesses correspond.

19.18 *Apex Game*
Try a vector of the form $(1-4\alpha,\alpha,\alpha,\alpha,\alpha)$. Equating the excesses of $N\setminus\{1\}$ with the excesses of coalitions of the form $\{1,j\}$ for $j\neq 1$ gives $\alpha=1/7$. Hence, the (pre-)nucleolus is $(3/7,1/7,1/7,1/7,1/7)$. This can easily be verified using the Kohlberg criterion.

19.19 *Landlord Game*
(a) By anonymity, each worker is assigned $\frac{1}{2}[f(n)-f(n-1)]$. By computing the excesses, it follows that among all coalitions containing the landlord, with this payoff vector the maximal excesses are reached by the coalitions containing $n-1$ workers, and further also by the coalitions consisting of a single worker and not the landlord. By the Kohlberg criterion this immediately implies that the given vector is the (pre-)nucleolus. For the Shapley value, see Problem 18.4.
(b) Compute the excesses for the payoff vector $\frac{f(n)}{n+1}\mathbf{e}^{\{0,1,\ldots,n\}}$, and apply the Kohlberg criterion.

19.20 *Game in Sect. 19.1*
The first linear program is: minimize α subject to the constraints $x_i+\alpha\geq 4$ for $i=1,2,3$, $x_1+x_2+\alpha\geq 8$, $x_1+x_3+\alpha\geq 12$, $x_2+x_3+\alpha\geq 16$, $x_1+x_2+x_3=24$. The program has optimal value $\alpha=-2$, reached for $x_1=6$ and $x_2,x_3\geq 6$.

In the second program x_1 has been eliminated. This program reduces to: minimize α subject to $x_2+\alpha\geq 4$, $x_2\leq 12+\alpha$, $x_2+x_3=18$. This has optimal value $\alpha=-4$, reached for $x_2=8$, $x_3=10$.

19.21 *The Prekernel*
For $i,j\in N$ denote by $\mathcal{T}_{ij}$ those coalitions that contain player i and not player j. For a payoff vector $\mathbf{x}$ denote by $s_{ij}(\mathbf{x},v)$ the maximum of $e(S,\mathbf{x},v)$ over all $S\in\mathcal{T}_{ij}$.

Let now $\mathbf{x}$ be the pre-nucleolus and suppose, contrary to what has to be proved, that there are two distinct players k,ℓ such that $s_{k\ell}(\mathbf{x},v)>s_{\ell k}(\mathbf{x},v)$. Let $\delta=(s_{k\ell}(\mathbf{x},v)-s_{\ell k}(\mathbf{x},v))/2$ and define $\mathbf{y}$ by $y_k=x_k+\delta$, $y_\ell=x_\ell-\delta$, and $y_i=x_i$ for all $i\neq k,\ell$. Denote $\mathcal{S}=\{S\in 2^N\setminus\mathcal{T}_{k\ell}\mid e(S,\mathbf{x},v)\geq s_{kl}(\mathbf{x},v)\}$ and $s=|\mathcal{S}|$. Then $\theta_{s+1}(\mathbf{x})=s_{k\ell}(\mathbf{x},v)$. For $S\in 2^N\setminus(\mathcal{T}_{k\ell}\cup\mathcal{T}_{\ell k})$, we have $e(S,\mathbf{x},v)=e(S,\mathbf{y},v)$. For $S\in\mathcal{T}_{k\ell}$ we have $e(S,\mathbf{y},v)=e(S,\mathbf{x},v)-\delta$. Finally, for $S\in\mathcal{T}_{\ell k}$ we have

$$e(S,\mathbf{y},v)=e(S,\mathbf{x},v)+\delta\leq s_{\ell k}(\mathbf{x},v)+\delta=s_{k\ell}(\mathbf{x},v)-\delta\ .$$

Thus, $\theta_t(\mathbf{y}) = \theta_t(\mathbf{x})$ for all $t \leq s$ and $\theta_{s+1}(\mathbf{y}) < s_{k\ell}(\mathbf{x}, v) = \theta_{s+1}(\mathbf{x})$. Hence $\theta(\mathbf{x}) \succ_{\text{lex}} \theta(\mathbf{y})$, a contradiction.

Problems of Chapter 20

20.1 *The Dentist Game*
(20.2) and (20.3) are equivalent since each $\pi \in \Pi(S)$ corresponds to some collection $\{x_{ij} \in \{0,1\}\}$ where $\sum_{j \in N} x_{ij} = 1_S(i)$ for all $i \in S$, $\sum_{i \in N} x_{ij} = 1_S(j)$ for all $j \in S$. For the 'dentist game' of Sect. 1.3.4 the numbers k_{ij} are given by Table 1.3, i.e., by

	Pl.1	Pl.2	Pl.3
Pl.1	2	4	8
Pl.2	10	5	2
Pl.3	10	6	4

20.2 *Example 20.3*
The coalition $\{1,2\}$ can generate 3 by the permutation that exchanges 1 and 2, hence $a_{12} = 3$. Similarly, $a_{13} = 3$. Hence, $M = \{1\}$ and $P = \{2,3\}$. Thus, in the associated assignment game w we have $w(N) = \max\{3,3\} = 3 \neq 4 = v(N)$.

20.3 *Subgames of Permutation Games*
That a subgame of a permutation game is again a permutation game follows immediately from the definition: in (20.3) the worth $v(S)$ depends only on the numbers k_{ij} for $i, j \in S$. By a similar argument (consider (20.1)) this also holds for assignment games.

20.4 *A Flow Game*
(a) The coalitions $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, $\{1,3,4\}$, and $\{2,3,4\}$ have worth 1, N has worth 2, all other coalitions have worth 0.
(b) $C(v) = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_3 \geq 1, x_2 + x_3 \geq 1, x_1 + x_2 + x_3 + x_4 = 2, x_i \geq 0 \forall i\}$.
(c) $(1,1,0,0)$, corresponding to the minimum cut through e_1 and e_2; $\{(0,0,1+\alpha, 1-\alpha) \mid 0 \leq \alpha \leq 1\}$, corresponding to the minimum cut through e_3 and e_4.

20.5 *Every Nonnegative Balanced Game is a Flow Game*
Let v be a nonnegative balanced game, and write (following the hint to the problem) $v = \sum_{r=1}^{k} \alpha_r w_r$, where $\alpha_r > 0$ and w_r a balanced simple game for each $r = 1, \ldots, k$. Consider the controlled capacitated network with two vertices, the source and the sink, and k edges connecting them, where each edge e_r has capacity α_r and is controlled by w_r. Then show that the associated flow game is v.

20.6 *On Theorem 20.6 (1)*

(a) This follows straightforwardly from the proof of Theorem 20.6.
(b) Coalitions $\{1,3\}$ and $\{2,4\}$ have worth 1, $\{1,2\}$ has worth 10, all three-person coalitions have worth 11, the grand coalition has worth 21, and all other coalitions have worth 0. E.g., each player receiving $5\frac{1}{4}$ is a core element.

20.7 *On Theorem 20.6 (2)*
In any core element, player should 1 receive at least 1 and player 2 also, but $v(N) = 1$. Hence the game has an empty core.

20.8 *Totally Balanced Flow Games*
This follows immediately from Theorem 20.6, since every dictatorial game is balanced, i.e., has veto players.

20.9 *If-part of Theorem 20.9*

We show that the Banzhaf value satisfies 2-EFF, the other properties are obvious. With notations as in the formulation of 2-EFF, we have

$$\begin{aligned}\psi_p(v_p) &= \sum_{S\subseteq (N\setminus p)\cup\{p\}:\ p\notin S} \frac{1}{2^{|N|-2}}\left[v_p(S\cup\{p\}) - v_p(S)\right]\\ &= \sum_{S\subseteq N\setminus\{i,j\}} \frac{1}{2^{|N|-2}}\left[v(S\cup\{ij\}) - v(S)\right]\\ &= \sum_{S\subseteq N\setminus\{i,j\}} \frac{1}{2^{|N|-1}}\left[2v(S\cup\{ij\}) - 2v(S)\right]\ .\end{aligned}$$

The term in brackets can be written as

$$\begin{aligned}&[v(S\cup\{i,j\}) - v(S\cup\{i\}) + v(S\cup\{j\}) - v(S)]\\ +\ &[v(S\cup\{i,j\}) - v(S\cup\{j\}) + v(S\cup\{i\}) - v(S)]\ ,\end{aligned}$$

hence $\psi_p(v_p) = \psi_j(v) + \psi_i(v)$.
Note that we cannot weaken DUM to NP. For instance, the value ψ defined by $\psi_i(v) := \sum_{S\subset N:\ i\notin S} \frac{1}{2^{|N|}}[v(S\cup i) - v(S)]$ satisfies 2-EFF (by a similar argument as above), SYM, NP, and SMON.
For the three-person glove game $v(13) = v(23) = v(123) = 1$, $v(S) = 0$ otherwise, the Banzhaf value is $(1/4, 1/4, 1)$, which is not efficient.

Problems of Chapter 21

21.1 *Anonymity and Symmetry*
Let F be anonymous and $(S,\mathbf{d})$ symmetric. Let $S' := \{(x_2,x_1) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\} = S$, and $\mathbf{d}' := (d_2,d_1) = \mathbf{d}$. By Anonymity, $F_1(S,\mathbf{d}) = F_1(S',\mathbf{d}') = F_2(S,\mathbf{d})$.
An example of a symmetric but not anonymous solution is as follows. To symmetric problems, assign the point in $W(S)$ with equal coordinates; otherwise,

assign the point of S that is lexicographically (first player 1, then player 2) maximal.

21.2 *Revealed Preference*

Suppose F is represented by a binary relation $\succeq$. Let $S, T \in B_0$ with $S \subseteq T$ and $F(T) \in S$. Then $F(T) \succeq \mathbf{y}$ for all $\mathbf{y} \in T$, so $F(T) \succeq \mathbf{y}$ for all $\mathbf{y} \in S$. Since $\{F(S)\} = \{\mathbf{x} \in S \mid \mathbf{x} \succeq \mathbf{y}$ for all $\mathbf{y} \in S\}$, we have $F(S) = F(T)$.
Suppose F satisfies IIA. Define $\succeq$ on $\mathbb{R}^2$ by $\mathbf{x} \succeq \mathbf{y} :\Leftrightarrow \exists S \in B_0 : \mathbf{x} = F(S), \mathbf{y} \in S$. Let $S \in B_0$ arbitrary. By definition, $F(S) \succeq \mathbf{y}$ for all $\mathbf{y} \in S$. Suppose also $\mathbf{z} \in S$, $\mathbf{z} \succeq \mathbf{y}$ for all $\mathbf{y} \in S$. Let $T \in B_0$ such that $F(T) = \mathbf{z}$ and $F(S) \in T$. Then $S \cap T \in B_0$. By IIA, $F(S \cap T) = F(T) = \mathbf{z}$ and also $F(S \cap T) = F(S)$. Hence $F(S) = \mathbf{z}$, so $\{\mathbf{x} \in S \mid \mathbf{x} \succeq \mathbf{y}$ for all $\mathbf{y} \in S\} = \{F(S)\}$.

21.3 *The Nash Solution is Well-defined*
The function $\mathbf{x} \mapsto (x_1 - d_1)(x_2 - d_2)$ is continuous on the compact set $\{\mathbf{x} \in S \mid \mathbf{x} \geq \mathbf{d}\}$ and hence attains a maximum on this set. We have to show that this maximum is attained at a unique point. In general, consider two points $\mathbf{z}, \mathbf{z}' \in \{\mathbf{x} \in S \mid \mathbf{x} \geq \mathbf{d}\}$ with $(z_1 - d_1)(z_2 - d_2) = (z_1' - d_1)(z_2' - d_2) = \alpha$. Then one can show that at the point $\mathbf{w} = \frac{1}{2}(\mathbf{z} + \mathbf{z}') \in S$ one has $(w_1 - d_1)(w_2 - d_2) > \alpha$. This implies that the maximum is attained at a unique point.

21.4 $(a) \Rightarrow (b)$ *in Theorem 21.1*
WPO and IIA are straightforward by definition, and SC follows from an easy computation. For SYM, note that if $N(S, \mathbf{d}) = \mathbf{z}$ for a symmetric problem $(S, \mathbf{d})$, then also $(z_2, z_1) = N(S, \mathbf{d})$ by definition of the Nash bargaining solution. Hence, $z_1 = z_2$ by uniqueness.

21.5 *Geometric Characterization of the Nash Bargaining Solution*
Let $(S, \mathbf{d}) \in B$ and $N(S, \mathbf{d}) = \mathbf{z}$. The slope of the tangent line ℓ to the graph of the function $x_1 \mapsto (z_1 - d_1)(z_2 - d_2)/(x_1 - d_1) + d_2$ (which describes the level set of $\mathbf{x} \mapsto (x_1 - d_1)(x_2 - d_2)$ through $\mathbf{z}$) at $\mathbf{z}$ is equal to $-(z_2 - d_2)/(z_1 - d_1)$, i.e., the negative of the slope of the straight line through d and z. Clearly, ℓ supports S at $\mathbf{z}$: this can be seen by invoking a separating hyperplane theorem, but also as follows. Suppose there were some point $\mathbf{z}'$ of S on the other side of ℓ than $\mathbf{d}$. Then there is a point $\mathbf{w}$ on the line segment connecting $\mathbf{z}'$ and $\mathbf{z}$ (hence, $\mathbf{w} \in S$) with $(w_1 - d_1)(w_2 - d_2) > (z_1 - d_1)(z_2 - d_2)$, contradicting $\mathbf{z} = N(S, \mathbf{d})$. The existence of such a point $\mathbf{w}$ follows since otherwise the straight line through $\mathbf{z}'$ and $\mathbf{z}$ would also be a tangent line to the level curve of the Nash product at $\mathbf{z}$.

For the converse, suppose that at a point $\mathbf{z}$ there is a supporting line of S with slope $-(z_2 - d_2)/(z_1 - d_1)$. Clearly, this line is tangent to the graph of the function $x_1 \mapsto (z_1 - d_1)(z_2 - d_2)/(x_1 - d_1) + d_2$ at $\mathbf{z}$. It follows that $\mathbf{z} = N(S, \mathbf{d})$.

21.6 *Strong Individual Rationality*
The implication $(a) \Rightarrow (b)$ is straightforward. For $(b) \Rightarrow (a)$, if F is also weakly Pareto optimal, then $F = N$ by Theorem 21.1. So it is sufficient to show that,

if F is not weakly Pareto optimal then $F = D$. Suppose that F is not weakly Pareto optimal. Then there is an $(S, \mathbf{d}) \in B$ with $F(S, \mathbf{d}) \notin W(S)$. By IR, $F(S, \mathbf{d}) \geq \mathbf{d}$. Suppose $F(S, \mathbf{d}) \neq \mathbf{d}$. By SC, we may assume w.l.o.g. $\mathbf{d} = (0, 0)$. Let $\alpha > 0$ be such that $F(S, (0, 0)) \in W((\alpha, \alpha)S)$. Since $F(S, (0, 0)) \notin W(S)$, $\alpha < 1$. So $(\alpha, \alpha)S \subseteq S$. By IIA, $F((\alpha, \alpha)S, (0, 0)) = F(S, (0, 0))$, so by SC, $F((\alpha, \alpha)S, (0, 0)) = (\alpha, \alpha)F(S, (0, 0)) = F(S, (0, 0))$, contradicting $\alpha < 1$. So $F(S, (0, 0)) = (0, 0)$. Suppose $F(T, (0, 0)) \neq (0, 0)$ for some $(T, (0, 0)) \in B$. By SC we may assume $(0, 0) \neq F(T, (0, 0)) \in S$. By IIA applied twice, $(0, 0) = F(S \cap T, (0, 0)) = F(T, (0, 0)) \neq (0, 0)$, a contradiction. Hence, $F = D$.

21.7 *(a) ⇒ (b) in Theorem 21.2*
Straightforward. Note in particular that in a symmetric game the utopia point is also symmetric, and that the utopia point is 'scale covariant'.

21.8 *Restricted Monotonicity*
(a) Follows from applying IM twice.
(b) For $(S, \mathbf{d})$ with $\mathbf{d} = (0, 0)$ and $u(S, \mathbf{d}) = (1, 1)$, let $F(S, \mathbf{d})$ be the lexicographically (first player 1, then player 2) maximal point of $S \cap \mathbb{R}^2_+$. Otherwise, let F be equal to R. This F satisfies RM but not IM.

21.9 *Global Individual Monotonicity*
It is straightforward to verify that G satisfies WPO, SYM, SC, and GIM. For the converse, suppose that F satisfies these four axioms, let $(S, \mathbf{d}) \in B$ and $\mathbf{z} := G(S, \mathbf{d})$. By SC, w.l.o.g. $\mathbf{d} = (0, 0)$ and $g(S) = (1, 1)$. Let $\alpha \leq 0$ such that $S \subseteq \widetilde{S}$ where $\widetilde{S} := \{\mathbf{x} \in \mathbb{R}^2 \mid (\alpha, \alpha) \leq \mathbf{x} \leq \mathbf{y} \text{ for some } \mathbf{y} \in S\}$. In order to prove $F(S, (0, 0)) = G(S, (0, 0))$ it is sufficient to prove that $F(\widetilde{S}, (0, 0)) = G(\widetilde{S}, (0, 0))$ (in view of GIM and WPO). Let $T = \operatorname{conv}\{\mathbf{z}, (\alpha, g_2(\widetilde{S})), (g_1(\widetilde{S}), \alpha)\} = \operatorname{conv}\{\mathbf{z}, (\alpha, 1), (1, \alpha)\}$. By SYM and WPO, $F(T, (0, 0)) = \mathbf{z}$. By GIM, $F(\widetilde{S}, (0, 0)) \geq F(T, (0, 0)) = \mathbf{z} = G(S, (0, 0)) = G(\widetilde{S}, (0, 0))$, so by WPO: $F(\widetilde{S}, (0, 0)) = G(\widetilde{S}, (0, 0))$. (Make pictures. Note that this proof is analogous to the proof of Theorem 21.2.)

21.10 *Monotonicity and (Weak) Pareto Optimality*
(a) Take $\mathbf{d} = (0, 0)$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (2, 1)$, $S = \operatorname{conv}\{\mathbf{d}, \mathbf{a}, \mathbf{b}\}$. By WPO, $F(\operatorname{conv}\{\mathbf{d}, \mathbf{a}\}, \mathbf{d}) = \mathbf{a} = (1, 2)$, $F(\operatorname{conv}\{\mathbf{d}, \mathbf{b}\}, \mathbf{d}) = \mathbf{b} = (2, 1)$. By MON and WPO, $F(S, \mathbf{d}) = (1, 2)$ but also $F(S, \mathbf{d}) = (2, 1)$, a contradiction.
(b) For the first question the argument is almost similar as in (a), take the 'comprehensive hulls' of the three bargaining problems there. Further, the egalitarian solution E satisfies MON and WPO on B_0.

21.11 *The Egalitarian Solution (1)*
Straightforward.

21.12 *The Egalitarian Solution (2)*
Let $\mathbf{z} := E(S, \mathbf{d}) + E(T, \mathbf{e})$. Then it is straightforward to derive that $z_1 - (d_1 + e_1) = z_2 - (d_2 + e_2)$. Since $E(S + T, d + e)$ is the maximal point $\mathbf{x}$ such that $x_1 - (d_1 + e_1) = x_2 - (d_2 + e_2)$, it follows that $E(S + T, d + E) \geq \mathbf{z}$.

21.13 *Independence of Axioms*
Theorem 21.1:

WPO, SYM, SC: $F = R$; WPO, SYM, IIA: $F = L$, where $L(S, \mathbf{d})$ is the point of $P(S)$ nearest to the point $\mathbf{z} \geq \mathbf{d}$ with $z_1 - d_1 = z_2 - d_2$ measured along the boundary of S; WPO, SC, IIA: $F = D^1$, where $D^1(S, \mathbf{d})$ is the point of $\{\mathbf{x} \in P(S) \mid \mathbf{x} \geq \mathbf{d}\}$ with maximal first coordinate; SYM, SC, IIA: $F = D$ (disagreement solution).

Theorem 21.2:

WPO, SYM, SC: $F = N$; WPO, SYM, IM: $F = L$; WPO, SC, IM: if $\mathbf{d} = (0, 0)$ and $u(S, \mathbf{d}) = (1, 1)$, let F assign the point of intersection of $W(S)$ and the line segment connecting $(1/4, 3/4)$ and $(1, 1)$ and, otherwise, let F be determined by SC; SYM, SC, IM: $F = D$.

Theorem 21.3:

WPO, MON, SYM: $F(S, \mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$ if $\mathbf{d} = (1, 0)$, $F(S, \mathbf{d}) = E(S, \mathbf{d})$ otherwise; WPO, MON, TC: $F(S, \mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$; WPO, SYM, TC: $F = N$; MON, SYM, TC: $F = D$.

21.14 *Nash and Rubinstein*
(b) The Nash bargaining solution outcome is $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, hence $(\frac{1}{3}\sqrt{3}, 1 - \frac{1}{3}\sqrt{3})$ is the resulting distribution of the good.
(c) The Rubinstein bargaining outcome is $\left(\sqrt{\frac{1-\delta}{1-\delta^3}}, \frac{\delta - \delta^3}{1-\delta^3}\right)$.
(d) The outcome in (c) converges to the outcome in (b) if δ converges to 1.

Problems of Chapter 22

22.1 *Convex Sets*
The only-if part is obvious. For the if-part, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in Z the condition implies that $\frac{k}{2^m}\mathbf{x} + \frac{2^m - k}{2^m}\mathbf{y} \in Z$ for every $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^m\}$. By closedness of Z, this implies that $\operatorname{conv}\{\mathbf{x}, \mathbf{y}\} \subseteq Z$, hence Z is convex.

22.2 *Proof of Lemma 22.3*
Suppose that both systems have a solution, say $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$, $A\mathbf{y} + \mathbf{z} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. Then $\mathbf{x}A\mathbf{y} + \mathbf{x} \cdot \mathbf{z} = \mathbf{x}(A\mathbf{y} + \mathbf{z}) = 0$, hence $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ since $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. This contradicts $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$.

22.3 *Rank of AA^T*
We have to prove that the rank of AA^T is equal to k. It is sufficient to prove that the null space of AA^T is equal to the null space of A^T for then, by the Rank Theorem, we have $\operatorname{rank}(AA^T) = m - \dim \operatorname{Ker}(AA^T) = m - \dim \operatorname{Ker}(A^T) = \operatorname{rank}(A^T) = \operatorname{rank}(A) = k$. (Here, Ker denotes the null space.) Let $\mathbf{x} \in \mathbb{R}^n$. Clearly, if $A^T\mathbf{x} = \mathbf{0}$ then $AA^T\mathbf{x} = \mathbf{0}$. Conversely, if $AA^T\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^T AA^T\mathbf{x} = 0$, hence $A^T\mathbf{x} \cdot A^T\mathbf{x} = 0$. The last equality implies $A^T\mathbf{x} = \mathbf{0}$.

22.4 *Proof of Lemma 22.5*
Suppose that both systems have a solution, say $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A = \mathbf{b}$, $A\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$. Then $\mathbf{x}A\mathbf{y} < 0$, contradicting $\mathbf{x} > \mathbf{0}$ and $A\mathbf{y} \geq \mathbf{0}$.

22.5 *Proof of Lemma 22.7*
(a) If $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}A \leq \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b} \cdot \mathbf{y} < 0$ then $\mathbf{x}A\mathbf{y} \leq \mathbf{b} \cdot \mathbf{y} < 0$, so $A\mathbf{y} \not\geq \mathbf{0}$. This shows that at most one of the two systems has a solution.
(b) Suppose the system in (a) has no solution. Then also the system $\mathbf{x}A + \mathbf{z}I = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$ has no solution. Hence, by Farkas' Lemma the system $\begin{pmatrix} A \\ I \end{pmatrix} \mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$ has a solution. Therefore, the system in (b) has a solution.

22.6 *Extreme Points*
The implication (b) $\Rightarrow$ (a) follows by definition of an extreme point.
For the implication (a) $\Rightarrow$ (c), let $x, y \in C \setminus \{e\}$ and $0 < \lambda < 1$. Let $z = \lambda x + (1-\lambda)y$. If $z \neq e$ then $z \in C \setminus \{e\}$ by convexity of C. Suppose now that $z = e$. W.l.o.g. let $\lambda \geq 1/2$. Then $e = \lambda x + (1-\lambda)y = (1/2)x + (1/2)[\mu x + (1-\mu)y]$ for $\mu = 2\lambda - 1$. Since $\mu x + (1-\mu)y \in C$, this implies that e is not an extreme point of C. This proves the implication (a) $\Rightarrow$ (c).
For the implication (c) $\Rightarrow$ (b), let $x, y \in C$, $x \neq y$, and $0 < \alpha < 1$. If $x = e$ or $y = e$ then clearly $\alpha x + (1-\alpha)y \neq e$. If $x \neq e$ and $y \neq e$ then $\alpha x + (1-\alpha)y \in C \setminus \{e\}$ by convexity of $C \setminus \{e\}$, hence $\alpha x + (1-\alpha)y \neq e$ as well.

22.7 *Affine Subspaces*
Let $A = a + L$ be an affine subspace, $x, y \in A$, and $\lambda \in \mathbb{R}$. Write $x = a + \bar{x}$ and $y = a + \bar{y}$ for $\bar{x}, \bar{y} \in L$, then $\lambda x + (1-\lambda)y = a + \lambda\bar{x} + (1-\lambda)\bar{y} \in A$ since $\lambda\bar{x} + (1-\lambda)\bar{y} \in L$ (L is a linear subspace).
Conversely, suppose that A contains the straight line through any two of its elements. Let a be an arbitrary element of A and let $L := \{x - a \mid x \in A\}$. Then it follows straightforwardly that L is a linear subspace of V, and thus $A = a + L$ is an affine subspace.

22.8 *The set of Sup-points of a Linear Function on a Convex Set*
In general, linearity of f implies that, if $f(\mathbf{x}) = f(\mathbf{y})$, then $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = f(\mathbf{x}) = f(\mathbf{y})$ for any two points of C and $0 < \lambda < 1$. It follows, in particular, that the set D is convex.
Let $\mathbf{e} \in \operatorname{ext}(D)$ and suppose $\mathbf{e} = (1/2)\mathbf{x} + (1/2)\mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in C$. Then by linearity of f, $f(\mathbf{e}) = (1/2)f(\mathbf{x}) + (1/2)f(\mathbf{y})$, hence $\mathbf{x}, \mathbf{y} \in D$ since $\mathbf{e} \in D$. So $\mathbf{e} = \mathbf{x} = \mathbf{y}$ since $\mathbf{e}$ is an extreme point of D. Thus, $\mathbf{e}$ is also an extreme point of C.