Solution Manual
Game Theory: An Introduction

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ABSTRACT This Slution Manual is incomplete. It will be updated every 2-3 weeks to add the solutions to problems as they become available. A complete version is expected by March 15, 2013.

Contents

Part I

Rational Decision Making

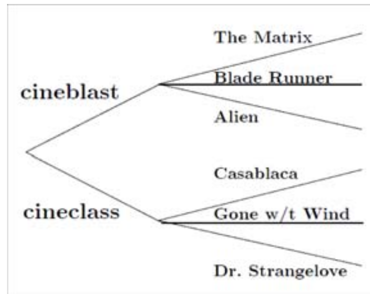
1
The Single-Person Decision Problem

1. Think of a simple decision you face regularly and formalize it as a decision problem, carefully listing the actions and outcomes without the preference relation. Then, assign payoffs to the outcomes, and draw the decision tree.

2. **Going to the Movies:** There are two movie theatres in your neighborhood: Cineclass, which is located one mile from your home, and Cineblast, located 3 miles from your home, each showing three films. Cineclass is showing *Casablanca*, *Gone with the Wind* and *Dr. Strangelove*, while Cineblast is showing *The Matrix*, *Blade Runner* and *Aliens*. Your problem is to decide which movie to go to.

   (a) Draw a decision tree that represents this problem without assigning payoff values.

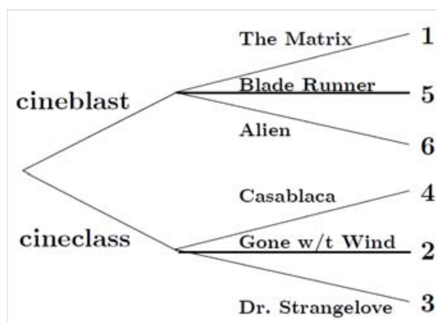
   Answer:



■

(b) Imagine that you don't care about distance and that your preferences for movies is alphabetic (i.e., you like *Aliens* the most and *The Matrix* the least.) Using payoff values 1 through 6 complete the decision tree you drew in part (a). What option would you choose?

Answer:



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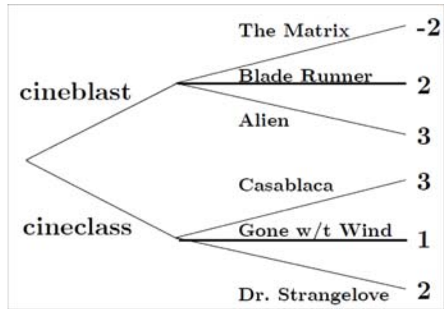
(c) Now imagine that your car is in the shop, and the cost of walking each mile is equal to one unit of payoff. Update the payoffs in the decision tree. Would your choice change?

Answer:



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3. **Fruit or Candy:** A banana costs \$0.50 and a candy costs \$0.25 at the local cafeteria. You have \$1.25 in your pocket and you value money. The money-equivalent value (payoff) you get from eating your first banana is \$1.20, and that of each additional banana is half the previous one (the second banana gives you a value of \$0.60, the third 0.30, etc.). Similarly, the payoff you get from eating your first candy is \$0.40, and that of each additional candy is half the previous one (\$0.20, 0.10, etc.). Your value from eating bananas is not affected by how many candies you eat and vice versa.

   (a) What is the set of possible actions you can take given your budget of \$1.25?

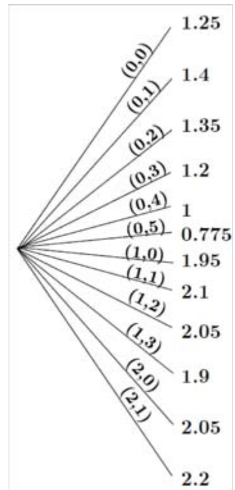
   Answer: You can buy any combination of bananas and candies that sum up to no more than \$1.25. If we denote by (b, c) the choice to buy b bananas and c candies, then the set of possible actions is

   $$A = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1)\}$$

   ■

   (b) Draw the decision tree that is associated with this decision problem.

   Answer: For each choice you need to calculate the final net value. For example, if you buy one banana and 2 candies then you get 1.2 worth from the banana, 0.4 from the first candy and 0.2 from the second which totals 1.8. To this we need to add the \$0.25 you have left (the cost was only \$1) so the net final value you have is 2.05.



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(c) Should you spend all your money at the cafeteria? Justify your answer with a rational choice argument.

Answer: Yes. The highest net final value if from buying two bananas and one candy. ■

(d) Now imagine that the price of a candy increased to \$0.30. How many possible actions do you have? Does your answer to (c) above change?

Answer: Of the 12 options above, three are no longer possible: $(0, 5)$, $(1, 3)$ and $(2, 1)$. Also, the net final values change because each candy is 5 cents more expensive. The highest net final value is 2.05 which can be obtained from one of two choices: $(1, 1)$ and $(2, 0)$, both leaving some money in the decision maker's pocket. ■

4. **Alcohol Consumption:** Recall the example in which you needed to choose how much to drink. Imagine that your payoff function is given by $\theta a - 4a^2$, where θ is a parameter that depends on your physique. Every person may have a different value of θ, and it is known that in the population (i) the smallest θ is 0.2; (i) the largest θ is 6; and (iii) larger people have higher θ's than smaller people.

(a) Can you find an amount of drinking that no person should drink?

Answer: The utility from drinking 0 is equal to 0. If a decision maker drinks $a = 2$ then, if he has the largest $\theta = 6$, his payoff is $v = 6 \times 2 - 4 \times (2)^2 = -4$ and it is easy to see that decision makers with smaller values of θ will obtain an even more negative payoff from consuming $a = 2$. Hence, no person should choose $a = 2$. ■

(b) How much should you drink if your $\theta = 1$? If $\theta = 4$?

Answer: The optimal solution is obtained by maximizing the payoff function $v(a) = \theta a - 4a^2$. The first-order maximization condition is $\theta - 8a = 0$ implying that $a = \frac{\theta}{8}$ is the optimal solution. For $\theta = 1$ the solution is $a = \frac{1}{8}$ and for $\theta = 4$ it is $a = \frac{1}{2}$. ■

(c) Show that in general, smaller people should drink less than larger people.

Answer: This follows from the solution in part (b) above. For every type of person θ, the solution is $a(\theta) = \frac{\theta}{8}$ which is increasing in θ, and larger people have higher values of θ. ■

(d) Should any person drink more than one bottle of wine?

Answer: No. Even the largest type of person with $\theta = 6$ should only consume $a = \frac{3}{4}$ of a bottle of wine. ■

5. **Buying a Car:** You plan on buying a used car. You have \$12,000, and you are not eligible for any loans. The prices of available cars on the lot are given as follows:

Make, Model & Year	Price
Toyota Corolla 2002	\$9,350
Toyota Camry 2001	\$10,500
Buick LeSabre 2001	\$8,825
Honda Civic 2000	\$9,215
Subaru Impreza 2000	\$9,690

For *any given year*, you prefer a Camry to an Impreza, an Impreza to a Corolla, a Corolla to a Civic and a Civic to a LeSabre. For *any given year*, you are willing to pay up to \$999 to move form a car to the next preferred car. For example, if the price of a Corolla is \$$z$, then you are willing to buy it over a Civic if the Civic costs more that \$$(z-999)$, but you would prefer buying the Civic if it costs less than this amount. Similarly, you prefer the Civic at \$$z$ to a Corolla that costs more than \$$(z+1000)$ but you prefer the Corolla if it costs less. For *any given car*, you are willing to move to a model a year older if it is cheaper by at least \$500. For example, if the price of a 2003 Civic is \$$z$, then you are willing to buy it over a 2002 Civic if the 2002 Civic costs more that \$$(z-500)$, but you would prefer buying the 2002 Civic if it costs less than this amount.

(a) What is your set of possible alternatives?

Answer: Given that you have \$12,000, which is more than the price of any car, you have six alternatives: any one of the five cars or buying nothing. ■

(b) What is your preference relation between the alternatives in (a) above?

Answer: To answer this we need use the information on willingness to pay given in the question, together with the prices. The least valued car would be a 2000 LeSabre. Assume that the value of owning that car is given by x. From the information above, a 2000 Civic is valued at $x+999$, a 2000 Corolla is valued at $x+1,998$, and so on up to a 2000 Camry valued at $x+3,996$. Similarly, each of these cars for the year 2001 is valued at 500 more than the 2000 model, and the 2002 model is valued at 1,000 more than the 2000 model. Hence, we can write the table of values as follows:

Make and Model	year 2000	year 2001	year 2002
Toyota Camry	$x+3,996$	$x+4,496$	$x+4,996$
Subaru Impreza	$x+2,997$	$x+3,497$	$x+3,997$
Toyota Corolla	$x+1,998$	$x+2,498$	$x+2,998$
Honda Civic	$x+999$	$x+1,499$	$x+1,999$
Buick LeSabre	$x+0$	$x+500$	$x+1,000$

Now, to see what the net value from each purchase would be we must deduct the price of the car from the value. Using the five prices given above and the values we just calculated we have net payoffs as (e.g., for the 2002 Corolla, the net payoff is $x + 2,998 - 9,350 = x - 6,352$),

Make, Model & Year	Price
Toyota Corolla 2002	$x - 6,352$
Toyota Camry 2001	$x - 6,004$
Buick LeSabre 2001	$x - 8,325$
Honda Civic 2000	$x - 8,216$
Subaru Impreza 2000	$x - 6,693$

Assuming that x is large enough to want to buy any car, the ranking of the alternatives is, Toyota Camry 2001, followed by Toyota Corolla 2002, followed by Subaru Impreza 2000, followed by Honda Civic 2000 and last being the Buick LeSabre 2001. ■

(c) Draw a decision tree an assign payoffs to the terminal nodes associated with the possible alternatives. What would you choose?

Answer: This follows directly from the analysis in (b) above: you should choose the Toyota Camry 2001 (with six branches, including no purchase.) ■

(d) Can you draw a decision tree with different payoffs that represents the same problem?

Answer: Because we left x as undetermined, we can find many values of x that will represent this problem. Notice that if x is small enough (less than 6,004) then the best choice would be not to buy a car. ■

6. **Fruit Trees:** You have room for up to two fruit bearing trees in your garden. The fruit trees that can grow in your garden are either apple, orange or pear. The cost of maintenance is \$100 for an apple tree, \$70 for an orange tree and \$120 for a pear tree. Your food bill will be reduced by \$130 for each apple tree you plant, by \$145 for each pear tree you plant and by \$90 for each

orange tree you plant. You care only about your total expenditure in making any planting decisions.

(a) What is the set of possible actions and related outcomes?

Answer: You have two "slots" that can be left empty, or have one of 3 possible trees planted in each slot. Hence, you have 10 possible choices.[1] The outcomes will just be the choices of what to plant. ■

(b) What is the payoff of each action/outcome?

Answer: To calculate the payoffs from each choice it is convenient to use a table as follows:

Choice	cost	food savings	net payoff
nothing	0	0	0
one apple tree	100	130	30
one orange tree	70	90	20
one pear tree	120	145	25
two apple trees	200	260	60
two orange trees	140	180	40
two pear trees	240	290	50
apple and orange	170	220	50
apple and pear	220	275	55
pear and orange	190	235	45

(c) Which actions are dominated?

Answer: All but choosing two apple trees are dominated. ■

(d) Draw the associated decision tree. What will a rational player choose?

Answer: The tree will have ten branches with the payoffs associated with the table above, and the optimal choice is two apple trees. ■

[1] This is a probem of choosing 2 items out of 4 possibilities with replacement, which is equal to $\binom{4+2-1}{2} = \frac{(4+2-1)!}{2!(4-1)!} = \frac{5\times 4}{2} = 10$.

(e) Now imagine that the food bill reduction is half for the second tree of the same kind (you like variety). That is, the first apple still reduces your food bill by \$130, but if you plant two apple trees your food bill will be reduced by \$130 + \$65 = \$195. (Similarly for pear and orange trees.) What will a rational player choose now?

Answer: An apple tree is still the best choice for the first tree, but now the second tree should be a pear tree. ■

7. **City Parks:** A city's mayor has to decide how much money to spend on parks and recreation. City codes restrict this spending to be no more than 5% of the budget, and the yearly budget of the city is \$20,000,000. He wants to please his constituents who have diminishing returns from parks. The money-equivalent benefit from spending \$$c$ on parks is $v(c) = \sqrt{400c} - \frac{1}{80}c$.

(a) What is the action set of the city's mayor?

Answer: The limit on spending is \$1 million, so the actions set is $c \in [0, 1000000]$. ■

(b) How much should the mayor spend?

Answer: The maximization problem is

$$\max_{c \in [0,1000000]} \sqrt{400c} - \frac{1}{80}c \ ,$$

and taking the derivative for the first-order condition we obtain,

$$\frac{10}{\sqrt{c}} - \frac{1}{80} = 0 \ ,$$

or $c = \$640,000$. The second order derivative is $-5c^{-\frac{3}{2}} < 0$ so this is indeed a maximum. ■

(c) The movie *An Inconvenient Truth* has shifted public opinion and now people are more willing to pay for parks. The new preferences of the people are given by $v(c) = \sqrt{1600c} - \frac{1}{80}c$. What now is the action set of the mayor, and how much spending should he choose to cater to his constituents?

Answer: The first-order condition is now,

$$\frac{20}{\sqrt{c}} - \frac{1}{80} = 0 ,$$

or $c = \$2,560,000$. This exceeds the budget and hence the optimal solution is to spend \$1 million. ■

2

Introducing Uncertainty and Time

1. **Getting an MBA:** Recall the decision problem in Section 2.3.1, and now assume that the probability of a strong labor market is p, of an average labor market is 0.5 and of a weak labor market is $0.5 - p$. All the other values are the same.

   (a) For which values of p will you decide not to get an MBA?

   Answer: The expected payoffs from each choice are given by,

$$\begin{aligned} v(\text{Get MBA}) &= p \times 22 + 0.5 \times 6 + (0.5 - p) \times 2 = 20p + 4 \\ v(\text{Don't get MBA}) &= p \times \ 12 + 0.5 \times \ 8 + (0.5 - p) \times \ 4 = 8p + 6 \end{aligned}$$

   which implies that getting an MBA is worthwhile if and only if

$$20p + 4 \geq 8p + 6 \ ,$$

   or, $p \geq \frac{1}{6}$. ■

   (b) If $p = 0.4$, what is the highest price the university can charge for you to be willing to go ahead and get an MBA?

Answer: If $p = 0.4$ then the payoffs are,

$$\begin{aligned} v(\text{Get MBA}) &= 0.4 \times 22 + 0.5 \times 6 + 0.1 \times 2 = 12 \\ v(\text{Don't get MBA}) &= 0.4 \times 12 + 0.5 \times 8 + 0.1 \times 4 = 9.2 \end{aligned}$$

which implies that an extra charge of up to 2.8 can be charged by the university and you would still be willing to get an MBA. ■

2. **Recreation Choices:** A player has three possible venues to choose from: going to a football game, going to a boxing match, or going for a hike. The payoff from each of these alternatives will depend on the weather. The following table gives the agent's payoff in each of the two relevant weather events:

Alternative	payoff if Rain	payoff if Shine
Football game	1	2
Boxing Match	3	0
Hike	0	1

For Let p denote the probability of rain.

(a) Is there an alternative that a rational player will never take regardless of p? (i.e., it is dominated for any $p \in [0, 1]$.)

Answer: For this decision maker choosing the hike is always worse (dominated) by going to the football game, and he should never go on a hike. ■

(b) What is the optimal decision, or best response, as a function of p.

Answer: The expected payoffs from each of the remaining two choices are given by,

$$\begin{aligned} v(\text{Football}) &= p \times 1 + (1-p) \times 2 = 2 - p\,, \\ v(\text{Boxing}) &= p \times 3 + (1-p) \times 0 = 3p\,, \end{aligned}$$

which implies that football is a better choice if and only if

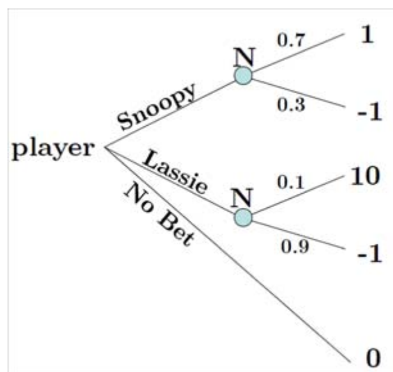
$$2 - p \geq 3p\,,$$

or, $p \leq \frac{1}{2}$, and boxing is better otherwise. ■

3. **At the Dog Races:** You're in Las Vegas, and you can decide what to do at the dog-racing bet room. You can choose not to participate, or you bet on one of two dogs as follows. Betting on Snoopy costs \$1, and you will be paid \$2 if he wins. Betting on Lassie costs \$1, and you will be paid \$11 if she wins. You believe that Snoopy has probability 0.7 of winning and that Lassie has probability 0.1 of winning (there are other dogs that you are not considering betting on). Your goal is to maximize the expected monetary return of your action.

   (a) Draw the decision tree of this problem.

   Answer:

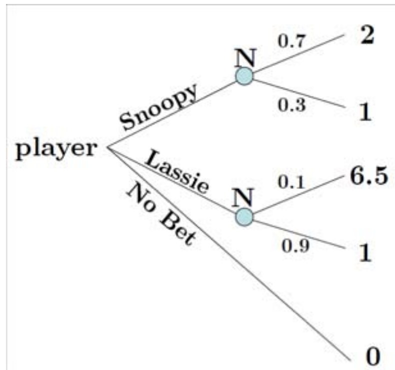


   (b) What is your best course of action, and what is your expected value?

   Answer: The expected payoff from betting on Snoopy is $0.7 - 0.3 = 0.4$ while betting on Lassie yields $1 - 0.9 = 0.1$, so betting on Snoopy is the best action. ■

   (c) Someone comes and offers you gambler's anti-insurance to which you can agree or not. If you agree to it, you get paid \$2 up front and you agree to pay back 50% of any winnings you receive. Draw the new decision tree, and find the optimal action.

Answer:



The best action is still to bet on Snoopy with an expected payoff of 1.7 versus 1.55 from betting on Lassie. ■

4. **Drilling for Oil:** An oil drilling company must decide whether or not to engage in a new drilling activity before regulators pass a law that bans drilling at that site. The cost of drilling is \$1,000,000. After drilling is completed and the drilling costs are incurred, then the company will learn if there is oil or not. If there is oil, operating profits generated are estimated at \$4,000,000. If there is no oil, there will be no future profits.

   (a) Using p to denote the likelihood that drilling results in oil, draw the decision tree of this problem.

   Answer: Two decision branches: drill or not drill. Following drilling, Nature chooses oil with probability p, with the payoff of \$3 million (4 minus the initial investment). With probability $1-p$ Nature chooses no-oil with a payoff $\$-1$ million. ■

   (b) The company estimates that $p = 0.6$. What is the expected value of drilling? Should the company go ahead and drill?

   Answer: The expected payoff (in millions) from drilling is $p \times 3 - (1 - p) \times 1 = 4p - 1 = 0.6$, which means that the company should drill. ■

   (c) To be on the safe side, the company hires a specialist to come up with a more accurate estimate of p. What is the minimum vale of p for which it would be the company's best response to go ahead and drill?

Answer: The minimum value of p for which drilling causes no expected loss is calculated by solving $p \times 3 - (1 - p) \times 1 \geq 0$, or $p \geq \frac{1}{4}$. ■

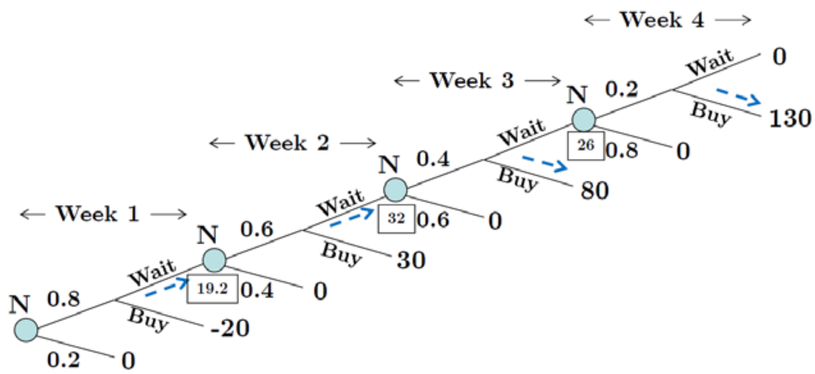
5. **Discount Prices:** A local department store puts out products at an initial price, and every week the product goes unsold, its price is discounted by 25% of the original price. If it is not sold after 4 weeks, it is sent back to the regional warehouse. There is a set of butcher knives that was just put out for the price of \$200. Your willingness to pay for the knives (your dollar value) is \$180, so if you buy them at a price P, your payoff is $u = 180 - P$. If you don't buy the knives, the chances that they are sold to someone else conditional on not selling in the week before are given in the following table:

week 1	0.2
week 2	0.4
week 3	0.6
week 4	0.8

For example, if you do not buy it during the first two weeks, the likelihood that it is available at the beginning of the third week is the likelihood that it does not sell in either weeks 1 and 2, which is $0.8 \times 0.6 = 0.48$.

(a) Draw your decision tree for the 4 weeks after the knives are put out for sale.

Answer: We can draw each week as having nature move first to determine whether someone else bought the knives, and if they did not, then our player can buy or wait. The tree therefore will be,



where the numbers in the squares next to Nature's nodes mark the expected value from choosing wait before that node. ■

(b) At the beginning of which week, if any, should you run to buy the knives?

Answer: We solve this backward. In week 4 the player will buy the knives of they are there. Waiting in week 3 gives an expected payoff of only $0.2 \times (180 - 50) = 26$, while buying in week 3 gives a payoff of $180 - 100 = 80 > 26$, so buying in week 3 beats waiting. Moving back to week 2, waiting gives an expected payoff of $0.4 \times 80 = 32$ while buying yields $180 - 150 = 30 < 32$ so waiting beats buying, and moving back to week 1 makes waiting even more valuable compared to buying (buying in week 1 is dominated by not buying. Hence, the player will wait till week 3 and then try to buy the knives. ■

(c) Find a willingness to pay for the knives that would make it optimal to buy at the beginning of the first week.

Answer: Waiting is risky so intuitively, to make an early purchase valuable, the willingness to pay must be very high. Set the willingness to pay at 1000. In week 4 the player will buy the knives. Waiting in week 3 yields $0.2 \times (1000 - 50) = 190$, while buying in week 3 gives a payoff of $1000 - 100 = 900 > 190$, so buying in week 3 beats waiting. Moving back to week 2, waiting gives an expected payoff of $0.4 \times 190 = 76$ while buying yields $1000 - 150 = 850 > 76$ so buying beats waiting. Moving back to week 1, waiting gives an expected payoff of $0.6 \times 850 = 510$ while buying yields $1000 - 200 = 800 > 510$ so buying in the first week is the optimal decision. ■

(d) Find a willingness to pay that would make it optimal to buy at the beginning of the fourth week.

Answer: Similarly to (c) above, to make a late purchase valuable, the willingness to pay must be quite low. Set the willingness to pay at 100. In any week but week 4 the price is above the willingness to pay, so the optimal decision is to wait for week 4 and then buy the knives if they are available. ■

6. **Real Estate Development:** A real estate developer wishes to build a new development. Regulations impose an environmental impact study that will yield an "impact score," which is an index number based on the impact the development will likely have on traffic, air quality, sewage and water usage, etc. The developer, who has lots of experience, knows that the score will be no less than 40, and no more than 70. Furthermore, he knows that any score between 40 and 70 is as likely as any other score between 40 and 70 (use continuous values). The local government's past behavior implies that there is a 35% chance that it will approve the development if the impact score is less than 50, a 5% chance that it will approve the development if the score is between 50 and 55, and if the score is greater than 55 then the project will surely be halted. The value of the development to the developer is $20,000,000. Assuming that the developer is risk neutral, what is the maximum cost of the impact study such that it is still worthwhile for the developer to have it conducted?

Answer: Observe that there is a $\frac{1}{3}$ probability of getting a score between 40 and 50 given that 40 to 50 is one-third of the range 40 to 70. There is a $\frac{1}{6}$ probability of getting a score between 50 and 55 given that 50 to 55 is one-sixth of the range 40 to 70. Hence, the expected value of doing a study is

$$
\begin{aligned}
& \frac{1}{3} \times .35 \times \$20,000,000 + \frac{1}{6} \times .05 \times \$20,000,000 + \frac{1}{2} \times 0 \times \$20,000,000 \\
= \quad & \$2,500,000
\end{aligned}
$$

Hence, the most the developer should pay for the study is \$2,500,000. ■

7. **Toys:** WakTek is a renowned manufacturer of electronic toys, with a specialty in remote-controlled (RC) miniature vehicles. WakTek is considering the introduction of a new product, an RC Hovercraft called WakAtak. Preliminary designs have already been produced at a cost of \$2 million. To introduce a marketable product requires the building of a dedicated product line at a cost of \$12 million. Also, before the product can be launched a prototype needs to be built and tested for safety. The prototype can be crafted even in the absence of a production line, at a cost of \$0.5 million, but if the prototype is built after the production line then its cost is negligible.[1] There is uncertainty over what safety rating WakAtak will get. This could have a large impact on demand, as a lower safety-rating will increase the minimum age required from users. The safety-testing costs \$1 million. The outcome of the safety-test is estimated to have a 65% chance of resulting in a minimum age of 8 years, a 30% chance of minimum age 15 years, and a 5% chance of being declared unsafe in which case it could not be sold at all. (The cost of improving the safety status of a finished design is deemed prohibitive.) After successful safety-testing the product could be launched at a cost of \$1.5 million.

   There is also uncertainty over demand, which will have a crucial impact on the eventual profits. Currently the best estimate is that the finished product, if available to the $8-14$ demographic, has a $50-50$ chance of resulting in profits of either \$10 million or \$5 million from that demographic. Similarly there is a $50-50$ chance of either \$14 million or \$6 million profit from the 15-or-above demographic. These demand outcomes are independent across the demographics. The profits do not take into account the costs defined above;

[1] "Negligible" means you can treat it as zero.

they are measured in expected present-value terms so they are directly comparable with the costs.

(a) What is the optimal plan of action for WakTek? What is currently the expected economic value of the WakAtak project?

Answer: The optimal plan is to build the prototype first and then do the safety test, then build the production line and launch the product only if the safety test results in the "safe for 8 years and above" status. The expected economic profits from this plan are \$1.1 million. For justification of this answer, consider the following decision tree:

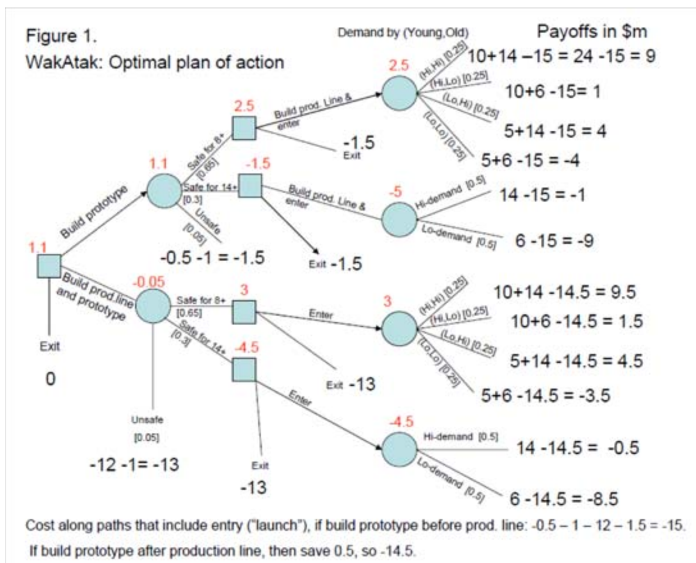


Figure 1.
WakAtak: Optimal plan of action

Cost along paths that include entry ("launch"), if build prototype before prod. line: -0.5 – 1 – 12 – 1.5 = -15.
If build prototype after production line, then save 0.5, so -14.5.

Notice that the cost of the preliminary design is sunk (cannot be recovered) and should be ignored. ■

(b) Suddenly it turns out that the original estimate of the cost of safety-testing was incorrect. Analyze the sensitivity of WakTek's optimal plan of action to the cost of safety-testing.

Answer: If the cost of safety-testing is too high, then the expected value becomes negative and the optimal plan is to exit the project. To find out the threshold cost of safety-testing above which exit becomes optimal, notice that the cost of safety-testing is incurred for sure under the optimal plan of action which brings expected profits of \$1.1 million. Therefore, if the cost of safety-testing is increased by \$1.1 million or more (bringing it to \$2.1 million or more) then the decision should be changed to "exit." ■

(c) Suppose WakTek has also the possibility of conducting a market survey, which would tell exactly which demand scenario is true. This market research costs \$1.5 million if done simultaneously for both demographics, and \$1 million if done for one demographic only. How, if at all, is the answer to part a) affected?

Answer: First examine the decision tree from part a) to see whether we can simplify the effect of the market research, by eliminating some logically possible alternatives. Which alternatives to eliminate from the tree as "obviously irrelevant" is partly a matter of taste. For example, there are points in the tree where the opportunity to exit is irrelevant (e.g. after we've found out that demand is high for the "young"[2]) because the profits will clearly be higher by not exiting. You can always just include all alternatives, although that can lead to a very large tree; the final answer is of course unaffected. Eliminations that are not obvious but that were used in simplifying the decision trees are justified by logic as follows:
(i) We can completely ignore the possibility of building a production line before the safety test. We already established in part (a) that doing the safety test first achieves expected profits that are $(1.1-(-0.05)=1.15)$ million higher than doing the production line first. The only potential benefit of doing the production line first is the saved \$0.5 million prototype cost. Thus no information could ever change the difference in payoffs to the advantage of a "production line first" plan by more than this \$0.5 million. Since research always costs at least \$1 million, "prod. line first" can not become optimal due to the possibility of doing market

[2] For brevity, the 8-14 demographic is henceforth referred to as the "young," and the 14+ demographic as the "old."

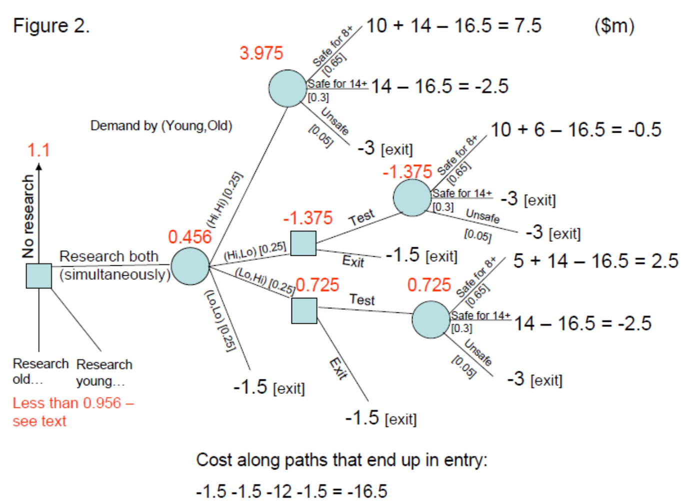
research.

(ii) It is never profitable to do research after the safety test. If the result were "safe for both groups" then the only case where info is useful (i.e. changes the decision to enter into exit) is if both groups have low demand. (See Figure 1: exiting payoff -1.5 is better than the -4 of Low-Low demand scenario, but less than the payoff under the other three demand scenarios). This demand scenario could be ruled out by researching either group. The expected payoff would be $\frac{1}{4}(9+1+4-1.5)-1 < 2.5$, i.e., not worth it after paying for the cost of research. Research after finding out that WakAtak is only "safe for old" is obviously not profitable, since even if the information caused the decision to change (from "exit" to "enter," if demand is high) this results only in a payoff of -1 before the research cost, while exit guarantees -1.5; since research is more costly than the 0:5 difference it cannot be worthwhile.

(iii) The potential benefit of research is that it allows WakTek to save the cost of production line under unfavorable demand conditions, so there would be no point in plans of action where research is conducted after the production line is built.

Consider a plan where both groups are researched simultaneously.

Figure 2.

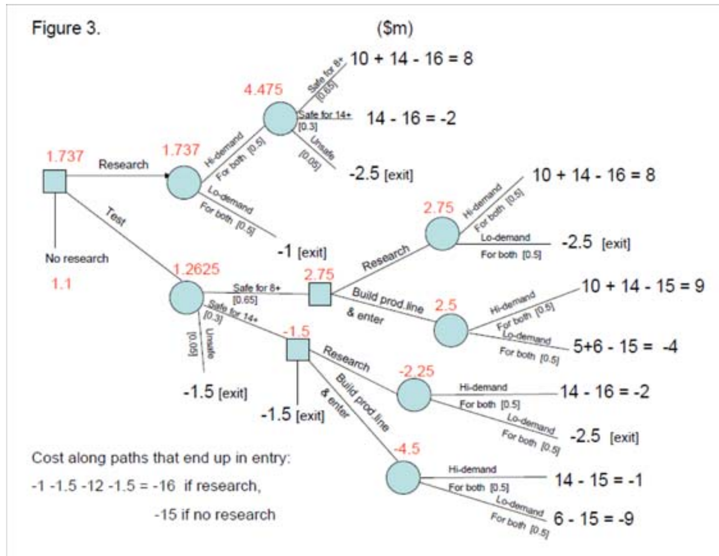


This would lead to expected value of \$0.456 million, so not doing research is better than researching both simultaneously. We can now deduce that researching only one of the groups cannot be optimal either. The reason is that it is less informative than researching both, so the expected payoff could not be higher than \$0.456 million for any other reason than the fact that it is cheaper by $\$(1.5 - 1 = 0.5)$ million. This means that the expected value (EV) of a plan where only one group is researched must be lower than $(\$0.456 + \$0.5 = \$0.956)$ million. Thus the \$1.1 million value from no research is still the highest. Similarly, consider the possibility of researching both groups sequentially. This is, at best, equally informative as researching both groups simultaneously. It offers the added option of stopping the research after finding out the results for one group, and thus potentially a saving of \$0.5 million compared to the cost of researching both simultaneously. Again, this cost-saving could not increase the EV to above \$0.956, so the optimal plan of action for part a) is not affected.

(d) Suppose that demand is not independent across demographics after all, but instead is perfectly correlated (i.e., if demand is high in one demographic, then it is for sure high in the other one as well). How, if at all, would that change your answer to part c)?

Answer: Now researching either one of the demographic groups is just as informative as researching both (but cheaper, at \$1 million); it tells WakTek whether the demand is high for both groups or low for both groups. In this case the optimal decision would be to research one (doesn't matter which) group, and do the safety testing if the demand is high for both groups, then build the production line and launch the product unless deemed unsafe; This results in EV of \$1.7375 million. The following figure shows the decision tree.[3]

[3] Note that expected values are not directly affected by the correlation so the EV of no research is still 1.1. However, the correlation of demands is good for WakTek, not just because it makes market research cheaper. For example, compared to the case (in part c) where WakTek researches both groups simultaneously, one added benefit here is that WakTek will never have to "waste" the cost of safety-testing in the event where the result turns out to be "safe for old only," which leads to exit.



8. **Juice:** Bozoni is a renowned Swiss maker of fruit and vegetable juice, whose products are sold at specialty stores around Western Europe. Bozoni is considering whether to add cherimoya juice to its line of products. "It would be one of our more difficult varieties to produce and distribute," observes Johann Ziffenboeffel, Bozoni's CEO. "The cherimoya would be flown in from New Zealand in firm, unripe form, and it would need its own dedicated ripening facility here in Europe." Three successful steps are absolutely necessary for the new cherimoya variety to be worth producing. The industrial ripening process must be shown to allow the delicate flavors of the cherimoya to be preserved; the testing of the ripening process requires the building of a small-scale ripening facility. Market research in selected small regions around Europe must show that there is sufficient demand among consumers for cherimoya juice. And cherimoya juice must be shown to withstand the existing tiny gaps in the cold chain between the Bozoni plant and the end consumers (these gaps would be prohibitively expensive to fix). Once these three steps have been completed, there are about €2,500,000 worth of expenses in launching the new variety of juice. A successful new variety will then yield profits, in expected present-value terms, of €42.5 million.

The three absolutely necessary steps can be done in parallel or sequentially in any order. Data about these three steps is given in Table 1. "Probability of success" refers to how likely it is that the step will be successful. If it is not successful, then that means that cherimoya juice cannot be sold at a profit. All probabilities are independent of each other (i.e., whether a given step is successful or not does not affect the probabilities that the other steps will be successful). "Cost" refers to the cost of doing this step (regardless of whether it is successful or not).

(a) Suppose Mr. Ziffenboeffel calls you and asks your advice about the project. In particular, he wants to know (i) should he do the three necessary steps in parallel (i.e., all at once) or should he do them sequentially; and (ii) if sequentially, what's the right order for the steps to be done? What answers do you give him?

Answer: Bozoni should do the steps sequentially in this order: first test the cold chain, then the ripening process, then do the test-marketing. The expected value of profits is 1.84 million. Observe that it would not be profitable to launch the product if Bozoni had to do all the steps simultaneously. This is an example of real options—by sequencing the steps, Bozoni creates options to switch out of a doomed project before too much money gets spent. ■

(b) Mr. Ziffenboeffel calls you back. Since Table 1 was produced (see below), Bozoni has found a small research firm that can perform the necessary tests for the ripening process at a lower cost than Bozoni's in-house research department.

Table 1: Data on launching the Cherimoya juice

Step	Probability of success	Cost
Ripening process	0.7	€1,000,000
Test marketing	0.3	€5,000,000
Cold chain	0.6	€500,000

At the same time, the EU has raised the criteria for getting approval for new food producing facilities, which raises the costs of these tests.

Mr. Ziffenboeffel would, therefore, like to know how your answer to (a) changes as a function of the cost of the ripening test. What do you tell him?

Answer: This is sensitivity analysis for the cost of testing the ripening process. This can be done by varying the cost for ripening, and seeing which expected payoff (highlighted yellow) is highest for which values of the cost. For example, whenever we set the cost below 375,000 it turns out that the payoff from the sequence $R \to C \to T$ gives the highest payoff among the six possible sequences. (Excel's GoalSeek is a particularly handy way for finding the threshold values quickly).
Specifically, the optimal sequence is

i) $R \to C \to T$ if the cost of $R \leq 375,000$
ii) $C \to R \to T$ if the cost of $375,000 \leq R \leq 2,142,857$
iii) $C \to T \to R$ if the cost of $2,142,857 \leq R \leq 8,640,000$
iv) don't launch if R costs more than $8,640,000$

where "R" stands for the ripening process, "C" stands for the cold chain, and "T" stands for test marketing. ■

(c) Mr. Ziffenboeffel calls you back yet again. The good news is the EU regulations and the outsourcing of the ripening tests "balance" each other out, so the cost of the test remains €1,000,000. Now the problem is that his marketing department is suggesting that the probability that the market research will result in good news about the demand could be different in light of some recent data on the sales of other subtropical fruit products. He would, therefore, like to know how your answer to (a) changes as a function of the probability of a positive result from the market research. What do you tell him?

Answer: This can be found by varying the probability of success for test marketing (highlighted by blue in the excel sheet) between 0 and 1. The optimal sequence turns out to be

i) don't launch if $p \leq 0.1905$
ii) $C \to R \to T$ if $p > 0.1905$

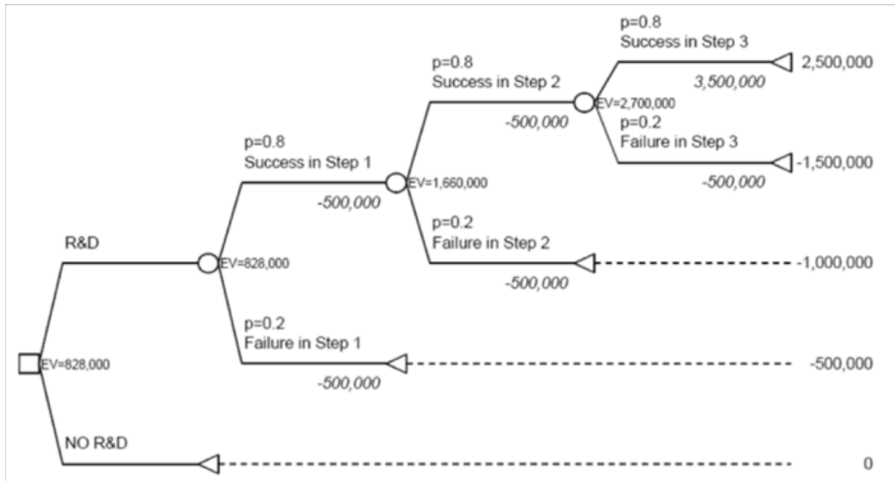
where p is the probability that the test marketing will be successful.
■

9. **Steel:** AK Steel Holding Corporation is a producer of flat-rolled carbon, stainless and electrical steels and tubular products through its wholly owned subsidiary, AK Steel Corporation. The recent surge in the demand for steel significantly increased AK's profits,[4] and it is now engaged in a research project to improve its production of rolled steel. The research involves three distinct steps, each of which must be successfully completed before the firm can implement the cost-saving new production process. If the research is completed successfully, it will save the firm \$4 million. Unfortunately, there is a chance that one or more of the research steps might fail, in which case the project is worthless. The three steps are done sequentially, so that the firm knows whether one step was successful before it has to invest in the next step. Each step has a 0.8 probability of success and each step costs \$500,000. The risks of failure in the three steps are uncorrelated with one another. AK Steel is a risk neutral company. (In case you are worried about such things, the interest rate is zero).

   (a) Draw the decision tree for the firm.

   Answer:

[4] See "Demand Sends AK Steel Profit Up 32%," *New York Time,* 07/23/2008. http://www.nytimes.com/2008/07/23/business/23steel.html?partner=rssnyt&emc=rss



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(b) If the firm proceeds with this project, what is the probability that it will succeed in implementing the new production process?

Answer: For the project to be successful, each of the three independent steps must be completed. Since the probability of success in each stage is 0.8 and the probabilities are independent, the probability of three successes is $pr = 0.8 \cdot 0.8 \cdot 0.8 = 0.8^3 = 0.512$, just over one-half. ■

(c) If the research were costless, what would be the firm's expected gain from it before the project began?

Answer: $\mathbf{E}[gain] = 0.512 \cdot \$4,000,000 + 0.488 \cdot 0 = \$2,048,000.$■

(d) Should the firm begin the research, given that each step costs $500, 000?

Answer: The expected cost of the project is

$$0.2 \cdot \$500,000 + 0.8 \cdot 0.2 \cdot \$1,000,000 + 0.8 \cdot 0.8 \cdot \$1,500,000 = \$1,220,000.$$

The first term is the probability times cost of a failure in the first step. The second term is the probability times cost of success in the first step and failure in the second step. The third term is the probability times cost of success in the first step and success in the second step (success or failure in the third step does not affect the cost of the project, just the gain from it). The expected cost is less than the expected gain (by \$828,000). Since the company is not risk averse, it should begin the project. Note that this is not the only way to do the calculation. An alternate approach would be to aggregate the costs and benefits of each possible outcome:

$$\begin{aligned} & 0.8 \cdot 0.8 \cdot 0.8 \cdot (4,000,000 - 500,000 - 500,000 - 500,000) \\ & +0.8 \cdot 0.8 \cdot 0.2(-500,000 - 500,000 - 500,000) \\ & +0.8 \cdot 0.2(-500,000 - 500,000) + 0.2(-500,000) \\ = \; & \$828,000. \end{aligned}$$

Either way, the expected net gain is $\$828,000$. ■

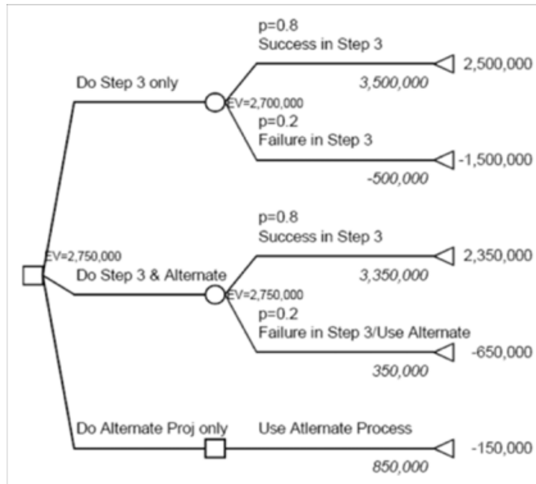
(e) Once the research has begun, should the firm quit at any point even if it has had no failures? should it ever continue the research even if it has had a failure?

Answer: NO to both. Obviously, if one stage fails, then the project cannot be completed successfully, so any more expenditures on it are a waste. If no stage has failed and at least one has succeeded, then the benefit/cost comparison of going forward with the project is even more favorable than when the project began. ■

After the firm has successfully completed steps one and two, it discovers an alternate production process that would cost $\$150,000$ and would lower production costs by $\$1,000,000$ with certainty. This process, however, is a substitute for the three-step cost-saving process; they cannot be used simultaneously. Furthermore, to have this process available, the firm must spend the $\$150,000$ before it knows if it will successfully complete step three of the three-step research project.

(f) Draw the augmented decision tree that includes the possibility of pursuing this alternate production process.

Answer:



(g) If the firm continues the three-step project, what is the chance it would get *any* value from also developing the alternate production process?

Answer: The alternate process would be used only if step three of the current project failed, which has a 0.2 probability. ■

(h) If developing the alternate production process were costless and if the firm continues the three-step project, what is the expected value that it would get from having the alternate production process available (at the beginning of research step 3)? (This is known as the *option value* of having this process available.)

Answer: There is a 0.2 probability that the alternate process would be used and a $1,000,000 value if it is used, so the option value of having the alternate process available is $200,000. ■

(i) Should the firm:

i. Pursue only the third step of the three-step project

ii. Pursue only alternate production process

iii. Pursue both the third step of the three-step project and the alternate process

Answer: Since the option value of the alternate process is greater than the cost of having this option, the alternate process should be developed if one continues with the three-step project. The net value of developing this option is $\$200,000 - \$150,000 = \$50,000$. Of course, the alternate process would also be developed if the three-step project were unavailable, since it will be used with certainty and the net value of the alternate process would then be \$850,000. The remaining question is whether AK should drop the three-step project rather than attempting the third step. Given that the alternate process will be developed, the extra (or *marginal*) value of successfully completing the three-step project would be \$3,000,000, because it would save \$3,000,000 more than the alternate process. The expected value of attempting the third step is then $0.8 \cdot \$3,000,000 = \$2,400,000$. This is greater than the \$500,000 cost of the third step, so AK should proceed with the three-step project as well as the alternate process, i.e., take strategy (iii). ■

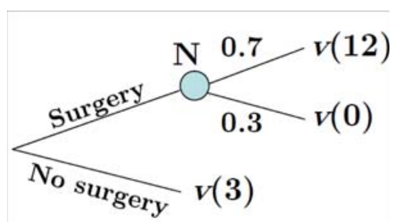
(j) If the firm had known of the alternate production process before it began the three-step research project, what should it have done?

Answer: We know that AK should pursue the alternate process: It was worth doing after successful completion of steps one and two (see (i)) and would have greater expected value if the probability of the three-step project failing were higher. In fact, the option value of the alternative process declines with each step of success in the three-step project. At the beginning of step three AK would pay up to \$200,000 for the alternate process. Convince yourself that it would be willing to pay up to \$360,000 for the alternate process at the beginning of step two and up to \$488,000 for the alternate process at the beginning of step one, assuming in each case that it couldn't wait to develop the alternate later. In fact, the option to wait until the beginning of the third period to develop the alternate process could itself be valuable, but it isn't in this case, when the process costs \$150,000. The other question is whether AK should pursue the three-step project given that it will have the alternate process available with certainty. As in (i), the marginal value of successfully completing the three-step project would be \$3,000,000, because it would save \$3,000,000 more than the alternate process. The expected value of attempting the three-step project is then $0.512 \cdot \$3,000,000 = \$1,536,000$. This is greater than the the expected cost of pursuing the three-step project, which is $0.2 \cdot 500,000 + 0.8 \cdot 0.2 \cdot 1,000,000 + 0.8 \cdot 0.8 \cdot 1,500,000 = 1,220,000$, so AK should proceed with the three-step project as well as the alternate process. This is the same calculation as in (c) and (d) except the benefit of success is now \$3,000,000 instead of \$4,000,000. ■

10. **Surgery:** A patient is very sick, and will die in 6 months if he goes untreated. The only available treatment is risky surgery. The patient is expected to live for 12 months if the surgery is successful, but the probability that the surgery fails and the patient dies immediately is 0.3.

    (a) Draw a decision tree for this decision problem.

    Answer: Using $v(t)$ to denote the value of living t more months, the following is the decision tree:



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    (b) Let $v(x)$ be the patient's payoff function, where x is the number of months till death. Assuming that $v(12) = 1$ and $v(0) = 0$, what is the lowest payoff the patient can have for living 3 months so that having surgery is a best response?

    Answer: The expected value of the surgery given the payoffs above is

$$E[v(\text{surgery})] = 0.7v(12) + 0.3v(0) = 0.7$$

which implies that if $v(3) < 0.7$ then the surgery should be performed. ■

For the rest of the problem, assume that $v(3) = 0.8$.

(c) A test is available that will provide some information that predicts whether or not surgery will be successful. A positive test implies an increased likelihood that the patient will survive the surgery as follows:
True-positive rate: The probability that the results of this test will be positive if surgery is to be successful is 0.90.
False-positive rate: The probability that the results of this test will be positive if the patient will not survive the operation is 0.10.
What is the probability of a successful surgery if the test is positive?

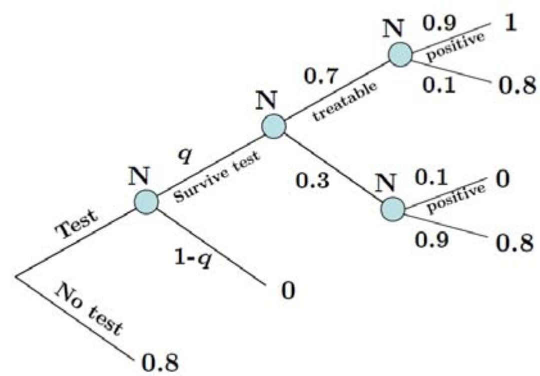
Answer: The easiest way to think about this is to imagine that the original 0.7 probability of success is true because for 70% of the sick population, call these the "treatable" patients, the surgery is successful, while for the other 30% ("untreatable") it is not, and previously the patient did not know which population he belongs to. The test can be thought of as detecting which population the patient belongs to. The above description means that if the patient is treatable then the test will claim he is treatable with probability 0.9, while if the patient is untreatable then the test will claim he is treatable with probability 0.1. Hence, 63% of the population are treatable and detected as such (0.7×0.9), while 3% of the population are *untreatable* but are detected as treatable (0.3×0.1). Hence, of the population of people for whom the test is positive, the probability of successful surgery is $\frac{63}{63+3} = 0.955$. ■

(d) Assuming that the patient has the test done, at no cost, and the result is positive, should surgery be performed?

Answer: The value from not having surgery is $v(3) = 0.8$, and a positive test updates the probability of success to 0.955 with the expected payoff being 0.955×1 so the patient should have surgery done. ■

(e) It turns out that the test may have some fatal complications, i.e., the patient may die during the test. Draw a decision tree for this revised problem.

Answer: Given the data above, we know that without taking the test the patient will not have surgery because the expected value of surgery is 0.7 while the value of living 3 months is 0.8. Also, we showed above that after a positive test the patient will choose to have surgery, and it is easy to show that after a negative test he won't (the probability of a successful outcome is $\frac{7}{7+27} = 0.206$.) Hence, the decision tree can be collapsed as follows 9the decision to have surgery have been collapsed to the relevant payoffs):



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(f) If the probability of death during the test is 0.005, should the patient opt to have the test prior to deciding on the operation?

Answer: From the decision tree in part (e), the expected value conditional on surviving the test is equal to

$$0.7(0.9 \times 1 + 0.1 \times 0.8) + 0.3(0.1 \times 0 + 0.9 \times 0.8) = 0.902$$

which implies that if the test succeeds with probability 0.995 then the expected payoff from taking the test is

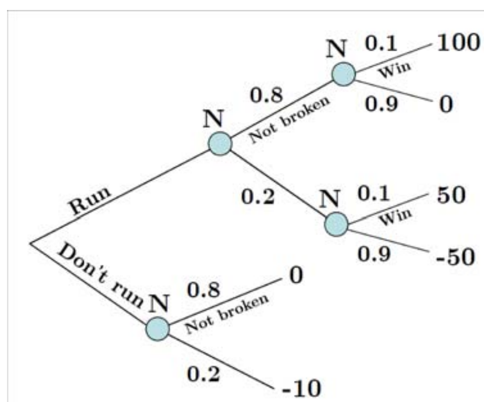
$$0.995 \times 0.902 + 0.005 \times 0 = 0.897$$

which implies that the test should be taken because $0.897 > 0.8$. ■

11. **To Run or not to Run:** You're a sprinter, and in practice today you fell and hurt your leg. An x-ray suggests that it's broken with probability 0.2. Your problem is whether you should participate in next week's tournament. If you run, you think you'll win with probability 0.1. If your leg is broken and you run, then it will be further damaged and your payoffs are as follows:
+100 if you win the race and your leg isn't broken;
+50 if you win and your leg is broken;
0 if you lose and your leg isn't broken;
−50 if you lose and your leg is broken;
−10 if you don't run and if your leg is broken;
0 if you don't run and your leg isn't broken.

    (a) Draw the decision tree for this problem.

    Answer:



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    (b) What is your best choice of action and its expected payoff?

    Answer: The expected payoff from not running is

$$E[v(\text{not run})] = 0.8 \times 0 + 0.2(-10) = -2,$$

and the expected payoff from running is

$$E[v(\text{run})] = 0.8 \times (0.1 \times 100 + 0.9 \times 0) + 0.2 \times (0.1 \times 50 + 0.9 \times (-50)) = 0,$$

so the best choice is to run and have an expected payoff of 0. ■

You can gather some more information by having more tests, and you can gather more information about whether you'll win the race by talking to your coach.

(c) What is the value of perfect information about the state of your leg?

Answer: If you knew your leg is broken then running yields an expected payoff of $0.1 \times 50 + 0.9 \times (-50) = -40$, while not running yields a payoff of -10, so you would not run and get -10. If you knew your leg is not broken then the expected payoff from running is $0.1 \times 100 + 0.9 \times 0 = 10$, while the payoff from not running is 0, and hence you would run and get 10. Before getting the information you know your leg is broken with probability 0.2, so before getting the perfect information, your expected payoff from being able to then act on the perfect information is $0.2(-10) + 0.8 \times 10 = 6$. Recall from (b) that the expected payoff of not having perfect information is 0, so the value of being able to obtain the perfect information is $6 - 0 = 6$. ■

(d) What is the value of perfect information about whether you'll win the tournament?

Answer: In this case we know that you will run if you know you will win and you will not if you know you will lose. Hence, with probability 0.1 you will learn that you'll win and your expected payoff (depending on the state of your leg) is $0.2 \times 100 + 0.8 \times 50 = 60$.Similarly, with probability 0.9 you learn that you'll lose in which case your expected payoff is $0.2 \times (-10) + 0.8 \times 0 = -2$. Before getting the information you know you will win with probability 0.1, so before getting the perfect information, your expected payoff from being able to then act on the perfect information is $0.1 \times 60 + 0.9(-2) = 4.2$. Recall from (b) that the expected payoff of not having perfect information is 0, so the value of being able to obtain the perfect information is $4.2 - 0 = 4.2$. ■

(e) As stated above, the probability that your leg is broken and the probability that you will win the tournament are independent. Can you use a decision tree in the case that the probability that you will win the race depends on whether your leg is broken?

Answer: Yes. All you need to do is have different probabilities of winning that depend on whether or not your leg is broken. ■

12. **More Oil:** Chevron, the No. 2 US oil company, is facing a tough decision. The new oil project dubbed "Tahiti" is scheduled to produce its first commercial oil in mid-2008, yet it is still unclear how productive it will be. "Tahiti is one of Chevron's five big projects," told Peter Robertson, vice chairman of the company's board to the *Wall Street Journal*.[5] Still, it was unclear whether the project will result in the blockbuster success Chevron is hoping for. As of June 2007, $4-billion has been invested in the high-tech deep sea platform, which suffices to perform early well tests. Aside from offering information on the type of reservoir, the tests will produce enough oil to just cover the incremental costs of the testing (beyond the $4 billion investment). Following the test wells, Chevron predicts one of three possible scenarios. The optimistic one is that Tahiti sits on one giant, easily accessible oil reservoir, in which case the company expects to extract 200,000 barrels a day after expending another $5 billion in platform setup costs, with a cost of extraction at about $10 a barrel. This will continue for 10 years, after which the field will have no more economically recoverable oil. Chevron believes this scenario has a 1 in 6 chance of occurring. A less rosy scenario, that is twice as likely as the optimistic one, is that Chevron would have to drill two more wells at an additional cost of $0.5 billion each (above and beyond the $5 billion set-up costs), and in which case production will be around 100,000 barrels a day with a cost of extraction at about $30 a barrel, and the field will still be depleted after 10 years. The worst case scenario involves

[5] "Chevron's Tahiti Facility Bets Big on Gulf Oil Boom." Jun 27, 2007. pg. B5C. http://proquest.umi.com/pqdweb?did=1295308671&sid=1&Fmt=3&clientId=1566&RQT=309&VName=PQD

the oil tucked away in numerous pockets, requiring expensive water injection techniques which would include up-front costs of another \$4 billion (above and beyond the \$5 billion set-up costs), extraction costs of \$50 a barrel, and production is estimated to be at about 60,000 barrels a day, for 10 years. Bill Varnado, Tahiti's project manager, was quoted giving this least desirable outcome odds of 50-50.
The current price of oil is \$70 a barrel. For simplicity, assume that the price of oil and all costs will remain constant (adjusted for inflation) and that Chevron's faces a 0% cost of capital (also adjusted for inflation).

(a) If the test-wells would not produce information about which one of three possible scenarios will result, should Chevron invest the set-up costs of \$5 billion to be prepared to produce at whatever scenario is realized?

Answer: We start by noticing that the \$2 billion that were invested are a sunk cost and hence irrelevant. Also, since the cost of capital is just about the same as the projected increase in oil prices, we do not need to discount future oil revenues to get the net present value (NPV) sine the two effects (price increase and time discounting) will cancel each other out. If the company invests the \$2.5 billion dollars, then they will be prepared to act upon whatever scenario arises (great with probability $\frac{1}{6}$, ok with probability $\frac{1}{3}$, or bad with probability $\frac{1}{2}$). Notice from the table below that in each scenario the added costs of extraction that Chevron needs to invest (once it becomes clear which scenario it is) is worthwhile (e.g., even in the bad scenario, the profits are \$2.19 billion, which covers the added drilling costs of \$2 billion in this case.) Hence, Chevron would proceed to drill in each of the three scenarios, and the expected profits including the initial \$2.5 billion investment would be,

$$E\pi = \frac{1}{6}\times(\$21B)+\frac{1}{3}\times(\$7.3B-\$0.5B)+\frac{1}{2}\times(\$2.19B-\$2B)-\$2.5B = \$3,511,666,667.$$

(b) If the test-wells do produce accurate information about which of three possible scenarios is true, what is the added value of performing these tests?

Answer: Now, if the test drilling will reveal the scenario ahead of time, then in the event of the bad scenario the revenues would not cover the total investment of \$4.5 billion (\$2.5 billion initially, and another \$2 billion for the bad scenario.) In the great and ok scenarios, however, the revenues cover all the costs. Hence, with the information Chevron would not proceed with the investments at all when the bad scenario happens (probability $\frac{1}{2}$), and proceed only when the scenario is great or ok, yielding an expected profit of

$$E\pi = \frac{1}{6}\times(\$21B - 2.5B) + \frac{1}{3}\times(\$7.3B - \$2.5B - \$0.5B) + \frac{1}{2}\times 0 = \$4,666,666,667.$$

Hence, the added value of performing the tests is,

$$v_{\text{info}} = \$4,666,666,667 - \$3,511,666,667 = \$1,155,000,000$$

	great	ok	bad
quantity/day	200000	100000	60000
price	\$70	\$70	\$70
extr. Costs	\$10	\$30	\$50
gross profits/day	\$12,000,000	\$4,000,000	\$1,200,000
gross profits 5 yr	\$21,900,000,000	\$7,300,000,000	\$2,190,000,000
added drilling costs	\$0	\$500,000,000	\$2,000,000,000
net profits from extraction	\$21,900,000,000	\$6,800,000,000	\$190,000,000
probability	0.166666667	0.333333333	0.5
Expected profits w/o initial investment:	\$6,011,666,667		
initial investment:	\$2,500,000,000		
Expected net pofits	\$3,511,666,667		
Expected net profits with added info	\$4,666,666,667		
value of information:	\$1,155,000,000		

■

13. **Today, Tomorrow or the Day after:** A player has \$100 today that need to be consumed over the next three periods, $t = 1, 2, 3$. The utility over consuming $\$x_t$ in period t is given by the utility function $u(x) = \ln(x)$, and at period $t = 1$, the player values his net present value from all consumption as $u(x_1) + \delta u(x_2) + \delta^2 u(x_3)$, where $\delta = 0.9$.

(a) How will the player plan to spend the \$100 over the three periods of consumption?

Answer: The player will maximize

$$\max_{x_1,x_2} \ln(x_1) + \delta \ln(x_2) + \delta^2 \ln(100 - x_1 - x_2)$$

which yields the following two first-order equations:

$$\begin{aligned} \frac{1}{x_1} - \frac{\delta^2}{100 - x_1 - x_2} &= 0, \\ \frac{\delta}{x_2} - \frac{\delta^2}{100 - x_1 - x_2} &= 0 . \end{aligned}$$

From these two equations conclude that

$$\frac{1}{x_1} = \frac{\delta}{x_2} ,$$

or $x_2 = \delta x_1$. We can then then substitute x_2 with δx_1 in the first equation above to obtain,

$$\frac{1}{x_1} - \frac{\delta^2}{100 - x_1 - \delta x_1} = 0$$

or

$$100 - x_1 - \delta x_1 - \delta^2 x_1 = 0$$

and the solution is

$$x_1 = \frac{100}{\delta + \delta^2 + 1}$$

and in turn

$$x_2 = \frac{\delta 100}{\delta + \delta^2 + 1}, \text{ and } x_3 = \frac{\delta^2 100}{\delta + \delta^2 + 1}. \blacksquare$$

(b) Imagine that the player knows that in period $t = 2$ he will receive an additional gift of \$20. How will he choose to allocate his original \$100 initially, and how will he spend the extra \$20?

Answer: After spending $x_1 \leq 100$, the player has $100 - x_1 + 20$ in the beginning of the second period. We can now solve this backward and assume that the player has $100 - x_1 + 20$ in the beginning of period 2 and has to choose between x_2 and x_3 so that he solve,

$$\max_{x_2} \ln(x_2) + \delta \ln(120 - x_1 - x_2)$$

with the first order condition

$$\frac{1}{x_2} - \frac{\delta}{120 - x_1 - x_2} = 0$$

which yields,

$$x_2 = \frac{120 - x_1}{1 + \delta} \text{ and } x_3 = \frac{\delta(120 - x_1)}{1 + \delta}.$$

Now we can step back to the first period and solve the optimal choice of x_1 given the way x_2 and x_3 will be chosen later. The player solves,

$$\max_{x_1, x_2} \ln(x_1) + \delta \ln(\frac{120 - x_1}{1 + \delta}) + \delta^2 \ln(\frac{\delta(120 - x_1)}{1 + \delta})$$

and the first order condition is,

$$\frac{1}{x_1} - \frac{\delta(1 + \delta)}{120 - x_1} \times \frac{1}{1 + \delta} - \frac{\delta^2(1 + \delta)}{120 - x_1} \times \frac{1}{1 + \delta} = 0$$

or

$$x_1 = \frac{120}{1 + \delta + \delta^2}.$$

Notice, however, that as δ drops, x_1 increases, and for a small enough δ this equation will call for $x_1 > 100$. In particular, the value of δ for which $x_1 = 100$ can be solved as follows,

$$100 = \frac{120}{1 + \delta + \delta^2},$$

or, $\delta = \frac{3}{10}\sqrt{5} - \frac{1}{2} \approx 0.17$. However, $x_1 > 100$ is not possible, so the solution is,

$$x_1 = \begin{matrix} \frac{120}{1+\delta+\delta^2} & \text{if } \delta \geq 0.17 \\ 100 & \text{if } \delta < 0.17 \end{matrix}$$

and from the calculations earlier,

$$x_2 = \frac{120 - x_1}{1+\delta} \text{ and } x_3 = \frac{\delta(120 - x_1)}{1+\delta}.$$

■

Part II

Static Games of Complete Information

3
Preliminaries

1. **eBay:** Hundreds of millions of people bid on eBay auctions to purchase goods from all over the world. Despite being done online, in spirit these auctions are similar to those conducted centuries ago. Is an auction a game? Why or why not?

   Answer: An auction is indeed a game. A bidder's payoff depends on his bid and on the bid of other bidders, and hence there are players, actions (which are bids) and payoffs that depend on all the bids. The winner gets the item and pays the price (which on eBay is the second highest bid plus the auction increment), while the losers all pay nothing and get nothing. ■

2. **Penalty Kicks:** Imagine a kicker and a goalie who confront each other in a penalty kick that will determine the outcome of the game. The kicker can kick the ball left or right, while the goalie can choose to jump left or right. Because of the speed of the kick, the decisions need to be made simultaneously. If the goalie jumps in the same direction as the kick, then the goalie wins and the kicker loses. If the goalie jumps in the opposite direction of the kick then the kicker wins and the goalie loses. Model this as a normal form game and write down the matrix that represents the game you modeled.

Answer: There are two players, 1 (kicker) and 2 (goalie). Each has two actions, $a_i \in \{L, R\}$ to denote left or right. The kicker wins when they choose opposite directions while the goalie wins if they choose the same direction. Using 1 to denote a win and -1 to denote a loss, we can write $v_1(L,R) = v_1(R,L) = v_2(L,L) = v_2(R,R) = 1$ and $v_1(L,L) = v_1(R,R) = v_2(L,R) = v_2(R,L) = -1$. The matrix is therefore,

		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

■

3. **Meeting Up:** Two old friends plan to meet at a conference in San Francisco, and agreed to meet by the tower. When arriving in town, each realizes that there are two natural choices: Sutro Tower or Coit Tower. Not having cell phones, each must choose independently which tower to go to. Each player prefers meeting up to not meeting up, and neither cares where this would happen. Model this as a normal form came, and write down the matrix form of the game.

   Answer: There are two players, 1 and 2. Each has two actions, $a_i \in \{S, C\}$ to denote Sutro or Coit. Both players are happy if they choose the same tower and unhappy if they don't. Using 1 to denote happy and 0 to denote unhappy, we can write $v_i(S,S) = v_i(C,C) = 1$ and $v_i(S,C) = v_i(C,S) = 0$ for $i \in \{1, 2\}$. The matrix is therefore,

		Player 2	
		S	C
Player 1	S	$1, 1$	$-1, -1$
	C	$-1, -1$	$1, 1$

■

4. **Hunting:** Two hunters, players 1 and 2, can each choose to hunt a stag, which provides a rather large and tasty meal, or hunt a hare, also tasty, but much less filling. Hunting stags is challenging and requires mutual cooperation. If either hunts a stag alone, then the stag will get away, while hunting the stag together guarantees that the stag is caught. Hunting hares is an individualistic enterprise that is not done in pairs, and whoever chooses to hunt a hare will catch one. The payoff from hunting a hare is 1, while the payoff to each from hunting a stag together is 3. The payoff from an unsuccessful stag-hunt is 0. Represent this game as a matrix.

   Answer: This is the famous "stag hunt" game. Using S for stag and H for hare, the matrix is,

		Player 2	
		S	H
Player 1	S	$3,3$	$0,1$
	H	$1,0$	$1,1$

   ■

5. **Matching Pennies:** Players 1 and 2 both put a penny on a table simultaneously. If the two pennies come up the same side (heads or tails) then player 1 gets both pennies, otherwise player 2 gets both pennies. Represent this game as a matrix.

   Answer: Letting H denote a choice of heads and T a choice of tails, and letting winning give a payoff of 1 while losing gives -1, the matrix is therefore,

		Player 2	
		H	T
Player 1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

   ■

6. **Price Competition:** Imagine a market with demand $p(q) = 100 - q$. There are two firms, 1 and 2, and each firm i has to simultaneously choose it's price

p_i. If $p_i < p_j$, then firm i gets all of the market while no one demands the good of firm j. If the prices are the same then both firms equally split the market demand. Imagine that there are no costs to produce any quantity of the good. (These are two large dairy farms, and the product is manure.) Write down the normal form of this game.

Answer: The players are $N = \{1, 2\}$ and the strategy sets are $S_i = [0, \infty]$ for $i \in \{1, 2\}$ and firms choose prices $p_i \in S_i$. To calculate payoffs, we need to know what the quantities will be for each firm given prices (p_1, p_2). Given the assumption on ties, the quantities are given by,

$$q_i(p_i, p_j) = \begin{cases} 100 - p_i \text{ if } p_i < p_j \\ 0 \text{ if } p_i > p_j \\ \frac{100-p_i}{2} \text{ if } p_i = p_j \end{cases}$$

which in turn means that the payoff function is given by quantity times price (there are no costs):

$$v_i(p_i, p_j) = \begin{cases} (100 - p_i)p_i & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j \\ \frac{100-p_i}{2}p_i & \text{if } p_i = p_j \end{cases}$$

■

7. **Public Good Contribution:** Three players live in a town and each can choose to contribute to fund a street lamp. The value of having the street lamp is 3 for each player and the value of not having one is 0. The Mayor asks each player to either contribute 1 or nothing. If at least two players contribute then the lamp will be erected. If one or less people contribute then the lamp will not be erected, in which case any person who contributed will not get their money back. Write down the normal form of this game.

Answer: The set of players is $N = \{1, 2, 3\}$ and each has an strategy set $S_i = \{0, 1\}$ where 0 is not to contribute and 1 is to contribute. The payoffs of player i from a profile of strategies (s_1, s_2, s_3) is given by,

$$v_i(s_1, s_2, s_3) = \begin{cases} 0 & \text{if } s_i = 0 \text{ and } s_j = 0 \text{ for some } j \neq i \\ 3 & \text{if } s_i = 0 \text{ and } s_j = 1 \text{ for both } j \neq i \\ -1 & \text{if } s_i = 1 \text{ and } s_j = 0 \text{ for both } j \neq i \\ 2 & \text{if } s_i = 1 \text{ and } s_j = 1 \text{ for some } j \neq i \end{cases}$$

■

4

Rationality and Common Knowledge

1. Prove Proposition **??**: If the game $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ has a strictly dominant strategy equilibrium s^D, then s^D is the unique dominant strategy equilibrium.

   Answer: Assume not. That is, there is some other strategy profile $s^* \neq s^D$ that is also a strictly dominant strategy equilibrium. But this implies that for every i, $s_i^* > s_i^D$, which contradicts that s^D is a strictly dominant strategy equilibrium. ■

2. **Weak dominance.** We call the strategy profile $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$.

   (a) Provide an example of a game in which there is no weakly dominant strategy equilibrium.

   Answer:

		Player 2	
		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

■

(b) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

Answer: In the following game each player is indifferent between his strategies and so each one is weakly dominated by the other. This means that any outcome is a weakly dominant strategy equilibrium.

		Player 2	
		H	T
Player 1	H	$1,1$	$1,1$
	T	$1,1$	$1,1$

■

3. **Discrete first-price auction:** An item is up for auction. Player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either $0,\ 1$ or 2. If player i bids more than player j then i win's the good and pays his bid, while the loser does not pay. If both players bid the same amount then a coin is tossed to determine who the winner is, who gets the good and pays his bid while the loser pays nothing.

(a) Write down the game in matrix form.

Answer: We need to determine what the payoffs are if the bidders tie. The one who wins the coin toss bids his bid and the loser gets and pays nothing. Hence, we can just calculate the expected payoff as a 50:50 lottery between getting nothing and winning. For example, if both players bid 2 then player 1 gets $3-2=1$ unit of payoff with probability $\frac{1}{2}$ and player 2 gets $5-2=3$ units of payoff with probability $\frac{1}{2}$, so the pair of payoffs is $(\frac{1}{2}, \frac{3}{2})$.

Player 1 \ Player 2	0	1	2
0	$\frac{3}{2}, \frac{5}{2}$	$0, 4$	$0, 3$
1	$2, 0$	$1, 2$	$0, 3$
2	$1, 0$	$1, 0$	$\frac{1}{2}, \frac{3}{2}$

(b) Does any player have a strictly dominated strategy?

Answer: Yes - for player 2 bidding 0 is strictly dominated by bidding 2. ■

(c) Which strategies survive IESDS?

Answer: After removing the strategy 0 of player 2, player 1's strategy of 0 is dominated by 2, so we can remove that too. But then, in the remaining 2×2 game where both players can choose 1 or 2, bidding 1 is strictly dominated by bidding 2 for player 2, and after this round, bidding 1 is strictly dominated by bidding 2 for player 1. Hence, the unique strategy that survives IESDS is $(2, 2)$ yielding expected payoffs of $(\frac{1}{2}, \frac{3}{2})$. ■

4. **eBay's recommendation:** It is hard to imagine that anyone is not familiar with eBay©, the most popular auction website by far. The way a typical eBay auction works is that a good is placed for sale, and each bidder places a "proxy bid", which eBay keeps in memory. If you enter a proxy bid that is lower than the current highest bid, then your bid is ignored. If, however, it is higher, then the current bid increases up to one increment (say, 1 cent) above the *second highest* proxy bid. For example, imagine that three people placed bids on a used laptop of \$55, \$98 and \$112. The current price will be at \$98.01, and if the auction ended the player who bid \$112 would win at a price of \$98.01. If you were to place a bid of \$103.45 then the who bid \$112 would still win, but at a price of \$103.46, while if your bid was \$123.12 then

you would win at a price of \$112.01.

Now consider eBay's historical recommendation that you think hard about your value of the good, and that you enter your true value as your bid, no more, no less. Assume that the value of the good for each potential bidder is independent of how much other bidders value it.

(a) Argue that bidding more than your valuation is weakly dominated by actually bidding your valuation.

Answer: If you put in a bid $b_i = b'_i > v_i$ where v_i is your valuation, then only the three following cases can happen: (i) All other bids are below v_i. In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll win at the same price. (ii) Some bid is above b'_i. In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll lose to a higher bid. (iii) No bids are above b'_i and some bid b^*_j is in between v_i and b'_i. In this case bidding b'_i will cause you to win in and pay $b^*_j > v_i$ which means that your payoff is negative, while if you would have bid $b_i = v_i$ then you would lose and get nothing. Hence, in cases (i) and (ii) bidding v_i would do as well as bidding b'_i, and in case (iii) it would do strictly better, implying that bidding more than your valuation is weakly dominated by actually bidding your valuation. ■

(b) Argue that bidding less than your valuation is weakly dominated by actually bidding your valuation.

Answer: If you put in a bid $b_i = b'_i < v_i$ where v_i is your valuation, then only the three following cases can happen: (i) Some other bid are above v_i. In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll lose to a higher bid. (ii) All other bids are below b'_i. In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll win at the same price. (iii) No bids are above v_i and some bid b^*_j is in between b'_i and v_i. In this case bidding b'_i will cause you to lose and get nothing, while if you would have bid $b_i = v_i$ then you would win and get a positive payoff of $v_i - b^*_j$. Hence, in cases (i) and (ii) bidding v_i would do as well as bidding b'_i, and in case (iii) it would do strictly better, implying that bidding less than your valuation is weakly dominated by actually bidding your valuation. ■

(c) Use your analysis above to make sense of eBay's recommendation. Would you follow it?

Answer: The recommendation is indeed supported by an analysis of rational behavior.[1]

5. In the following normal-form game, which strategy profiles survive iterated elimination of strictly dominated strategies?

		Player 2		
		L	C	R
	U	$6,8$	$2,6$	$8,2$
Player 1	M	$8,2$	$4,4$	$9,5$
	D	$8,10$	$4,6$	$6,7$

Answer: First, U is dominated by M for player 1. In the remaining game, C is dominated by R for player 2. No more strategies are strictly dominated, and hence $(M,L),(M,R),(D,L)$ and (D,R) all survive IESDS. (Note: after the last stage above, D is *weakly* dominated by M for player 1, after which L is dominated by R for player 1, so that (M,R) would be the only strategy profile that would survive iterated elimination of weakly dominated strategies. ■

6. **Roommates**: Two roommates need to each choose to clean their apartment, and each can choose an amount of time $t_i \geq 0$ to clean. If their choices are t_i and t_j, then player i's payoff is given by $(10-t_j)t_i - t_i^2$. (This payoff function implies that the more one roommate cleans, the less valuable is cleaning for the other roommate.)

[1] Those familiar with eBay know about sniping, which is bidding in the last minute. It still is a weakly dominated strategy to bid your valuation at that time, and waiting for the last minute may be a "best response" if you believe other people may respond to an early bid. More on this is discussed in chapter 13.

(a) What is the best response correspondence of each player i?

Answer: Player i maximizes $(10-t_j)t_i - t_i^2$ given a belief about t_j, and the first-order optimality condition is $10 - t_j - 2t_i = 0$ implying that the best response is $t_i = \frac{10-t_j}{2}$. ■

(b) Which choices survive one round of IESDS?

Answer: The most player i would choose is $t_i = 5$, which is a BR to $t_j = 0$. Hence, any $t_i > 5$ is dominated by $t_i = 5$.[2] Hence, $t_i \in [0, 5]$ are the choices that survive one round of IESDS.

(c) Which choices survive IESDS?

Answer: The analysis follows the same ideas that were used for the Cournot duopoly in section 4.2.2. In the second round of elimination, because $t_2 \leq 5$, the best response $t_i = \frac{10-t_j}{2}$ implies that firm 1 will choose $t_1 \geq 2.5$, and a symmetric argument applies to firm 2. Hence, the second round of elimination implies that the surviving strategy sets are $t_i \in [2.5, 5]$ for $i \in \{1, 2\}$. If this process were to converge to an interval, and not to a single point, then by the symmetry between both players, the resulting interval for each firm would be $[t_{\min}, t_{\max}]$ that simultaneously satisfy two equations with two unknowns: $t_{\min} = \frac{10-t_{\max}}{2}$ and $t_{\max} = \frac{10-t_{\min}}{2}$. However, the only solution to these two equations is $t_{\min} = t_{\max} = \frac{10}{3}$. Hence, the unique pair of choices that survive IESDS for this game are $t_1 = t_2 = \frac{10}{3}$. ■

7. **Campaigning:** Two candidates, 1 and 2, are running for office. They each have one of three choices in running their campaign: focus on the positive aspects of one's own platform, call this a positive campaign (or P), focus on the positive aspects of one's own platform while attacking one's opponent's

[2] This can be shown directly: The payoff from choosing $t_i = 5$ when the opponent is choosing t_j is $v(5, t_j) = (10 - t_j)5 - 25 = 25 - 5t_j$. The payoff from choosing $t_i = 5 + k$ where $k > 0$ when the opponent is choosing t_j is $v(5+k, t_j) = (10-t_j)(5+k)-(5+k)^2 = 25-5t-k^2-t_jk_j$, and because $k > 0$ it follows that $v(5+k, t_j) < v(5, t_j)$.

campaign, call this a balanced campaign (or B), and finally, focus only on attacking one's opponent, call this a negative campaign (or N). All a candidate cares about is the probability of winning, so assume that if a candidate expects to win with probability $\pi \in [0,1]$, then his payoff is π. The probability that a candidate wins depends on his choice of campaign and his opponent's choice. The probabilities of winning are given as follows:

- – If both choose the same campaign, each wins with probability 0.5.
  – If candidate i uses a positive campaign while $j \neq i$ uses a balanced one, then i loses for sure.
  – If candidate i uses a positive campaign while $j \neq i$ uses a negative one, then i wins with probability 0.3.
  – If candidate i uses a negative campaign while $j \neq i$ uses a balanced one, then i wins with probability 0.6.

(a) Model this story as a normal form game. (It suffices to be specific about the payoff function of one player, and explaining how the other player's payoff function is different and why.)

Answer: There are two players $i \in \{1,2\}$, each has three strategies $S_i = \{P, B, N\}$ and the payoffs are $v_i(P,P) = v_i(B,B) = v_i(N,N) = 0.5$; $v_1(B,P) = v_2(P,B) = 1$; $v_2(B,P) = v_1(P,B) = 0$; $v_1(P,N) = v_2(N,P) = 0.3$; $v_2(P,N) = v_1(N,P) = 0.7$; $v_1(N,B) = v_2(B,N) = 0.6$; and $v_2(N,B) = v_1(B,N) = 0.4$. ■

(b) Write the game in matrix form.

Answer:

		Player 2		
		P	B	N
	P	$0.5, 0.5$	$0, 1$	$0.3, 0.7$
Player 1	B	$1, 0$	$0.5, 0.5$	$0.4, 0.6$
	N	$0.7, 0.3$	$0.6, 0.4$	$0.5, 0.5$

■

(c) What happens at each stage of elimination of strictly dominated strategies? Will this procedure lead to a clear prediction?

Answer: Notice that for each player B strictly dominates P. In the remaining 2×2 game without the strategies P, N strictly dominates B for each player. Hence, the unique clear prediction is that both candidates will engage in negative campaigns. ■

8. Consider the p-Beauty contest presented in section 4.3.5.

(a) Show that if player i believes that everyone else is choosing 20 then 19 is not the only best response for any number of players n.

Answer: If everyone else is choosing 20 and if player i chooses 19 then $\frac{3}{4}$ of the average will be somewhere below 15, and 19 is closer to that number, and therefore is a best response. But the same argument holds for any choice of player i that is between 15 and and 20 regardless of the number of players. (In fact, you should be able to convince yourself that this will be true for any choice of i between 10 and 20.) ■

(b) Show that the set of best response strategies to everyone else choosing the number 20 depends on the number of players n.

Answer: Imagine that $n = 2$. If one player j is choosing 20, then any number s_i between 0 and 19 will beat 20. This follows because the target number ($\frac{3}{4}$ of the average) is equal to $\frac{3}{4} \times \frac{20+s_i}{2} = \frac{15}{2} + \frac{3}{8}s_i$, the distance between 20 and the target number is $\frac{25}{2} - \frac{3}{8}s_i$ (this will always be positive because the target number is less than 20) while the distance between s_i and the target number is $\left|\frac{5}{8}s_i - \frac{15}{2}\right|$. The latter will be smaller than the former if and only if $\left|\frac{5}{8}s_i - \frac{15}{2}\right| < \frac{25}{2} - \frac{3}{8}s_i$, or $-20 < s_i < 20$. Given the constraints on the choices, $BR_i \in \{0, 1, ...19\}$. Now imagine that $n = 5$. The target number is equal to $\frac{3}{4} \times \frac{80+s_i}{5} = 12 + \frac{3}{20}s_i$, the distance between 20 and the target number is $8 - \frac{3}{20}s_i$ while the distance between s_i and the target number is $\left|\frac{17}{20}s_i - 12\right|$. The latter will be smaller than the former if and only if $\left|\frac{17}{20}s - 12\right| < 8 - \frac{3}{20}s$, or $\frac{40}{7} < s_i < 20$. Hence, $BR_i = \{6, 7, ..., 19\}$. You should be able to convince yourself that as $n \to \infty$, if everyone but i chooses 20 then i's best response will converge to $BR_i = \{10, 11, ..., 19\}$. ■

9. Consider the p-Beauty contest presented in section 4.3.5. Show that if the number of players $n > 2$ then the choices $\{0, 1\}$ for each player are both Rationalizable, while if $n = 2$ then only the choice of $\{0\}$ by each player is Rationalizable.

   Answer: We start with $n = 2$. If player 2 chooses 0 then player 1's best response is clearly 0. Now imagine that player 2 is choosing 1. If player 1 chooses $s_1 = 1$ then they tie and he wins with probability 0.5, while if he chooses $s_1 = 0$ then the target number is $\frac{3}{8}$ and he wins for sure. Hence, 0 is a best reply to 1 and only the choice of 0 by both players is Rationalizable. Now assume that $n > 2$. If all player's but i choose 0, then i's best response is 0, and hence choosing 0 is Rationalizable. Now assume that everyone but i chooses 1. If player 1 chooses $s_1 = 1$ then he ties. If he chooses $s_1 = 0$ then the target number is $\frac{3}{4} \times \frac{n}{n+1} \geq \frac{1}{2}$ because $n \geq 2$ (it is equal to $\frac{1}{2}$ when $n = 2$ and greater when $n > 2$). Hence, for $n \geq 2$ the set of Rationalizable choices is $\{0, 1\}$. The analysis in the text shows that no other choice is Rationalizable when $p = \frac{3}{4}$. ■

10. **Popsicle stands:** There are five lifeguard towers lined along a beach, where the left-most tower is number 1 and the right most tower is number 5. Two vendors, players 1 and 2, each have a popsicle stand that can be located next to one of five towers. There are 25 people located next to each tower, and each person will purchase a popsicle from the stand that is closest to him or her. That is, if player 1 locates his stand at tower 2 and player 2 at tower 3, then 50 people (at towers 1 and 2) will purchase from player 1, while 75 (from towers 3,4 and 5) will purchase from vendor 2. Each purchase yields a profit of \$1.

(a) Specify the strategy set of each player. Are there any strictly dominated strategies?

Answer: The strategy sets for each player are $S_i = \{1, 2, ..., 5\}$ where each choice represents a tower. To see whether there are any strictly dominated strategies it is useful to construct the matrix representation of this game. Assume that if a group of people are indifferent between the two places (equidistant) then they will split between the two vendors (e.g., if the vendors are at the same tower then their payoffs will be 62.5 each, while if they are located at towers 1 and 3 then they split the people from tower 2 and their payoffs are 37.5 and 87.5 respectively.) Otherwise they get the people closest to them, so payoffs are:

		Player 2				
		1	2	3	4	5
	1	$62.5, 62.5$	$25, 100$	$37.5, 87.5$	$50, 75$	$62.5, 62.5$
Player 1	2	$100, 25$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$75, 50$
	3	$87.5, 37.5$	$75, 50$	$62.5, 62.5$	$75, 50$	$87.5, 37.5$
	4	$75, 50$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$100, 25$
	5	$62.5, 62.5$	$50, 75$	$37.5, 87.5$	$25, 100$	$62.5, 62.5$

Notice that the choices of 1 and 5 are strictly dominated by any other choice for both players 1 and 2. ■

(b) Find the set of strategies that survive Rationalizability.

Answer: Because the strategies 1 and 5 are strictly dominated then they cannot be a best response to any belief (Proposition 4.3). In the reduced game in which these strategies are removed, both strategies 2 and 4 are dominated by 3, and therefore cannot be a best response in this second stage. Hence, only the choice $\{3\}$ is rationalizable. ■

5

Pinning Down Beliefs: Nash Equilibrium

1. Prove Proposition **??**.

   Answer: (1) Assume that s^* is a strict dominant strategy equilibrium. This implies that for any player i, s_i^* is a best response to any choice of his opponents including s_{-i}^*, which in turn implies that s^* is a Nash equilibrium.
   (2) Assume that s^* is the unique survivor of IESDS. By construction of the IESDS procedure, there is no round in which s_i^* is strictly dominated against the surviving strategies of i's opponents, an in particular, against s_{-i}^*, implying that s_i^* is a best response to s_{-i}^*, which in turn implies that s^* is a Nash equilibrium.
   (3) Assume that s^* is the unique Rationalizable strategy profile. By construction of the Rationalizability procedure, any strategy of player i that survives a round of rationalizability can be a best response to some strategy of i's opponents that survives that round. Hence, by definition, s_i^* is a best response to s_{-i}^*, which in turn implies that s^* is a Nash equilibrium. ■

2. A strategy $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i^W, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$. Provide an example of a game

for which there is a weakly dominant strategy equilibrium, as well as another Nash equilibrium.

Answer: Consider the following game:

		Player 2	
		L	R
Player 1	U	$1,1$	$1,1$
	D	$1,1$	$2,2$

In this game, (D, R) is a weakly dominant strategy equilibrium (and of course, a Nash equilibrium), yet (U, L) is a Nash equilibrium that is not a weakly dominant strategy equilibrium. ■

3. Consider a 2 player game with m pure strategies for each player that can be represented by a $m \times m$ matrix. Assume that for each player no two payoffs in the matrix are the same.

   (a) Show that if $m = 2$ and the game has a unique pure strategy Nash equilibrium then this is the unique strategy profile that survives IESDS.

   Answer: Consider a general 2×2 game as follows,

		Player 2	
		s_{2a}	s_{2b}
Player 1	s_{1a}	v_1^{aa}, v_2^{aa}	v_1^{ab}, v_2^{ab}
	s_{1b}	v_1^{ba}, v_2^{ba}	v_1^{bb}, v_2^{bb}

   and assume without loss of generality that (s_{1a}, s_{2a}) is the unique pure strategy Nash equilibrium.[1] Two statements are true: first, because (s_{1a}, s_{2a}) is a Nash equilibrium and no two payoffs are the same for each player then $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$. Second, because (s_{1b}, s_{2b}) is *not* a Nash equilibrium then $v_1^{bb} > v_1^{ab}$ and $v_2^{bb} > v_2^{ba}$ cannot hold together (otherwise it would have been another Nash equilibrium). These

[1] The term "without loss of generality" means that we are choosing one particular strategy profile but there is nothing special about it and we could have chosen any one of the others using the same argument.

two statements imply that either (i) $v_1^{aa} > v_1^{ba}$ *and* $v_1^{ab} > v_1^{bb}\}$ in which case s_{1b} is dominated by s_{1a}, or (ii) $v_2^{aa} > v_2^{ab}$ *and* $v_2^{ba} > v_2^{bb}$ in which case s_{2b} is strictly dominated by s_{2a}. This implies that either s_{1b} or s_{2b} (or both) will be eliminated in the first round of IESDS, and from the fact that $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$ it follows that if only one of the strategies was removed in the first round of IESDS then the remaining one will be removed in the second and final round, leaving (s_{1a}, s_{2a}) as the unique strategy that survives IESDS. ■

(b) Show that if $m = 3$ and the game has a unique pure strategy equilibrium then it may not be the only strategy profile that survives IESDS.

Answer: Consider this following game:

		Player 2		
		L	M	R
Player 1	U	$7, 6$	$3, 0$	$6, 5$
	C	$1, 3$	$4, 4$	$0, 2$
	D	$8, 7$	$2, 1$	$5, 8$

Notice that for both players none of the strategies are strictly dominated implying that IESDS does not restrict any strategy profile survives IESDS. However, this game has a unique Nash equilibrium: (C, M). ■

4. **Splitting Pizza:** You and a friend are in an Italian restaurant, and the owner offers both of you an 8-slice pizza for free under the following condition. Each of you must simultaneously announce how many slices you would like; that is, each player $i \in \{1, 2\}$ names his desired amount of pizza, $0 \leq s_i \leq 8$. If $s_1 + s_2 \leq 8$ then the players get their demands (and the owner eats any leftover slices). If $s_1 + s_2 > 8$, then the players get nothing. Assume that you each care only about how much pizza you individually consume, and the more the better.

(a) Write out or graph each player's best-response correspondence.

Answer: Restrict attention to integer demands (more on continuous demands is below). If player j demands $s_j \in \{0, 1, ..., 7\}$ then i's best response is to demand the complement to 8 slices. If i asks for more then both get nothing while if i asks for less then he is leaving some slices unclaimed. If instead player j demands $s_j = 8$ then player i gets nothing regardless of his request so any demand is a best response. In summary,

$$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in \{0, 1, ..., 7\} \\ \{0, 1, ..., 8\} & \text{if } s_j = 8 \end{cases} .$$

Note: if the players can ask for amounts that are not restricted to integers then the same logic applies and the best response is

$$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in [0, 8) \\ [0, 8] & \text{if } s_j = 8 \end{cases} .$$

■

(b) What outcomes can be supported as pure-strategy Nash equilibria?

Answer: It is easy to see from the best response correspondence that any pair of demands that add up to 8 will be a Nash equilibrium, i.e., $(0, 8), (1, 7), ..., (8, 0)$. However, there is another Nash equilibrium: (8,8) in which both players get nothing. It is a Nash equilibrium because given that each player is asking for 8 slices, the other player gets nothing *regardless* of his request, hence he is indifferent between all of his requests including 8.
Note: The pair $s_j = 8$ and $s_i = s$ where $s \in \{1, 2, ..., 7\}$ is not a Nash equilibrium because even though player i is playing a best response to s_j, player j is not playing a best response to s_i because by demanding 8 player j received nothing, but if he instead demanded $8 - s > 0$ then he would get those amount of slices and get something. ■

5. **Public Good Contribution:** Three players live in a town and each can choose to contribute to fund a street lamp. The value of having the street

lamp is 3 for each player and the value of not having one is 0. The Mayor asks each player to either contribute 1 or nothing. If at least two players contribute then the lamp will be erected. If one or less people contribute then the lamp will not be erected, in which case any person who contributed will not get their money back.

(a) Write out or graph each player's best-response correspondence.

Answer: Consider player i with beliefs about the choices of players j and k. If neither j nor k contribute then player i does not want to contribute because the lamp would not be erected and he would lose his contribution. Similarly, if both j and k contribute then player i does not want to contribute because the lamp would be erected without his contribution so he can "free ride" on their contributions. The remaining cases is where only one of the players j and k contribute, in which case by contributing 1 player i receives 3, while by not contributing he receives 0, and hence contributing is a best response. In summary,

$$BR_i(s_j, s_k) = \begin{cases} 0 & \text{if } s_j = s_k \\ 1 & \text{if } s_j \neq s_k \end{cases} .$$

(b) What outcomes can be supported as pure-strategy Nash equilibria?

Answer: The best response correspondence described in (a) above implies that there are two kinds of Nash equilibria: one kind (which is unique) is where no player contributes, and the other kind has two of the three players contributing and the third free riding. Hence, either the lamp being erected with two players contributing or the lamp not being erected with no player contributing can be supported as Nash equilibria. ■

6. **Hawk-Dove:** The following game has been widely used in evolutionary biology to understand how "fighting" and "display" strategies by animals could coexist in a population. For a typical Hawk-Dove game there are resources to

be gained (i.e. food, mates, territories, etc.) denoted as v. Each of two players can chooses to be aggressive, called "Hawk" (H), or can be compromising, called "Dove" (D). If both players choose H then they split the resources, but loose some payoff from injuries, denoted as k. Assume that $k > \frac{v}{2}$. If both choose D then they split the resources, but engage in some display of power that a display cost d, with $d < \frac{v}{2}$. Finally, if player i chooses H while j chooses D, then i gets all the resources while j leaves with no benefits and no costs.

(a) Describe this game in a matrix

Answer:

		Player 2	
		H	D
Player 1	H	$\frac{v}{2}-k, \frac{v}{2}-k$	$v, 0$
	D	$0, v$	$\frac{v}{2}-d, \frac{v}{2}-d$

■

(b) Assume that $v = 10$, $k = 6$ and $d = 4$. What outcomes can be supported as pure-strategy Nash equilibria?[2]

Answer: The game is:

		Player 2	
		H	D
Player 1	H	$-1, -1$	$10, 0$
	D	$0, 10$	$1, 1$

and the two strategy profiles that can be supported as pure strategy Nash equilibria are (H, D) and (D, H), leading to outcomes $(10, 0)$ and $(0, 10)$ respectively. ■

[2] In the evolutionary biology literature, the analysis performed is of a very different nature. Instead of considering the Nash equilibrium analysis of a static game, the analysis is a dynamic analysis where successful strategies "replicate" in a large population. This analysis is part of a methodology called "evolutionary game theory." For more on this see Gintis (2000).

7. **The n player Tragedy of the Commons:** Suppose there are n players in the Tragedy of the Commons example in section 5.2.2.

   (a) Find the Nash equilibrium of this game. How does n affect the Nash outcome?

   Answer: The analysis in section 2 concluded that in the n-player game the best response of player i is given by

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2} \ .$$

   First, let's consider a symmetric Nash equilibrium where each player chooses the same level of consumption k^*. Because the best response must hold for each i and they all choose the same level k^* then in the Nash equilibrium the best response reduces to,

$$k^* = \frac{K - (n-1)k^*}{2} \ ,$$

   or,

$$k^* = \frac{K}{n+1} \ .$$

   The way in which n affects the outcome is that first, as there are more firms, each will consume less clean air. Second, as there are more firms, the sum of clean air consumed by the firms is $\frac{nK}{n-1}$, which increases with n.

   It is more subtle to show that there cannot be other Nash equilibria. To show this we will show that conditional on whatever is chosen by all but two players, the two players must choose the same amount in a Nash equilibrium. Assume that there is another asymmetric Nash equilibrium in which two players, i and j, choose two different equilibrium levels $k_i^* \neq k_j^*$. Let $\overline{k} = \sum_{m \neq i,j} k_m^*$ be the sum of all the other equilibrium choices of the players who are not i or l. Because we assumed that this is a Nash equilibrium, the best response function of both i and j must hold simultaneously, that is,

$$k_i^* = \frac{K - \overline{k} - k_j^*}{2} \ , \tag{5.1}$$

and

$$k_j^* = \frac{K - \overline{k} - k_i^*}{2} \ . \tag{5.2}$$

If we substitute (5.2) into (5.1) we obtain,

$$k_i^* = \frac{K - \overline{k} - \frac{K - \overline{k} - k_i^*}{2}}{2} \ ,$$

which implies that $k_i^* = \frac{K - \overline{k}}{3}$. If we substitute this back into (5.2) we obtain,

$$k_j^* = \frac{K - \overline{k} - \frac{K - \overline{k}}{3}}{2} = \frac{K - \overline{k}}{3} = k_i^* \ ,$$

which contradicts the assumption we started with, that $k_i^* \neq k_j^*$. Hence, the unique Nash equilibrium has all the players choosing the same level $k^* = \frac{K}{n+1}$. ■

(b) Find the socially optimal outcome with n players. How does n affect this outcome?

Answer: The socially optimal outcome is found my maximizing,

$$\max_{(k_1, k_2, \ldots, k_n)} \sum_{i=1}^{n} \ln(k_i) + n \ln(K - \sum_{i=1}^{n} k_i) \ .$$

The n first order conditions for this problem are,

$$\frac{1}{k_i} - \frac{n}{K - \sum_{j=1}^{n} k_j} = 0 \text{ for } i = 1, 2, ..., n \ .$$

Just as for the analysis of the Nash equilibrium in part (a), the solution here is also symmetric. Therefore the optimal solution, k^o, can be found using the following equation:

$$\frac{1}{k^o} - \frac{n}{K - nk^o} = 0 \ ,$$

or, $k^o = \frac{K}{2n}$, and the socially optimal total consumption of clean air will be equal to $\frac{K}{2}$ regardless of the number of players. This implies that the socially optimal solution is for the players to equally divide up half of the clean air. ■

(c) How does the Nash equilibrium outcome compare to the socially efficient outcome as n approaches infinity?

Answer: The Nash equilibrium outcome always has the firms consume too much clean air as compared to the total $\frac{K}{2}$ amount that social optimality requires. Furthermore, as n approaches infinity the Nash levels of consumption approach the total amount of clean air K and the payoffs of the players approaches $-\infty$. ■

8. **The n firm Cournot Model:** Suppose there are n firms in the Cournot oligopoly model. Let q_i denote the quantity produced by firm i, and let $Q = q_i + \cdots + q_n$ denote the aggregate production. Let $P(Q)$ denote the market clearing price (when demand equals Q) and assume that inverse demand function is given by $P(Q) = a - Q$ (where $Q < a$). Assume that firms have no fixed cost, and the cost of producing quantity q_i is cq_i (all firms have the same marginal cost, and assume that $c < a$).

(a) Model this as a Normal form game

Answer: The players are $N = \{1, 2, ..., n\}$, each player chooses $q_i \in S_i$ where the strategy sets are $S_i = [0, \infty)$ for all $i \in N$, and the payoffs of each player are given by,

$$v_i(q_i, q_{-i}) = \begin{cases} (a - \sum\limits_{j=1}^{n} q_j)q_i - cq_i & \text{if } \sum\limits_{j=1}^{n} q_j < a \\ -cq_i & \text{if } \sum\limits_{j=1}^{n} q_j \geq a \end{cases}$$

■

(b) What is the Nash (Cournot) Equilibrium of the game where firms choose their quantities simultaneously?

Answer: Let's begin by assuming that there is a symmetric "interior solution" where each firm chooses the same positive quantity as a Nash equilibrium, and then we will show that this is the only possible Nash equilibrium. Because each firm maximizes

$$v_i(q_i, q_{-i}) = (a - \sum_{j=1}^{n} q_j)q_i - cq_i \ ,$$

the first order condition is

$$a - \sum_{j \neq i} q_j - 2q_i - c = 0 \ ,$$

which yields the best response of player i to be

$$BR_i(q_{-i}) = \frac{a - \sum_{j \neq i} q_j - c}{2} \ .$$

Imposing symmetry in equilibrium implies that all n best response conditions will hold with the same values $q_i^* = q^*$ for all $i \in N$, and can be solved using the best response function as follows,

$$q^* = \frac{a - (n-1)q^* - c}{2} \, ,$$

which yields

$$q^* = \frac{a - c}{n+1} \ .$$

It is more subtle to show that there cannot be other Nash equilibria. To show this we will show that conditional on whatever is chosen by all but two players, the two players must choose the same amount in a Nash equilibrium. Assume that there is another asymmetric Nash equilibrium in which two players, i and j, choose two different equilibrium quantities $q_i^* \neq q_j^*$. Let $\overline{q} = \sum_{m \neq i,j} q_m^*$ be the sum of all the other equilibrium quantity choices of the players who are not i or l. Because we assumed that this is a Nash equilibrium, the best response function of both i and j must hold simultaneously, that is,

$$q_i^* = \frac{a - \overline{q} - q_j^* - c}{2} \ , \tag{5.3}$$

and

$$q_j^* = \frac{a - \overline{q} - q_i^* - c}{2} \ . \tag{5.4}$$

If we substitute (5.4) into (5.3) we obtain,

$$q_i^* = \frac{a - \overline{q} - \frac{a-\overline{q}-q_i^*-c}{2} - c}{2} \ ,$$

which implies that $q_i^* = \frac{a-\overline{q}-c}{3}$. If we substitute this back into (5.4) we obtain,

$$q_j^* = \frac{a - \overline{q} - \frac{a-\overline{q}-c}{3} - c}{2} = \frac{a - \overline{q} - c}{3} = k_i^* \ ,$$

which contradicts the assumption we started with, that $q_i^* \neq q_j^*$. Hence, the unique Nash equilibrium has all the players choosing the same level $q^* = \frac{a-c}{n+1}$. ■

(c) What happens to the equilibrium price as n approaches infinity? Is this familiar?

Answer: First consider the total quantity in the Nash equilibrium as a function of n,

$$Q^* = nq^* = \frac{n(a-c)}{n+1}$$

and the resulting limit price is

$$\lim_{n\to\infty} P(Q^*) = \lim_{n\to\infty}\left(a - \frac{n(a-c)}{n+1}\right) = c.$$

This means that as the number of firms grow, the Nash equilibrium price will also fall and will approach the marginal costs of the firms as the number of firms grows to infinity. Those familiar with a standard economics class know that in perfect competition price will equal marginal costs, which is what happens here when n approaches infinity. ■

9. **Tragedy of the Roommates:** You and your $n-1$ roommates each have 5 hours of free time you could spend cleaning your apartment. You all dislike cleaning, but you all like having a clean room: each person's payoff is the

total hours spent (by everyone) cleaning, minus a number c times the hours spent (individually) cleaning. That is,

$$v_i(s_1, s_2, \ldots, s_n) = -c \cdot s_i + \sum_{j=1}^{n} s_j$$

Assume everyone chooses simultaneously how much time to spend cleaning.

(a) Find the Nash equilibrium if $c < 1$.

Answer: The payoff function is linear in one's own time spent s_i and in the time spent by the other roommates s_j, and we can rewrite the payoff function as

$$v_i(s_i, s_{-i}) = s_i - cs_i + \sum_{j \neq i} s_j \ .$$

Considering this payoff function, if $c < 1$ then every additional amount ε of time that i spends cleaning gives him an extra payoff of $(1-c)\varepsilon > 0$ so that each player i would choose to spend all the 5 hours cleaning. Note that using a first-order condition would not work here because taking the derivative of $v_i(s_i, s_{-i})$ with respect to s_i will just yield $1 - c = 0$ which is not true for $c < 1$. This implies that there is a "corner" solution in the range $s_i \in [0, 5]$, in this case the Nash equilibrium is at the corner $s_i^* = 5$ for all $i = 1, 2, ..., n$. ■

(b) Find the Nash equilibrium if $c > 1$.

Answer: Similarly to (a) above, every additional amount ε of time that i spends cleaning gives him an extra payoff of $(1 - c)\varepsilon < 0$, so that each player i would choose to spend no time cleaning and the Nash equilibrium is $s_i^* = 0$ for all $i = 1, 2, ..., n$. ■

(c) Set $n = 5$ and $c = 2$. Is the Nash equilibrium Pareto efficient? If not, can you find an outcome where everyone is better off than at the Nash equilibrium outcome?

Answer: Following the analysis in part (b), the unique Nash equilibrium is where everyone chooses to spend no time cleaning and everyone's payoff is equal to zero. Consider the case where everyone is somehow forced to choose $s_i = 1$. Each player's payoff will be

$$\begin{aligned} v_i(s_i, s_{-i}) &= s_i - cs_i + \sum_{j \neq i} s_j \\ &= 1 - 2 \times 1 + 4 \times 1 = 3 > 0 \ . \end{aligned}$$

so that all the players will be better off if they all chose $s_i = 1$. In fact, each amount of time $\varepsilon > 0$ that player i chooses to clean cause him a personal loss of $\varepsilon - 2\varepsilon = \varepsilon$, but increases the payoff of each of the other players by ε. Hence, if we can get each player to increase his time cleaning by ε, this yields an increase of value for each player that equals his own loss, but the former is multiplied by the number of players. Hence, the best symmetric outcome is when each player chooses $s_i = 5$. ■

10. **Synergies:** Two division managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest. In particular, the payoff function for player i from effort levels (e_i, e_j) is $v_i(e_i, e_j) = (a + e_j)e_i - e_i^2$.

    (a) What is the best response correspondence of each player?

    Answer: If player i believes that player j chooses e_j then i's first order optimality condition for maximizing his payoff is,

    $$a + e_j - 2e_i = 0 \ ,$$

    yielding the best response function,

    $$BR_i(e_j) = \frac{a + e_j}{2} \text{ for all } e_j \geq 0.$$

    ■

(b) In what way are the best response correspondences different from those in the Cournot game? Why?

Answer: Here the best response function of player i is *increasing* in the choice of player j whereas in the Cournot model it is *decreasing* in the choice of player j. This is because in this game the choices of the two players are strategic complements while in the Cournot game they are strategic substitutes. ■

(c) Find the Nash equilibrium of this game and argue that it is unique.

Answer: We solve two equations with two unknowns,

$$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2},$$

which yield the solution $e_1 = e_2 = a$. It is easy to see that it is unique because it is the only point at which these two best response functions cross. ■

11. **Wasteful Shipping Costs**. Consider two countries, A and B, each with a monopolist that owns the only coal mine in the country, and it produces coal. Let firm 1 be the one located in country A, and firm 2 the one in country B. Let q_i^j, $i \in \{1, 2\}$ and $j \in \{A, B\}$ denote the quantity that firm i sells in country j. Consequently, let $q_i = q_i^A + q_i^B$ be the total quantity produced by firm $i \in \{1, 2\}$, and let $q^j = q_1^j + q_2^j$ be the total quantity sold in country $j \in \{A, B\}$. The demand for coal in countries A and B is given respectively by,

$$p^j = 90 - q^j,\ j \in \{A, B\},$$

and the costs of production for each firm is given by,

$$c_i(q_i) = 10q_i\,,\ i \in \{1, 2\}.$$

(a) Assume that the countries do not have a trade agreement and, in fact, imports in both countries are prohibited. This implies that $q_2^A = q_1^B = 0$ is set as a political constraint. What quantities q_1^A and q_2^B will both firms produce?

Answer: Each firm is a monopolist in its own country. Let and maximizes,

$$\max_{q_i^j \geq 0} (90 - q_i^j)q_i^j - 10q_i^j$$

where either $i = 1$ and $j = A$, or $i = 2$ and $j = B$ (so that $q_2^A = q_1^B = 0$ is set by assumption.) The first order maximization condition is $90 - 2q_i^j - 10 = 0$, which yields $q_1^A = q_2^B = 40$. The payoff for each firm is $1,600$. ■

Now assume that the two countries sign a free-trade agreement that allows foreign firms to sell in their countries without any tariffs. There are, however shipping costs. If firm i sells quantity q_i^j in the foreign country (i.e., firm 1 selling in B or firm 2 selling in A) then shipping costs are equal to $10q_i^j$. Assume further that *each firm* chooses a pair of quantities q_i^A, q_i^B simultaneously, $i \in \{1, 2\}$, so that a profile of actions consists of four quantity choices.

(b) Model this as a normal form game and find a Nash equilibrium of the game you described. Is it unique?

Answer: This game has two players, $i \in \{1, 2\}$, each choosing a strategy that consists of two non-negative quantities, $(q_i^A, q_i^B) \in \mathbb{R}_+^2$, and the payoff of the two players are given by,

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 10q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 10q_2^A, \end{aligned}$$

where the first term is the firm's revenue in market A, the second is the revenue in market B, the third is the total production cost and the last is the shipping cost. Given beliefs (q_2^A, q_2^B) about what firm 2 chooses to produce, firm 1's optimization requires two partial derivatives with respect to q_1^A and q_1^B as follows,

$$\begin{aligned}\frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^A} &= 90 - q_2^A - 2q_1^A - 10 = 0, \\ \frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^B} &= 90 - q_2^B - 2q_1^B - 20 = 0,\end{aligned}$$

which in turn lead to the two parts of firm 1's best response function,[3]

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.5}$$

$$q_1^B = \frac{70 - q_2^B}{2}. \tag{5.6}$$

It is easy to see that the objective of firm 2 is symmetric to that of firm 1 and hence we can directly write down firm 2's best responses as,

$$q_2^A = \frac{70 - q_1^A}{2}, \tag{5.7}$$

$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.8}$$

The Nash equilibrium is solved by finding a profile of strategies $(q_1^A, q_1^B, q_2^A, q_2^B)$ for which (5.5), (5.6), (5.7) and (5.8) all hold simultaneously. From (5.5) and (5.7) we obtain $q_1^A = 30$ and , $q_2^A = 20$. Similarly, from (5.6) and (5.8) we obtain $q_1^B = 20$ and , $q_2^B = 30$. The payoff of each firms would be equal to $1,300$.

Now assume that before the game you described in (b) is played, the research department of firm 1 discovered that shipping coal with the current ships causes the release of pollutants. If the firm would disclose this report to the World-Trade-Organization (WTO) then the WTO would prohibit the use of the current ships. Instead, a new shipping

[3] Because the payoff function has no interactions between the markets (i.e., it is *separable* in the two markets so that there are no interactions through the cost function) then q_1^A depends only on q_2^A and q_1^B depends only on q_2^B (and vice versa for firm 2). If costs were not linear then this would not be the case and the solution would involve solving four equations wit four unknowns simultaneously.

technology would be offered that would increase shipping costs to $40q_i^j$ (instead of $10q_i^j$ as above).

(c) Would firm 1 be willing to release the information to the WTO? Justify your answer with an equilibrium analysis.

Answer: To answer this we need to solve the Nash equilibrium with the more expensive shipping technology and compare the profits to that of the current cheaper technology. We know that a monopolist (or competitive firm) would never prefer a more expensive technology to a cheaper one, but here there may be interesting strategic effects: the more expensive shipping technology will dampen competition. The new payoff functions are

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 40q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 40q_2^A, \end{aligned}$$

and following the same arguments in part (b) above, the four equations that will define the best responses of both firms are,

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.9}$$

$$q_1^B = \frac{40 - q_2^B}{2}. \tag{5.10}$$

and,

$$q_2^A = \frac{40 - q_1^A}{2}, \tag{5.11}$$

$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.12}$$

From (5.9) and (5.11) we obtain $q_1^A = 40$ and , $q_2^A = 0$. Similarly, from (5.10) and (5.12) we obtain $q_1^B = 0$ and , $q_2^B = 40$. The payoff of each firms would be equal to $1,600$, as we calculated in part (a) above. Hence, the firm would like to disclose the information and let the WTO impose a ban that would effectively kill cross-border competition. ■

12. **Asymmetric Bertrand:** Consider the Bertrand game with $c_1(q_1) = q_1$ and $c_2(q_2) = 2q_2$, demand equal to $p = 100 - q$, and where firms must choose prices in increments of one cent. We have seen in section **??** that one possible Nash equilibrium is $(p_1^*, p_2^*) = (1.99, 2.00)$.

    (a) Show that there are other Nash equilibria for this game.

    Answer: Another Nash equilibrium is $(p_1', p_2') = (1.50, 1.51)$.In this equilibrium firm 1 fulfills market demand at a price of 1.50 and has no incentive to change the price in either direction. Firm 2 is indifferent between the current price and any higher price, and strictly prefers it to lower prices. ■

    (b) How many Nash equilibria does this game have?

    Answer: There are 100 Nash equilibria of this game starting with $(p_1, p_2) = (1.00, 1.01)$ and going all the way up with one-cent increases to $(p_1^*, p_2^*) = (1.99, 2.00)$. The same logic explains why each of these is a Nash equilibrium. ■

13. **Comparative Economics:** Two high tech firms (1 and 2) are considering a joint venture. Each firm i can invest in a novel technology, and can choose a level of investment $x_i \in [0, 5]$ at a cost of $c_i(x_i) = \frac{x_i^2}{4}$ (think of x_i as how many hours to train employees, or how much capital to buy for R&D labs). The revenue of each firm depends both on its investment, and of the other firm's investment. In particular, if firm i and j choose x_i and x_j respectively, then the **gross revenue** to firm i is

$$R(x_i, x_j) = \begin{cases} 0 & \text{if } x_i < 1 \\ 2 & \text{if } x_i \geq 1 \text{and } x_j < 2 \\ x_i \cdot x_j & \text{if } x_i \geq 1 \text{and } x_j \geq 2 \end{cases}$$

    (a) Write down mathematically, and draw the **profit function** (gross revenue minus costs) of firm i as a function of x_i for three cases: (i) $x_j < 2$, (ii) $x_j = 2$, and (iii) $x_j = 4$

Answer: For the case where $x_j < 2$ the payoff (profit) function of firm i is,

$$v_i(x_i, x_j) = \begin{cases} 0 - \frac{x_i^2}{4} & \text{if } x_i < 1 \\ 2 - \frac{x_i^2}{4} & \text{if } x_i \geq 1 \end{cases},$$

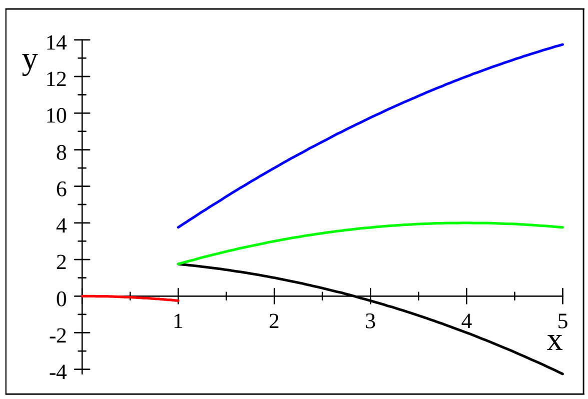
for the case where $x_j = 2$ the payoff function of firm i is,

$$v_i(x_i, x_j) = \begin{cases} 0 - \frac{x_i^2}{4} & \text{if } x_i < 1 \\ 2x_i - \frac{x_i^2}{4} & \text{if } x_i \geq 1 \end{cases},$$

and for the case where $x_j = 4$ the payoff function of firm i is,

$$v_i(x_i, x_j) = \begin{cases} 0 - \frac{x_i^2}{4} & \text{if } x_i < 1 \\ 4x_i - \frac{x_i^2}{4} & \text{if } x_i \geq 1 \end{cases}.$$

The three profit functions are depicted in the following figure:



All three share the same profits in the range $x_i \in [0, 1)$ which is the red line. The black line depicts the rest of the payoff function for the case of $x_j < 2$, the green line depicts the rest of the payoff function for the case of $x_j = 2$, and the blue line depicts the rest of the payoff function for the case of $x_j = 4$. ■

(b) What is the best response function of firm i ?

Answer: It is easy to see (and calculate) that when $x_j < 2$ then firm i's best response is to choose $x_i = 1$, and when $x_j < 4$ then firm i's best response is to choose $x_i = 5$ (a "corner" solution.) When $x_j = 2$ then firm i's best response solves

$$\max_{x_i \in [0,5]} \; 2x_i - \frac{x_i^2}{4} \; ,$$

and the first order optimality condition is $2 - \frac{x_i}{2} = 0$ which yields $x_i = 4$. More generally, as x_j grows above 2 the best response of firm i will grow above 4 until it hits the corner solution of $x_i = 5$. In the range in which player i's best response in between 4 and 5 he maximizes his payoff function which is,

$$\max_{x_i \in [0,5]} \; x_j x_i - \frac{x_i^2}{4} \; ,$$

and his best response is derived form the first order condition $x_j - \frac{x_i}{2} = 0$, which yields,

$$x_i(x_j) = 2x_j \; .$$

It is easy to see that for any $x_j \in [2, 2.5]$ the best response of firm i is within the range $[4, 5]$ and for any $x_j > 2.5$ the best response of i is "stuck" at the corner solution $x_i = 5$. Hence, we can write down the general best response function of firm i as,

$$x_i(x_j) = \begin{cases} 1 & \text{if } x_j < 2 \\ 2x_j & \text{if } x_j \in [2, 2.5] \\ 5 & \text{if } x_j > 2.5 \end{cases} \quad .$$

■

(c) It turns out that there are two *identical* pairs of such firms (that is, the technology above describes the situation for both pairs). One pair in Russia where coordination is hard to achieve and business people are very cautious, and the other pair in Germany where coordination is common and business people expect their partners to go the extra mile. You learn that the Russian firms are earning significantly less profits than the German firms, despite the fact that their technologies are identical. Can you use Nash equilibrium analysis to shed light on this dilemma? If so, be precise and use your previous analysis to do so.

Answer: The best response function described in part (b) leads to two Nash equilibria: in the first $(x_i^*, x_j^*) = (1,1)$ and $(x_i^{**}, x_j^{**}) = (5,5)$. In the first Nash equilibrium the profits of each firm are $v_i^* = 1.75$, while in the second Nash equilibrium $v_i^{**} = 18.75$. This is an example where "self fulfilling expectations" can lead to two Nash equilibria, one with high payoffs and one with low payoffs. This is an example of a game with strategic complements (see page 93) where the complementarity cause multiple equilibria. ■

14. **Negative Ad Campaigns:** Each one of two political parties can choose to buy time on commercial radio shows to broadcast negative ad campaigns against their rival. These choices are made simultaneously. Due to government regulation it is forbidden to buy more than 2 hours of negative campaign time so that each party cannot choose an amount of negative campaigning above 2 hours. Given a pair of choices (a_1, a_2), the payoff of party i is given by the following function: $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$.

    (a) What is the normal form representation of this game?

    Answer: Two players $N = \{1, 2\}$, for each player the strategy space is $S_i = [0, 2]$ and the payoff of player i is given by $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$. ■

    (b) What is the best response function for each party?

    Answer: Each player maximizes $v_i(a_1, a_2)$ resulting in the first order optimality condition $1 + a_j - 2a_i = 0$ resulting in the best response function,

$$a_i(a_j) = \frac{1 + a_j}{2} .$$

    ■

    (c) What is the pure strategy Nash equilibrium? is it unique?

Answer: Solving the two best response functions simultaneously,

$$a_1 = \frac{1 + a_2}{2} \text{ and } a_2 = \frac{1 + a_1}{2}$$

yields the Nash equilibrium $a_1 = a_2 = 1$, and this is the unique solution to these equations implying that this is the unique equilibrium. ■

(d) If the parties could sign a binding agreement on how much to campaign, what levels would they choose?

Answer: Both parties would be better off if they can choose not to spend money on negative campaigns. The payoffs for each player from the Nash equilibrium solved in part (c) are $v_i(1, 1) = -1$ while of they agreed not to spend anything they each would obtain zero. This is a variant of the Prisoners' Dilemma. ■

15. **Hotelling's Continuous Model:** Consider Hotelling's model where the citizens are a continuum of voters on the interval $A = [-a, a]$, with uniform distribution $U(a)$.

(a) What is the best response of candidate i if candidate j is choosing $a_j > 0$.

Answer: Player i's best response will depend on the position of a_j relative to the choice $\frac{1}{2}$. For example, if $a_j = -\varepsilon \in [-a, 0)$ then any choice $a_i \in (a_j, \varepsilon)$ will guarantee player i victory. This follows because player i will get *more* than $a - \varepsilon$ of the vote from the interval (a_i, a) on his "right" (it is more because $a_i < \varepsilon$), and he will split the inner interval $(-\varepsilon, a_i)$ with player j so that his total share of the vote is

$$v_i > a - \varepsilon + \frac{a_i - (-\varepsilon)}{2} = \frac{2a + a_i - \varepsilon}{2} ,$$

while the total share of player j is the interval $(-a, -\varepsilon)$ on his "left" plus splitting the inner interval $(-\varepsilon, a_i)$, which is,

$$v_j = -\varepsilon - (-a) + \frac{a_i - (-\varepsilon)}{2} = \frac{2a + a_i - \varepsilon}{2} ,$$

which implies that $v_i > v_j$ and i will win the vote. A symmetric argument works for any choice $a_j = \varepsilon \in (0, a]$. The remaining case is $a_j = 0$. In this case if $a_i \neq a_j$ then player i gets less tan half the vote while if $a_i = a_j$ then player i gets half the vote, making $a_i = a_j$ the best response to $a_j = 0$. We conclude that the best response correspondence is,

$$BR_i(a_j) = \begin{cases} (a_j, -a_j) & \text{if } a_j < 0 \\ 0 & \text{if } x_j = 0 \\ (-a_j, a_j) & \text{if } x_j > 0 \end{cases} .$$

■

(b) Show that the unique Nash equilibrium is $a_1 = a_2 = 0$.

Answer: This follows immediately from the best response correspondence in part (a) above: only at the pair $(a_i, a_j) = (0, 0)$ are both players playing a best response to each other. ■

(c) Show that for a general distribution $F(\cdot)$ over $[-a, a]$, the unique Nash equilibrium is where is candidate chooses the policy associated with the median voter.

Answer: This again follows from the analysis in part (a) above just that instead of 0 being the point at which half the vote is obtained, it is the median voter a^* for which half the vote is at or above a^* and half the vote is at or below a^*.[4] ■

16. **Hotelling's Price Competition:** Imagine a continuum of potential buyers, located on the line segment $[0, 1]$, with uniform distribution. (Hence, the

[4] If the distribution $F(a)$ is continuous the there will be some a^* such that $F(a^*) = \frac{1}{2}$ and that will be the median voter. If there are "jumps" in the distribution $F(a)$ then the median voter can be some a^* for which $F(a^*) > \frac{1}{2}$. For instance, if half the population is distributed Uniformly on $[-1, 1]$ and the other half are all located at the point $a^* = \frac{1}{2}$ then $\frac{3}{8}$ of the population are strictly below a^*, $\frac{1}{8}$ of the population are strictly above a^*, and $\frac{1}{2}$ of the population is exactly at a^*. In this case

$$F(a) = \begin{cases} \frac{1+a}{4} & \text{if } -1 \leq a < \frac{1}{2} \\ \frac{3+a}{4} & \text{if } \frac{1}{2} \leq a \leq 1 \end{cases} .$$

so that $F(a^*) = \frac{3}{4}$, but a^* is still the median voter.

"mass" or quantity of buyers in the interval $[a, b]$ is equal to $b - a$.) Imagine two firms, players 1 and 2 who are located at each end of the interval (player 1 at the 0 point and player 2 at the 1 point.) Each player i can choose its price p_i, and each customer goes to the vendor who offers them the highest value. However, price alone does not determine the value, but distance is important as well. In particular, each buyer who buys the product from player i has a net value of $v - p_i - d_i$ where d_i is the distance between the buyer and vendor i, and represents the transportation costs of buying from vendor i. Thus, buyer $a \in [0, 1]$ buys from 1 and not 2 if $v - p_1 - d_1 > v - p_2 - d_2$, and if buying is better than getting zero. (Here $d_1 = a$ and $d_2 = 1 - a$. The buying choice would be reversed if the inequality is reversed.) Finally, assume that the cost of production is zero.

(a) Assume that v is very large so that all the customers will be served by at least one firm, and that some customer $x^* \in [0, 1]$ is indifferent between the two firms. What is the best response function of each player?

Answer: Because customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, his indifference implies that

$$v - p_1 - x^* = v - p_2 - (1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1 + p_2 - p_1}{2} .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1 + p_2 - p_1}{2} , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1 + p_1 - p_2}{2} . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1 + p_2 - p_1}{2} \right) p_1$$

which yields the first order condition

$$1 + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

■

(b) Assume that $v = 1$. What is the Nash equilibrium? Is it unique?

Answer: If we use the best response functions calculated in part (a) above then we obtain a unique Nash equilibrium $p_1 = p_2 = 1$, and this implies that $x^* = \frac{1}{2}$ so that each firm gets half the market. However, when $v = 1$ then the utility of customer $x^* = \frac{1}{2}$ is $v - p_1 - \frac{1}{2} = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$, implying that he would prefer not to buy, and by continuity, an interval of customers around x^* would also prefer not to buy. his violated the assumptions we used to calculate the best response functions.[5] So, the analysis in part (a) is invalid when $v = 1$. It is therefore useful to start with the monopoly case when $v = 1$ and see how each firm would have priced if the other is absent. Firm 1 maximizes

$$\max_{p_1} \ (1 - p_1)p_1$$

which yields the solution $p_1 = \frac{1}{2}$ so that everyone in the interval $x \in [0, \frac{1}{2}]$ wished to buy from firm 1 and no other customer would buy. By symmetry, if firm 2 were a monopoly then the solution would be $p_2 = \frac{1}{2}$ so that everyone in the interval $x \in [\frac{1}{2}, 1]$ would buy from firm 2 and no other customer would buy. But this implies that if both firms set their

[5] We need $v \geq 1.5$ for customer $x^* = \frac{1}{2}$ to be just indifferent between buying and not buying when $p_1 = p_2 = 1$. All the other customers will strictly prefer buying.

monopoly prices $p_1 = p_2 = \frac{1}{2}$ then each would maximize profits ignoring the other firm, and hence this is the (trivially) unique Nash equilibrium. ■

(c) Now assume that $v = 1$ and that the transportation costs are $\frac{1}{2}d_i$, so that a buyer buys from 1 if and only if $v - p_1 - \frac{1}{2}d_1 > v - p_2 - \frac{1}{2}d_2$. Write the best response function of each player and solve for the Nash Equilibrium.

Answer: Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, and he is indifferent between buying from either, so his indifference implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + p_2 - p_1 \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + p_2 - p_1 \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + p_1 - p_2 \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + p_2 - p_1\right) p_1$$

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \frac{1}{2}$, and $x^* = \frac{1}{2}$. To verify that this is a Nash equilibrium notice that for customer x^*, the utility form buying from firm 1 is $v - p_1 - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$ implying that he is indeed indifferent between buying or not, which in turn implies that every other customer prefer buying over not buying. ■

(d) Following your analysis in (c) above, imagine that transportation costs are αd_i, with $\alpha \in [0, \frac{1}{2}]$. What happens to the Nash equilibrium as $\alpha \to 0$? What is the intuition for this result?

Answer: Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*)$$

$$v - p_1 - \alpha x = v - p_2 - \alpha(1 - x)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2) \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \right) p_1$$

which yields the first order condition

$$\frac{1}{2} + \frac{p_2}{2\alpha} - \frac{p_1}{\alpha} = 0 \ ,$$

implying the best response function

$$p_1 = \frac{\alpha}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{\alpha}{2} + \frac{p_1}{2}.$$

$$p_2 = \frac{\alpha}{2} + \frac{\frac{\alpha}{2} + \frac{p_2}{2}}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. From the analysis in (c) above we know that for any $\alpha \in [0, \frac{1}{2})$ customer x^* will strictly prefer to buy over not buying and so will every other customer. We see that as α decreases, so do the equilibrium prices, so that at the limit of $\alpha = 0$ the prices will be zero. The intuition is that the transportation costs d cause firms 1 and 2 to be differentiated, and this "softens" the Bertrand competition between the two firms. When the transportation costs are higher this implies that competition is less fierce and prices are higher, and the opposite holds for lower transportation costs.■

17. **To vote or not to vote:** Two candidates, D and R, are running for mayoral election in a town with n residents. A total of $0 < d < n$ residents support candidate D while the remainder $r = n - d$ support candidate R. The value for each resident for having their candidate win is 4, for having him tie is 2, and for having him lose is 0. Going to vote costs each resident 1.

    (a) Let $n = 2$ and $d = 1$. Write down this game as a matrix and solve for the Nash equilibrium.

    Answer: The game is between the residents as the candidates seem not to play a role and the question is whether to vote or not to vote. Letting Y denote "yes" vote and N denote "no" vote, the matrix representation of this two player game is

		Player 2	
		Y	N
Player 1	Y	$1,1$	$3,0$
	N	$0,3$	$2,2$

If both vote or both don't vote then there is a tie and the only difference is the cost of voting. If only one votes then his candidate wins and he exerts the voting costs, while the other gains and expends nothing. Voting is a dominant strategy so (Y, Y) is the unique Nash equilibrium. ■

(b) Let $n > 2$ be an even number and let $d = r = \frac{n}{2}$. Find all the Nash equilibria.

Answer: Observe that everyone voting is a Nash equilibrium. Like in part (a) there will be a tie and every player's payoff is 1, while if he chose not to vote then his candidate will lose and his payoff will be 0, hence it is a Nash equilibrium. We now show that no other profile of strategies is a Nash equilibrium in three steps. Let d' and r' denote the number of member of each group that plan on voting. (i) Assume that an identical number of voters from each side votes so that there is a tie but some voters are not voting, that is, $d' = r' < \frac{n}{2}$. In this case any one of the voters who is not voting would prefer to deviate, expend a voting cost of 1 and increase his payoff from 2 to 3 because he would tip the election. Hence, this cannot be a Nash equilibrium. (ii) Now assume that the number of supporters of candidate D is is at least 2 more than that of candidate R, that is, $d' \geq r' + 2$. (A symmetric argument will apply to the case of $r' \geq d' + 2$.) In this case any one of the D supporters who plans to vote knows that his vote is redundant, and hence he would prefer not to vote and save the voting costs. Hence, this cannot be a Nash equilibrium. (iii) Now assume that the number of supporters of candidate D is is exactly 1 more than that of candidate R, that is, $d' = r' + 1$. (A symmetric argument will apply to the case of $r' = d' + 1$.) In this case any one of the R supporters who does not plan to vote knows that his vote can turn a loss into a tie, and hence he would prefer to vote and change the election giving him a payoff of 1 instead of 0. Hence, this too cannot be a Nash equilibrium. This covers all the possible scenarios and shows that everyone voting is the unique Nash equilibrium. ■

(c) Assume now that the costs of voting are equal to 3. How does your answer to (a) and (b) change?

Answer: The two player game is now

		Player 2 Y	Player 2 N
Player 1	Y	$-1,-1$	$1,0$
	N	$0,1$	$2,2$

and the dominated strategy is voting, implying that the unique Nash equilibrium is for the players not to vote, (N, N). A similar argument to part (b) above shows that all players not voting is the unique Nash equilibrium. ■

18. **Political Campaigning:** Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on adds that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$. Each candidate values winning at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

    (a) Given two spend levels (s_1, s_2), write the expected payoff of a candidate i.

Answer: Player i's payoff function is

$$v_i(s_1, s_2) = \frac{s_i v}{s_1 + s_2} - s_i \ .$$

■

(b) What is the function that represents each player's best response function?

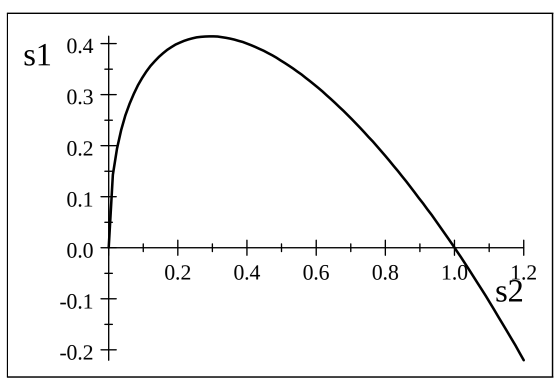
Answer: Player 1 maximizes his payoff $v_1(s_1, s_2)$ shown in (a) above and the first order optimality condition is,

$$\frac{v(s_1 + s_2) - s_1 v}{(s_1 + s_2)^2} - 1 = 0$$

and if we use $s_1(s_2)$ to denote player 1's best response function then it explicitly solves the following equality that is derived from the first-order condition,

$$[s_1(s_2)]^2 + 2s_1(s_2)s_2 + (s_2)^2 - vs_2 = 0 \ .$$

Because this is a quadratic equation we cannot write an explicit best response function (or correspondence). However, if we can graph $s_1(s_2)$ as shown in the following figure (the values correspond for the case of $v = 1$).



Similarly we can derive the symmetric function for player 2. ■

(c) Find the unique Nash equilibrium.

Answer: The best response functions are symmetric mirror images and have a symmetric solution where $s_1 = s_2$ in the unique Nash equilibrium. We can therefore use any one of the two best response functions and replace both variables with a single variable s,

$$s^2 + 2s^2 + s^2 - vs = 0 \ ,$$

or,

$$s = \frac{v}{4}$$

so that the unique Nash equilibrium has $s_1^* = s_2^* = \frac{v}{4}$.■

(d) What happens to the Nash equilibrium spending levels if v increases?

Answer: It is easy to see from part (c) that higher values of v cause the players to spend more in equilibrium. As the stakes of the prize rise, it is more valuable to fight over it. ■

(e) What happens to the Nash equilibrium levels if player 1 still values winning at v, but player 2 values winning at kv where $k > 1$?

Answer: Now the two best response functions are not symmetric. The best response function of player 1 remains as above, but that of player 2 will now have kv instead of v,

$$(s_1)^2 + 2s_1s_2 + (s_2)^2 - vs_2 = 0 \ . \qquad \text{((BR1))}$$

and

$$(s_2)^2 + 2s_1s_2 + (s_1)^2 - kvs_1 = 0 \ . \qquad \text{((BR2))}$$

Subtracting (BR2) from (BR1) we obtain,

$$ks_1 = s_2,$$

which implies that the solution will no longer be symmetric and, moreover, $s_2 > s_1$, which is intuitive because now player 2 cares more about the prize. Using $ks_1 = s_2$ we substitute for s_2 in (BR1) to obtain,

$$(s_1)^2 + 2k(s_1)^2 + k^2(s_1)^2 - kvs_1 = 0$$

which results in,

$$s_1 = \frac{kv}{1 + 2k + k^2} < \frac{v}{1 + 2k + k^2} < \frac{v}{4}$$

where both inequalities follow from the fact that $k > 1$. From $ks_1 = s_2$ above we have

$$s_2 = \frac{k^2 v}{1 + 2k + k^2} > \frac{k^2 v}{k^2 + 2k^2 + k^2} = \frac{v}{4}$$

where the inequality follows from $k > 1$. ■

6

Mixed Strategies

1. Use the best response correspondences in the Battle of the Sexes game to find all the Nash equilibria. (Follow the approach used for the example in section 6.2.3.)

   Answer: The Battle of the Sexes game is described by the following matrix,

		Player 2	
		O	F
player 1	O	$2,1$	$0,0$
	F	$0,0$	$1,2$

   Let p denote the probability that player 1 plays O and let q be the probability that player 2 plays O. The expected payoff of player 1 from playing O is $v_1(O,q) = 2q$ and of playing F is $v_1(F,q) = 1-q$. It is easy to see that O is better than F if and only if $2q > 1-q$, or $q > \frac{1}{3}$. Hence, the best response correspondence of player 1 is:

$$BR_1(q) = \begin{cases} p = 1 & \text{if } q > \frac{1}{3} \\ p \in [0,1] & \text{if } q = \frac{1}{3} \\ p = 0 & \text{if } q < \frac{1}{3} \end{cases} .$$

The best response of player 2 is derived analogously: $v_2(p, O) > v_2(p, F)$ if and only if $p > 2(1-p)$, or, $p > \frac{2}{3}$, implying that,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p > \frac{2}{3} \\ q \in [0, 1] & \text{if } p = \frac{2}{3} \\ q = 0 & \text{if } p < \frac{2}{3} \end{cases} .$$

It is now easy to see that there are three Nash equilibria: $(p, q) \in \{(1, 1), (\frac{2}{3}, \frac{1}{3}), (0, 0)\}$. ■

2. Let σ_i be a mixed strategy of player i that puts positive weight on one strictly dominated pure strategy. Show that there exists a mixed strategy σ_i' that puts no weight on any dominated pure strategy and that dominates σ_i.

   Answer: Let player i have L pure strategies $S_i = \{s_{i1}, s_{i2}, ..., s_{iL}\}$ and let s_{ik} be a pure strategy which is strictly dominated by $s_{ik'}$, that is, $v_i(s_{ik'}, s_{-i}) > v_i(s_{ik'}, s_{-i})$ for any strategy profile of i's opponents s_{-i}. Let $\sigma_i = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{iL})$ be a mixed strategy that puts some positive weight $\sigma_{ik} > 0$ on s_{ik} and let σ_i' be identical to σ_i except that it puts weight 0 on s_{ik} and diverts that weight over to $s_{ik'}$. That is, $\sigma_{ik}' = 0$ and $\sigma_{ik'}' = \sigma_{ik'} + \sigma_{ik}$, and $\sigma_{il}' = \sigma_{il}$ for all $l \neq k$ and $l \neq k'$. It follows that for all s_{-i},

$$v_i(\sigma_i', s_{-i}) = \sum_{l=1}^{L} \sigma_{il}' v_i(s_{il}, s_{-i}) > \sum_{l=1}^{L} \sigma_{il} v_i(s_{il}, s_{-i}) = v_i(\sigma_i', s_{-i})$$

   because $v_i(s_{ik'}, s_{-i}) > v_i(s_{ik'}, s_{-i})$ and the way in which σ_i' was constructed. Hence, σ_i is strictly dominated by σ_i'. ■

3. Consider the game used in section **??** as follows:

		Player 2		
		L	C	R
Player 1	U	$5, 1$	$1, 4$	$1, 0$
	M	$3, 2$	$0, 0$	$3, 5$
	D	$4, 3$	$4, 4$	$0, 3$

(a) Find a strategy different from $(\sigma_2(L), \sigma_2(C), \sigma_2(R)) = (0, \frac{1}{2}, \frac{1}{2})$ that strictly dominates the pure strategy L for player 2. Argue that you can find an infinite number of such strategies.

Answer: The expected payoff of any player in a matrix game is continuous in the probabilities of his mixed strategy (because it is a linear function of the probability weights), and hence if we "tweak" the strategy $(\sigma_2(L), \sigma_2(C), \sigma_2(R)) = (0, \frac{1}{2}, \frac{1}{2})$ just a little bit then the payoffs will be the same for any choice of player 1. For example, take $\sigma_2' = (\sigma_2'(L), \sigma_2'(C), \sigma_2'(R)) = (0, \frac{4}{10}, \frac{6}{10})$. The expected payoff of player 2 from σ_2' against any one of the three strategies of player 1 are,

$$\begin{aligned}
v_2(U, \sigma_2') &= 0.4 \times 4 + 0.6 \times 0 = 1.6 > 1 = v_2(U, L) \\
v_2(U, \sigma_2') &= 0.4 \times 0 + 0.6 \times 5 = 3 > 2 = v_2(U, L) \\
v_2(U, \sigma_2') &= 0.4 \times 4 + 0.6 \times 3 = 3.4 > 3 = v_2(U, L)
\end{aligned}$$

which shows that σ_2' strictly dominates L. It is therefore follows by the continuity of the expected payoff function that any one of the infinitely many mixed strategies that puts weights close to 0.5 on C and the remaining probability on R will dominate L.[1] ■

(b) Find a strategy different from $(\sigma_1(U), \sigma_1(M), \sigma_1(D)) = (0, \frac{1}{2}, \frac{1}{2})$ that strictly dominates the pure strategy U for player 1 in the game remaining after one stage of elimination. Argue that you can find an infinite number of such strategies.

Answer: This is an identical procedure as for part (a).

4. **Monitoring:** An employee (player 1) who works for a boss (player 2) can either work (W) or shirk (S), while his boss can either monitor the employee (M) or ignore him (I). Like most employee-boss relationships, if the employee is working then the boss prefers not to monitor, but if the boss is not

[1] A more elegant way 0f writing this would be to choose a mixed strategy $\sigma_2' = (0, \frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon)$ and show that for small enough values of ε it follows that σ_2' strictly dominates L, and it follows that there are infinitely many such values of ε.

monitoring then the employee prefers to shirk. The game is represented in the following matrix:

		Player 2	
		M	I
player 1	W	$1,1$	$1,2$
	S	$0,2$	$2,1$

(a) Draw the best response functions of each player.

Answer: Let p be the probability that player 1 chooses W and q the probability that player 2 chooses M. It follows that $v_1(W,q) > v_1(S,q)$ if and only if $1 > 2(1-q)$, or $q > \frac{1}{2}$, and $v_2(p,M) > v_2(p,I)$ if and only if $p + 2(1-p) > 2p + (1-p)$, or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

and for player 2,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases} .$$

Notice that these are identical to the best response functions for the matching pennies game (see Figure 6.3). ■

(b) Find the Nash equilibrium of this game. What kind of game does this game remind you of?

Answer: From the two best response correspondences the unique Nash equilibrium is $(p,q) = (\frac{1}{2}, \frac{1}{2})$ and the game's strategic forces are identical to those in the Matching Pennies game. ■

5. **Cops and Robbers:** Player 1 is a police officer who must decide whether to patrol the streets or to hang out at the coffee shop. His payoff from hanging out at the coffee shop is 10, while his payoff from patrolling the streets

depends on whether he catches a robber, who is player 2. If the robber prowls the streets then the police officer will catch him and obtain a payoff of 20. If the robber stays in his hideaway then the officer's payoff is 0. The robber must choose between staying hidden or prowling the street. If he stays hidden then his payoff is 0, while if he walks the street his payoff is (-10) if the officer is patrolling the streets, and it is 10 if the officer is at the coffee shop.

(a) Write down the matrix form of this game.

Answer: Let P denote patrol and C coffee shop for player 1, and S is the robber's choice of prowling while H is remaining hidden. The game is therefore

Player 2

player 1	S	H
P	$20, -10$	$0, 0$
C	$10, 10$	$10, 0$

■

(b) Draw the best response functions of each player.

Answer: Let p be the probability that player 1 chooses P and q the probability that player 2 chooses S. It follows that $v_1(P, q) > v_1(C, q)$ if and only if $20q > 10$, or $q > \frac{1}{2}$, and $v_2(p, S) > v_2(p, H)$ if and only if $-10p + 10(1 - p) > 0$, or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

and for player 2,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0, 1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases} .$$

Notice that these are identical to the best response functions for the matching pennies game (see Figure 6.3). ■

(c) Find the Nash equilibrium of this game. What kind of game does this game remind you of?

Answer: From the two best response correspondences the unique Nash equilibrium is $(p, q) = (\frac{1}{2}, \frac{1}{2})$ and the game's strategic forces are identical to those in the Matching Pennies game. ■

6. **Declining Industry:** Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form.

Answer: Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter.

		Player 2		
		E	T	N
Player 1	E	$0, 0$	$0, 2$	$0, 4$
	T	$2, 0$	$-1, -1$	$-1, 1$
	N	$4, 0$	$1, -1$	$-2, -2$

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: There are no strictly dominated strategies but there is a weakly dominated one: T. To see this note that choosing both E and

N with probability $\frac{1}{2}$ each yields the same expected payoff as choosing T against E or N, and a higher expected payoff against T, and hence $\sigma_i = (\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{2}, 0, \frac{1}{2})$ weakly dominates T. The reason there is no strictly dominated strategy is that, starting with σ_i, increasing the weight on E causes the mixed strategy to be worse than T against E, while increasing the weight on N causes the mixed strategy to be worse than T against N, implying it is impossible to find a mixed strategy that strictly dominates T. ■

(c) Find the pure strategy Nash equilibria.

Answer: Because T is weakly dominated, it is suspect of never being a best response. A quick observation should convince you that this is indeed the case: it is never a best response to any of the pure strategies, and hence cannot be part of a pure strategy Nash equilibrium. Removing T from consideration results in the reduced game:

Player 1 \ Player 2	E	N
E	$0, 0$	$0, 4$
N	$4, 0$	$-2, -2$

for which there are two pure strategy Nash equilibria, (E, N) and (N, E). ■

(d) Find the unique mixed strategy Nash equilibrium (hint: you can use your answer to (b) to make things easier.)

Answer: We start by ignoring T and using the reduced game in part (c) by assuming that the weakly dominated strategy T will never be part of a Nash equilibrium. We need to find a pair of mixed strategies, $(\sigma_1(E), \sigma_1(N))$ and $(\sigma_2(E), \sigma_2(N))$ that make both players indifferent between E and N. For player 1 the indifference equation is,

$$0 = 4\sigma_2(E) - 2(1 - \sigma_2(E))$$

which results in $\sigma_2(E) = \frac{1}{3}$, and for player 2 the indifference equation is symmetric, resulting in $\sigma_1(E) = \frac{1}{3}$. Hence, the mixed strategy Nash equilibrium of the original game is $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$. Notice that at this Nash equilibrium, each player is not only indifferent between E and N, but choosing T gives the same expected payoff of zero. However, choosing T with positive probability cannot be part of a mixed strategy Nash equilibrium. To prove this let player 2 play the mixed strategy $\sigma_2 = (\sigma_2(E), \sigma_2(T), \sigma_2(N)) = (\sigma_{2E}, \sigma_{2T}, 1 - \sigma_{2E} - \sigma_{2T})$. The strategy T for player 1 is at least as good as E if and only if,

$$0 \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2E} \geq \frac{1}{3}$. The strategy T for player 1 is at least as good as N if and only if,

$$4\sigma_{2E} - \sigma_{2T} - 2(1 - \sigma_{2E} - \sigma_{2T}) \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2T} \leq 1 - 3\sigma_{2E}$. But if $\sigma_{2E} \geq \frac{1}{3}$ (when T is as good as E) then $\sigma_{2T} \leq 1 - 3\sigma_{2E}$ reduces to $\sigma_{2T} \leq 0$, which can only hold when $\sigma_{2E} = \frac{1}{3}$ and $\sigma_{2T} = 0$ (which is the Nash equilibrium we found above). A symmetric argument holds to conclude that $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium. ■

7. **Grad School Competition:** Two students sign up for an honors thesis with a Professor. Each can invest time in their own project: either no time, one week, or two weeks (these are the only three options). The cost of time is 0 for no time, and each week costs 1 unit of payoff. The more time a student puts in the better their work will be, so that if one student puts in more time than the other there will be a clear "leader". If they put in the same amount of time then their thesis projects will have the same quality. The professor, however, will give out only one "A" grade. If there is a clear leader then he will get the A, while if they are equally good then the professor will toss a fair coin to decide who gets the A grade. The other student gets a "B". Since both wish to continue to graduate school, a grade of A is worth 3 while a grade of B is worth zero.

   (a) Write down this game in matrix form.

Answer: Let N denote no time, O denote one week, and T denote two weeks. The matrix game is,

		Player 2		
		N	O	T
Player 1	N	$1.5, 1.5$	$0, 2$	$0, 1$
	O	$2, 0$	$0.5, 0.5$	$-1, 1$
	T	$1, 0$	$1, -1$	$-0.5, -0.5$

The payoffs are derived by the fact that a tie is an equal chance of getting 3 so each player gets 1.5 in expectation. ■

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: Each one of the three strategies can be a strict best response: N is a best response to T, O is a best response to N, and T is a best response to O. Hence, no strategy is strictly or weakly dominated. ■

(c) Find the unique mixed strategy Nash equilibrium.

Answer: Let $\sigma_i = (\sigma_{iN}, \sigma_{iO}, 1 - \sigma_{iN} - \sigma_{iO})$ denote a mixed strategy for player i. Because the game is symmetric it suffices to solve the indifference conditions for one player. For player i to be indifferent between N and O,

$$1.5\sigma_{iN} = 2\sigma_{iN} + 0.5\sigma_{iO} - (1 - \sigma_{iN} - \sigma_{iO})$$

and for him to be indifferent between N and T,

$$1.5\sigma_{iN} = \sigma_{iN} + \sigma_{iO} - 0.5(1 - \sigma_{iN} - \sigma_{iO})$$

Solving these two equations with two unknowns yields $\sigma_{iN} = \sigma_{iO} = \frac{1}{3}$ implying that the unique mixed strategy Nash equilibrium has the players mixing between all three pure strategies with equal probability. ■

8. **Market entry:** There are 3 firms that are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$ where n is the number of firms that enter. The cost of entering is 62.

(a) Find all the pure strategy Nash equilibria.

Answer: The costs of entry are 62 so the benefits of entry must be at least that for a firm to choose to enter. Clearly, if a firm believes the other two are not entering then it wants to enter, and if it believes that the other firms are entering then it would stay out (it would only get 50). If a firm believes that only one other firm is entering then it prefers to enter and get 75. Hence, there are three pure strategy Nash equilibria in which two of the three firms enter and one stays out. ■

(b) Find the symmetric mixed strategy equilibrium where all three players enter with the same probability.

Answer: Let p be the probability that a firm enters. In order to be willing to mix the expected payoff of entering must be equal to zero. If a firm enters then with probability p^2 it will face two other entrants and receive $v_i = 50 - 62 = -12$, with probability $(1-p)^2$ it will face no other entrants and receive $v_i = 150 - 62 = 88$, and with probability $2(1-p)p$ it will face one other entrant and receive $v_i = 75 - 62 = 13$. Hence, to be willing to mix the expected payoff must be zero,$p^2 + 1 - p^2$

$$(1-p)^2 88 + 2(1-p)p13 - p^2 12 = 0$$

which results in the quadratic equation $25p^2 - 75p + 44 = 0$, and the relevant solution (between 0 and 1) is $p = \frac{4}{5}$. ■

9. **Discrete all pay auction:** In section 6.1.4 we introduced a version of an all pay auction that worked as follows: Each bidder submits a bid. The highest bidder gets the good, but *all bidders pay there bids*. Consider an auction in which player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1 or 2. The twist is that each player pays his bid regardless of whether he wins the good. If player i bids more than player j then i win's the good and both pay. If both players bid the same amount then a coin is tossed to determine who gets the good but again, both pay.

(a) Write down the game in matrix form. Which strategies survive IESDS?

Answer: Let Z denote zero, O denote one, and T denote two. The matrix game is,

Player 2

Player 1	Z	O	T
Z	$1.5, 2.5$	$0, 4$	$0, 3$
O	$2, 0$	$0.5, 1.5$	$-1, 3$
T	$1, 0$	$1, -1$	$-0.5, 0.5$

The payoffs are derived by the fact that a tie is an equal chance of winning so player 1 gets 1.5 and player 2 gets 2.5 in expectation. It is easy to see that for player 2, playing Z is dominated by playing T, so it is eliminated in the first stage of IESDS. In the second stage O is dominated by T for player 1 and we are left with the following reduced game that survives IESDS,

Player 2

Player 1	O	T
Z	$0, 4$	$0, 3$
T	$1, -1$	$-0.5, 0.5$

■

(b) Find the Nash equilibria of this game.

Answer: From the reduced game it is easy to see that there is no pure strategy Nash equilibrium. Let $\sigma_1 = (\sigma_{1Z}, \sigma_{1T})$ and $\sigma_2 = (\sigma_{2O}, \sigma_{2T})$ denote the mixed strategies for the players in the reduced game. For player 1 to be indifferent between Z and T,

$$0 = \sigma_{2O} - 0.5(1 - \sigma_{2O})$$

which yields $\sigma_{2O} = \frac{1}{3}$. For player 2 to be indifferent between O and T,

$$4\sigma_{1Z} - (1 - \sigma_{1Z}) = 3\sigma_{1Z} + 0.5(1 - \sigma_{1Z})$$

which yields $\sigma_{1Z} = 0.6$. Thus, the unique mixed strategy Nash equilibrium has the players mixing $\sigma_1 = (\frac{3}{5}, \frac{2}{5})$ and $\sigma_2 = (\frac{1}{3}, \frac{2}{3})$ in the reduced game, or $\sigma_1 = (\frac{3}{5}, 0, \frac{2}{5})$ and $\sigma_2 = (0, \frac{1}{3}, \frac{2}{3})$ in the original game. ■

10. **Continuous all pay auction:** Consider an all-pay auction for a good worth 1 to each of the two bidders. Each bidder can choose to offer a bid from the unit interval so that $S_i = [0, 1]$. Players only care about the expected value they will end up with at the end of the game (i.e., if a player bids 0.4 and expects to win with probability 0.7 then his payoff is $0.7 \times 1 - 0.4$).

    (a) Model this auction as a normal-form game.

    Answer: There are two players, $N = \{1, 2\}$, each has a strategy set $S_i = [0, 1]$, and assuming that the players are equally likely to get the good in case of a tie, the payoff to player i is given by

$$v_i(s_i, s_j) = \begin{cases} 1 - s_i & \text{if } s_i > s_j \\ \frac{1}{2} - s_i & \text{if } s_i = s_j \\ -s_i & \text{if } s_i < s_j \end{cases}$$

    (b) Show that this game has no pure strategy Nash Equilibrium.

    Answer: First, it cannot be the case that $s_i = s_j < 1$ because then each player would benefit from raising his bid by a tiny amount ε in order to win the auction and receive a higher payoff $1 - \varepsilon - s_i > \frac{1}{2} - s_i$. Second, it cannot be the case that $s_i = s_j = 1$ because each player would prefer to bid nothing and receive $0 > -\frac{1}{2}$. Last, it cannot be the case that $s_i > s_j \geq 0$ because then player i would prefer to lower his bid by ε while still beating player j and paying less money. Hence, there cannot be a pure strategy Nash equilibrium. ■

    (c) Show that this game cannot have a Nash Equilibrium in which each player is randomizing over a finite number of bids.

    Answer: Assume that a Nash equilibrium involves player 1 mixing between a finite number of bids, $\{s_{11}, s_{12,...}, s_{1K}\}$ where $s_{11} \geq 0$ is the lowest bid, $s_{1K} \leq 1$ is the highest, $s_{1k} < s_{1(k+1)}$ and each bid s_{1k} is being played with some positive probability σ_{1k}. Similarly assume that player 2 is mixing between a finite number of bids, $\{s_{21}, s_{22,...}, s_{2L}\}$ and each bid s_{2l} is being played with some positive probability σ_{2l}. (i) First observe that it cannot be true that $s_{1K} < s_{2L}$ (or the reverse by symmetry). If it were the case then player 2 will win for sure when he bids s_{2L} and pay his bid, while if he reduces his bid by some ε such that $s_{1K} < s_{2L} - \varepsilon$ then he will still win for sure and pay less, contradicting that playing s_{2L} was part of a Nash equilibrium. (ii) Second observe that when $s_{1K} = s_{2L}$ then the expected payoff of player 2 from bidding s_{2L} is

$$\begin{aligned} Ev_2 &= \Pr\{s_{1k} < s_{2L}\}(1 - s_{2L}) + \Pr\{s_{1k} = s_{2L}\}(\frac{1}{2} - s_{2L}) \\ &= (1 - \sigma_{1K})(1 - s_{2L}) + \sigma_{1K}(\frac{1}{2} - s_{2L}) \\ &= 1 - s_{2L} - \frac{1}{2}\sigma_K \geq 0 \end{aligned}$$

where the last inequality follows from the fact that $\sigma_{2L} > 0$ (he would not play it with positive probability if the expected payoff were negative.) Let $s'_{2L} = s_{2L} + \varepsilon$ where $\varepsilon = \frac{1}{4}\sigma_K$. If instead of bidding s_{2L} player 2 bids s'_{2L} then he wins for sure and his utility is

$$v_2 = 1 - s'_{2L} = 1 - s_{2L} - \frac{1}{4}\sigma_K > 1 - s_{2L} - \frac{1}{2}\sigma_K$$

contradicting that playing s_{2L} was part of a Nash equilibrium. ■

(d) Consider mixed strategies of the following form: Each player i chooses and interval, $[\underline{x}_i, \overline{x}_i]$ with $0 \leq \underline{x}_i < \overline{x}_i \leq 1$ together with a cumulative distribution $F_i(x)$ over the interval $[\underline{x}_i, \overline{x}_i]$. (Alternatively you can think of each player choosing $F_i(x)$ over the interval $[0, 1]$, with two values $\underline{x}_i$ and $\overline{x}_i$ such that $F_i(\underline{x}_i) = 0$ and $F_i(\overline{x}_i) = 1$.)

   i. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 = \overline{x}_2$.

Answer: Assume not. There are two cases: (a) $\underline{x}_1 \neq \underline{x}_2$: Without loss assume that $\underline{x}_1 < \underline{x}_2$. This means that there are values of $s_1' \in (\underline{x}_1, \underline{x}_2)$ for which $s_1' > 0$ but for which player 1 loses with probability 1. This implies that the expected payoff from this bid is negative, and player would be better off bidding 0 instead. Hence, $\underline{x}_1 = \underline{x}_2$ must hold. (b) $\overline{x}_1 \neq \overline{x}_2$: Without loss assume that $\overline{x}_1 < \overline{x}_2$. This means that there are values of $s_2' \in (\overline{x}_1, \overline{x}_2)$ for which $\overline{x}_1 < s_2' < 1$ but for which player 2 wins with probability 1. But then player 2 could replace s_2' with $s_2'' = s_2' - \varepsilon$ with ε small enough such that $\overline{x}_1 < s_2'' < s_2' < 1$, he will win with probability 1 and pay less than he would pay with s_2'. Hence, $\overline{x}_1 = \overline{x}_2$ must hold. ■

ii. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2 = 0$.

Answer: Assume not so that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$. This means that when player i bids $\underline{x}$ then he loses with probability 1, and get an expected payoff of $-\underline{x} < 0$. But instead of bidding $\underline{x}$ player i can bid 0 and receive 0 which is better than $-\underline{x}$, implying that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$ cannot be an equilibrium. ■

iii. Using the above, argue that if two such strategies are a mixed strategy Nash equilibrium then both players must be getting an expected payoff of zero.

Answer: As proposition 6.1 states, if a player is randomizing between two alternatives then he must be indifferent between them. Because both players are including 0 in the support of their mixed strategy, their payoff from 0 is zero, and hence their expected payoff from any choice in equilibrium must be zero. ■

iv. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\overline{x}_1 = \overline{x}_2 = 1$.

Answer: Assume not so that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$. From (iii) above the expected payoff from any bid in $[0, \overline{x}]$ is equal to zero. If one of the players deviates from this strategy and choose to bid $\overline{x}+\varepsilon<1$ then he will win with probability 1 and receive a payoff of $1-(\overline{x}+\varepsilon)>0$, contradicting that $\overline{x}_1=\overline{x}_2=\overline{x}<1$ is an equilibrium. ■

v. Show that $F_i(x)$ being uniform over $[0,1]$ is a symmetric Nash equilibrium of this game.

Answer: Imagine that player 2 is playing according to the proposed strategy $F_2(x)$ uniform over $[0,1]$. If player 1 bids some value $s_1\in[0,1]$ then his expected payoff is

$$\Pr\{s_1>s_2\}(1-s_1)+\Pr\{s_1<s_2\}(-s_1)=s_1(1-s_1)+(1-s_1)(-s_1)=0$$

implying that player 1 is willing to bid any value in the $[0,1]$ interval, and in particular, choosing a bid according to $F_1(x)$ uniform over $[0,1]$. Hence, this is a symmetric Nash equilibrium. ■

11. **Bribes:** Two players find themselves in a legal battle over a patent. The patent is worth 20 for each player, so the winner would receive 20 and the loser 0. Given the norms of the country they are in, it is common to bribe the judge of a case. Each player can offer a bribe secretly, and the one whose bribe is the largest is awarded the patent. If both choose not to bribe, or if the bribes are the same amount, then each has an equal chance of being awarded the patent. If a player does bribe, then bribes can be either a value of 9 or of 20. Any other number is considered to be very unlucky and the judge would surely rule against a party who offers a different number.

(a) Find the unique pure-strategy Nash equilibrium of this game.

Answer: The game is captured in the following two player matrix, where Z represents no payment, N represents a bribe of 9 and T a bribe of 20. For example, if both choose 9 then they have an equal chance of getting 20, so the expected payoff is $\frac{1}{2} \times 20 - 9 = 1$,

Player 2

Player 1	Z	N	T
Z	$10, 10$	$0, 11$	$0, 0$
N	$11, 0$	$1, 1$	$-9, 0$
T	$0, 0$	$0, -9$	$-10, -10$

It is easy to see that T is strictly dominated by N. In the remaining game, Z is strictly dominated by N, and hence (N, N) is the unique Nash equilibrium. ■

(b) If the norm were different so that a bribe of 15 were also acceptable, is there a pure strategy Nash equilibrium?

Answer: Now the game is as follows (where F denotes a bribe of 15),

Player 2

Player 1	Z	N	F	T
Z	$10, 10$	$\overline{0, 11}$	$\underline{0, 5}$	$\underline{0, 0}$
N	$\underline{11, 0}$	$1, 1$	$\overline{-9, 5}$	$-9, 0$
F	$\overline{5, 0}$	$\underline{5, -9}$	$-5, -5$	$\overline{-15, 0}$
T	$\overline{0, 0}$	$0, -9$	$\underline{0, -15}$	$-10, -10$

Using the best responses of each player it is easy to see that there is no pure strategy Nash equilibrium. ■

(c) Find the symmetric mixed-strategy Nash equilibrium for the game with possible bribes of $9, 15$ and 20.

Answer: Note first that T is weakly dominated by Z, so consider the game without T,

Player 2

Player 1	Z	N	F
Z	$10, 10$	$0, 11$	$0, 5$
N	$11, 0$	$1, 1$	$-9, 5$
F	$5, 0$	$5, -9$	$-5, -5$

Let $\sigma_i = (\sigma_{iZ}, \sigma_{iN}, \sigma_{iF})$ denote a mixed strategy for player i where $\sigma_{iF} = 1 - \sigma_{iZ} - \sigma_{iN}$. The game is symmetric so for player 1 to be indifferent between Z and T it must be that,

$$10\sigma_{2Z} = 11\sigma_{2Z} + \sigma_{2N} - 9(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{9}{10} - \sigma_{2Z}$. For player 1 to be indifferent between Z and F it must be that,

$$10\sigma_{2Z} = 5\sigma_{2Z} + 5\sigma_{2N} - 5(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{1}{2}$. Hence, the unique (mixed strategy) Nash equilibrium has each player play $\sigma_i = (\frac{2}{5}, \frac{1}{2}, \frac{1}{10})$. ■

12. **The Tax Man:** A citizen (player 1) must choose whether or not to file taxes honestly or whether to cheat. The tax man (player 2) decides how much effort to invest in auditing and can choose $a \in [0, 1]$, and the cost to the tax man of investing at a level a is $c(a) = 100a^2$. If the citizen is honest then he receives the benchmark payoff of 0, and the tax man pays the auditing costs without any benefit from the audit, yielding him a payoff of $(-100a^2)$. If the citizen cheats then his payoff depends on whether he is caught. If he is caught then his payoff is (-100) and the tax man's payoff is $100 - 100a^2$. If he is not caught then his payoff is 50 while the tax man's payoff is $(-100a^2)$. If the citizen cheats and the tax man chooses to audit at level a then the citizen is caught with probability a and is not caught with probability $(1 - a)$.

    (a) If the tax man believes that the citizen is cheating for sure, what is his best response level of a?

    Answer: The tax man maximizes $a(100 - 100a^2) + (1 - a)(0 - 100a^2) = 100a - 100a^2$. The first-order optimality condition is $100 - 200a = 0$, yielding $a = \frac{1}{2}$. ■

    (b) If the tax man believes that the citizen is honest for sure, what is his best response level of a?

Answer: The tax man maximizes $-100a^2$ which is maximized at $a = 0$. ■

(c) If the tax man believes that the citizen is honest with probability p, what is his best response level of a as a function of p?

Answer: The tax man maximizes $p(-100a^2) + (1-p)(100a - 100a^2) = 100(1-p)a - 100a^2$. The first-order optimality condition is $100(1-p) - 200a = 0$, yielding the best response function $a^*(p) = \frac{1-p}{2}$. ■

(d) Is there a pure strategy Nash equilibrium of this game? Why or why not?

Answer: There is no pure strategy Nash equilibrium. To see this, consider the best response of player 1 who believes that player 2 chooses some level $a \in [0, 1]$. His payoff from being honest is 0 while his payoff from cheating is $a(-100) + (1 - a)50 = 50 - 150a$. Hence, he prefers to be honest if and only if $0 \geq 50 - 150a$, or $a \geq \frac{1}{3}$. Letting $p^*(a)$ denote the best response correspondence of player 1 as the probability that he is honest, we have that

$$p^*(a) = \begin{cases} 1 & \text{if } a > \frac{1}{3} \\ [0,1] & \text{if } a = \frac{1}{3} \\ 0 & \text{if } a < \frac{1}{3} \end{cases}$$

and it is easy to see that there are no values of a and p for which both players are playing mutual best responses. ■

(e) Is there a mixed strategy Nash equilibrium of this game? Why or why not?

Answer: From (d) above we know that player 1 is willing to mix if and only if $a = \frac{1}{3}$, which must therefore hold true in a mixed strategy Nash equilibrium. For player 2 to be willing to play $a = \frac{1}{3}$ we use his best response from part (c), $\frac{1}{3} = \frac{1-p}{2}$, which yields, $p = \frac{1}{3}$. Hence, the unique mixed strategy Nash equilibrium has player 1 being honest with probability $\frac{1}{3}$ and player 2 choosing $a = \frac{1}{3}$. ■

Part III

Dynamic Games of Complete Information

7

Preliminaries

1. **Strategies:** Imagine an extensive form game in which player i has K information sets.

   (a) If the player has an identical number of m possible actions in each information set, how many pure strategies does he have?

   Answer: The player has m^K pure strategies.

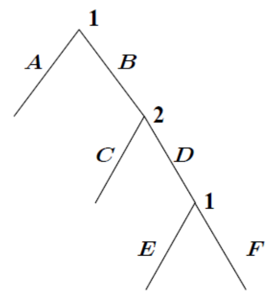
   (b) If the player has m_k actions in information set $k \in \{1, 2, ..., K\}$, how many pure strategies does the player have?

   Answer: The player has $m_1 \times m_2 \times \cdots \times m_K$ pure strategies. ■

2. **Strategies and equilibrium:** Consider a two player game in which player 1 can choose A or B. The game ends if he chooses A while it continues to player 2 if he chooses B. Player 2 can then choose C or D, with he game ending after C and continuing again with player 1 after D. Player 1 then can choose E or F, and the game ends after each of these choices.

   (a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?

Answer:



This game is a game of perfect information. ■

(b) How many terminal nodes does the game have? How many information sets?

Answer: The game has 4 terminal nodes (after choices A, C, E and F) and 3 information sets (one for each player. ■

(c) How many pure strategies does each player have?

Answer: Player 1 has 4 pure strategies and player 2 has 2. ■

(d) Imagine that the payoffs following choice A by player 1 are $(2, 0)$, following C by player 2 are $(3, 1)$, following E by player 1 are $(0, 0)$ and following F by player 1 are $(1, 2)$. What are the Nash equilibria of this game? Does one strike you as more "appealing" than the other? If so, explain why.

Answer: We can write down the matrix form of this game as follows (xy denotes a strategy for player 1 where $x \in \{A, B\}$ is what he does in his first information set and $y \in \{E, F\}$ in his second one),

		Player 2	
		C	D
Player 1	AE	$\overline{2,0}$	$\underline{\overline{2,0}}$
	AF	$\overline{2,0}$	$\underline{\overline{2,0}}$
	BE	$\underline{\overline{3,1}}$	$0,0$
	BF	$\underline{3,1}$	$\overline{1,2}$

It's easy to see that there are three pure strategy Nash equilibria: $(AE, D), (AF, D)$ and (BE, C). The equilibria $(AE, D), (AF, D)$ are Pareto dominated by the equilibrium (BE, C), and hence it would be tempting to argue that (BE, C) is the more "appealing" equilibrium. As we will see in Chapter 8 it is actually (AF, D) that has properties that are more appealing (sequential rationality). ■

3. **Tick-tack-toe:** The extensive form representation of a game can be cumbersome even for very simple games. Consider the game of Tick-tack-toe where 2 players mark "X" or "O" in a 3×3 matrix. Player 1 moves first, then player 2, and so on. If a player gets three of his kind in a row, column, or one of the diagonals then he wins, and otherwise it is a tie. For this question assume that even after a winner is declared, the players *must completely* fill the matrix before the game ends.

   (a) Is this a game of perfect or imperfect information? Why?

   Answer: This is a game of perfect information because each player knows exactly what transpired before he moves, and hence every information set contains one node. ■

   (b) How many information sets does player 2 have after player 1's first move?

   Answer: Player 2 has 9 information sets, one for each of the moves of player 1. ■

   (c) How many information sets does player 1 have after each of player 2's first move?

   Answer: Player 2 has 8 possible moves in his first turn, and this is true for each one of the 9 possible moves that player 1 has in hid first turn. Hence, player 1 has $9 \times 8 = 72$ information sets in his second move (after player 2's first move).

(d) How many information sets does each player have in total? (Hint: For this and the next part you may want to use a program like Excel.)

Answer: Continuing the logic of part (c), after player 1's second move, player 2 has $9 \times 8 \times 7 = 504$ information sets, then player 1 has $9 \times 8 \times 7 \times 6 = 3,024$ information sets, and so on ($15,120; 60,480; 181,440$ and $362,880$). We add the alternating numbers to get how many information set each player has, and we have to remember to add the root which belongs to player 1. Hence, player 1 has 426,457 information sets while player 2 has 197,073. ■

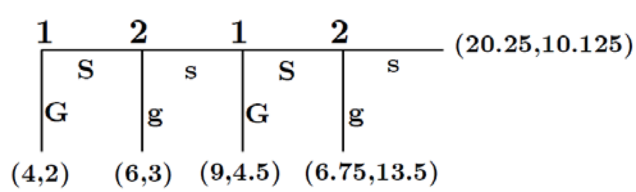
(e) How many terminal nodes does the game have?

Answer: The number of terminal nodes is equal to the number of information sets in player 1's last turn because at that pint he just has one move, to complete the tick-tack-toe matrix, which is $362,880$. ■

4. **Centipedes**: Imagine a two player game that proceeds as follows. A pot of money is created with \$6 in it initially. Player 1 moves first, then player 2, then player 1 again and finally player 2 again. At each player's turn to move, he has two possible actions: grab (G) or share (S). If he grabs, he gets $\frac{2}{3}$ of the current pot of money, the other player gets $\frac{1}{3}$ of the pot and the game ends. If he shares then the size of the current pot is multiplied by $\frac{3}{2}$ and the next player gets to move. At the last stage in which player 2 moves, if he chooses share then the pot is still multiplied by $\frac{3}{2}$, player 2 gets $\frac{1}{3}$ of the pot and player 1 gets $\frac{2}{3}$ of the pot.

(a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?

Answer:



This is a game of perfect information. Note that we draw the game from left to right (which is the common convention for "centipede games" of this sort.) We use capital letters for player 1 and lower case for player 2. ■

(b) How many terminal nodes does the game have? How many information sets?

Answer: The game has five terminal nodes and four information sets. ■

(c) How many pure strategies does each player have?

Answer: Each player has four pure strategies (2 actions in each of his 2 information sets). ■

(d) Find the Nash equilibria of this game. How many outcomes can be supported in equilibrium?

Answer: Using the convention of xy to denote a strategy of player where he chooses x in his first information set and y in his second, we can draw the following matrix representation of this game,

Player 1 \ Player 2	gg	gs	sg	ss
GG	$\overline{\underline{4,2}}$	$\overline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
GS	$\overline{\underline{4,2}}$	$\overline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
SG	$\overline{3,6}$	$\overline{3,6}$	$\underline{9,4.5}$	$9,4.5$
SS	$3,6$	$3,6$	$\overline{6.75,13.5}$	$\underline{20.25,10.125}$

We see that only one outcome can be supported as a Nash equilibrium: player 1 grabs immediately and the players' payoffs are $(4,2)$. ■

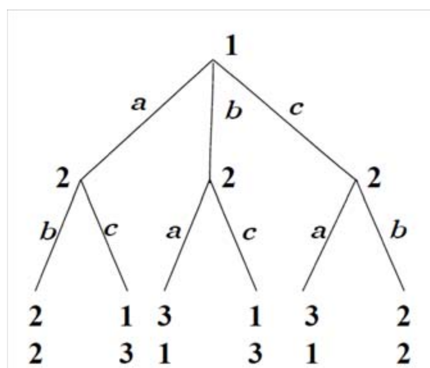
(e) Now imagine that at the last stage in which player 2 moves, if he chooses to share then the pot is equally split among the players. Does your answer to part (d) above change?

Answer: The answer does change because the payoffs from the pair of strategies (SS, ss) changes from $(20.25, 10.125)$ to $(15.1875, 15.1875)$ in which case player 2's best response to SS will be ss, and player 1's best response to ss remains SS, so that (SS, ss) is another Nash equilibrium in which they split 30.375 equally (the previous Nash equilibria are still equilibria). ■

5. **Veto Power:** Two players must choose between three alternatives, a, b and c. Player 1's preferences are given by $a \succ_1 b \succ_1 c$ while player 2's preferences are given by $c \succ_2 b \succ_2 a$. The rules are that player 1 moves first and can veto one of the three alternatives. Then, player two chooses which of the remaining two alternatives will be chosen.

   (a) Model this as an extensive form game tree (choose payoffs that represent the preferences).

   Answer: Assume that the payoff of the best option is 3, the second best 2 and the worst is 1. Player1's actions are which alternative to remove and player 2's which of the remaining two to choose.



   ■

   (b) How many pure strategies does each player have?

   Answer: Player 1 has three pure strategies while player 2 has eight (2 actions in each of three information sets.) ■

   (c) Find all the Nash equilibria of this game.

Answer: Let xyz be a strategy for player 2 where x is what he does following the removal of a, y for b and z for c so that we can use the following matrix,

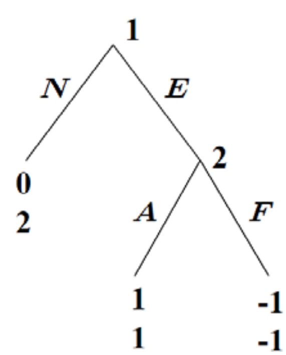
		Player 2							
		baa	bab	bca	bcb	caa	cab	cca	ccb
Player 1	a	$2,2$	$2,2$	$2,2$	$\underline{2,2}$	$\overline{1,3}$	$\overline{1,3}$	$\overline{1,3}$	$\overline{1,3}$
	b	$\underline{3,1}$	$\underline{3,1}$	$\overline{1,3}$	$\overline{1,3}$	$\underline{3,1}$	$\underline{3,1}$	$\overline{1,3}$	$\overline{1,3}$
	c	$\underline{3,1}$	$\overline{2,2}$	$\underline{3,1}$	$\underline{\overline{2,2}}$	$\underline{3,1}$	$\overline{2,2}$	$\underline{3,1}$	$\underline{\overline{2,2}}$

and we see that b is the only outcome that an be supported as an equilibrium via two Nash equilibria, (c, bcb) and (c, ccb). ■

6. **Entering an Industry:** A firm (player 1) is considering entering an established industry with one incumbent firm (player 2). Player 1 must choose whether to enter or to not enter the industry. If player 1 enters the industry then player 2 can either accommodate the entry, or fight the entry with a price war. Player 1's most preferred outcome is entering with player 2 not fighting, and his least preferred outcome is entering with player 2 fighting. Player 2's most preferred outcome is player 1 not entering, and his least preferred outcome is player 1 entering with player 2 fighting.

   (a) Model this as an extensive form game tree (choose payoffs that represent the preferences).

   Answer:



(b) How many pure strategies does each player have?

Answer: Each player has two pure strategies. ■

(c) Find all the Nash equilibria of this game.

Answer: There are two Nash equilibria which can be seen in the matrix,

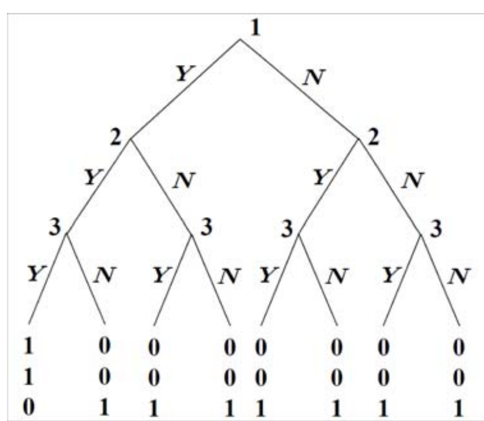
Player 1 \ Player 2	A	F
N	$\overline{0,2}$	$\underline{0,2}$
E	$\underline{\overline{1,1}}$	$-1,-1$

Both (N, F) and (E, A) are Nash equilibria of this game. ■

7. **Roommates Voting:** Three roommates need to vote on whether they will adopt a new rule and clean their room once a week, or stick to the current once a month rule. Each votes "yes" for the new rule or "no" for the current rule. Imagine that players 1 and 2 prefer the new rule while player 3 prefers the old rule.

    (a) Imagine that the players require a unanimous vote to adopt the new rule. Player 1 votes first, then player 2, and then player 3, each one observing the previous votes. Draw this as an extensive form game and find the Nash equilibria.

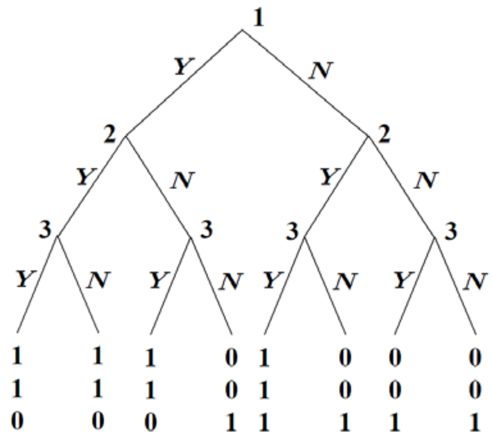
    Answer: The game is,



and there are many Nash equilibria. Player 1 has two pure strategies: Y and N. Player 2 has 4: $\{YY, YN, NY, NN\}$ (where the left entry corresponds to his left information set) and player 3 has 16 (again, with the natural left to right interpretation): $\{YYYY,$ $YYYN, YYNY, YYNN, YNYY, YNYN, YNNY, YNNN, NYYY,$ $NYYN, NYNY, NYNN, NNYY, NNYN, NNNY, NNNN\}$. Because a unanimous vote is needed, the only strategy profiles that are *not* a Nash equilibrium are those for which players 1 or 2 can change a "no" vote to a "yes" vote, or player 3 can change a "yes" vote to a "no" vote. These profiles are (i) $(N, Yx, Yuvw)$ where $x, u, v, w \in \{Y, N\}$ from which player 1 can profitably deviate from N; (ii) $(Y, Nx, Yuvw)$ from which player 2 can profitably deviate from Nx to Yx; and (iii) $(Y, Yx, Yuvw)$ from which player 3 can profitably deviate from $Yuvw$ to $Nuvw$. Thus, the Nash equilibria are profiles of strategies that belong to one of two classes: (i) player 3 votes $Nuvw$ and players 1 and 2 vote anything (a total of 64 strategy profiles which include 8 of player 3, 4 of player 2 and 2 of player 1); (ii) player 3 votes $Yuvw$, player 2 votes Nx and player 1 votes N (a total of 16 strategy profiles which include 8 of player 3, 2 of player 2 and 1 of player 1). All the outcomes have the current rule surviving. ■

(b) Imagine now that the players require a majority vote to adopt the new rule (at least two "yes" votes). Again, player 1 votes first, then player 2, and then player 3, each one observing the previous votes. Draw this as an extensive form game and find the Nash equilibria.

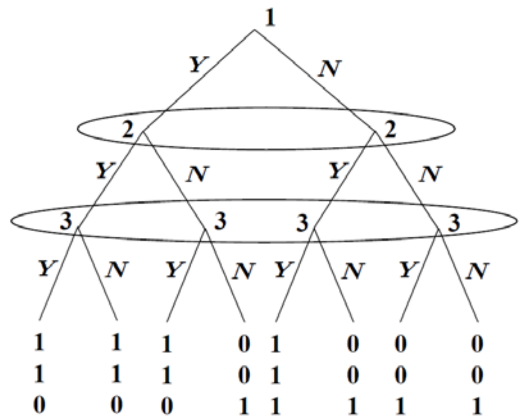
Answer: The game's payoff now change as follows,



Now only a majority vote is needed, but still, the only strategy profiles that are *not* a Nash equilibrium are those for which players 1 or 2 can change a "no" vote to a "yes" vote, or player 3 can change a "yes" vote to a "no" vote. These profiles are (i) $(N, Yx, uvwz)$ or $(N, Nx, uYwz)$ where $x, u, v, w, z \in \{Y, N\}$ from which player 1 can profitably deviate from N; (ii) $(Y, Nx, uvwz)$ or $(N, xN, uvYz)$ from which player 2 can profitably deviate from Nx to Yx or from xN to xY; and (iii) $(Y, Nx, Yvwz)$ or $(N, xY, uvYz)$ from which player 3 can profitably deviate from $Yvwz$ to $Nvwz$ or from $uvYz$ to $uvNz$. Thus, the Nash equilibria are profiles of strategies that belong to one of two classes: (i) player 3 votes $uvwz$, player 1 votes Y and 2 votes Yx (a total of 32 strategy profiles which include 16 of player 3, 2 of player 2 and 1 of player 1). These all have players 1 and 2 voting Y and support the new rule; (ii) player 3 votes $uNNz$, player 2 votes Nx and player 1 votes N (a total of 8 strategy profiles which include 4 of player 3, 2 of player 2 and 1 of player 1). These have players 1 and 3 or 1 and 2 (or all three) vote N and support the current rule surviving, and neither player 1 nor 2 can change the outcome by deviating unilaterally. ■

(c) Now imagine that the game is like in part (b), but the players put their votes in a hat, so that the votes of earlier movers are not observed by the votes of later movers, and at the end the votes are counted. Draw this as an extensive form game and find the Nash equilibria. In what way is this different from the result in (b) above?

Answer: This game is one of imperfect information where each player has one information set,



Like part (b), any strategy in which both players 1 and 2 are voting Y or one in which there are at least 2 no votes that cannot be changed to only one by players 1 and 2 will be an equilibrium, but the strategy sets are small because players 2 and 3 cannot condition their play on the history of what the "previous" players did. Hence, for each player the strategy set is $S_i = \{Y, N\}$ and the Nash equilibria are $(s_1, s_2, s_3) \in \{(Y, Y, Y), (Y, Y, N), (N, N, N)\}$. Like in part (b) both outcomes can be supported by a Nash equilibrium, just that now the strategy combinations that support it are fewer. ■

8. **Brothers**: Consider the following game that proceeds in two steps: In the first stage one brother (player 2) has two \$10 bills and can choose one of two options: he can give his younger brother (player 1) \$20, or give him one of the \$10 bills (giving nothing is inconceivable given the way they were raised.) This money will be used to buy snacks at the show they will see, and each one dollar of snack yields one unit of payoff for a player who uses it. The

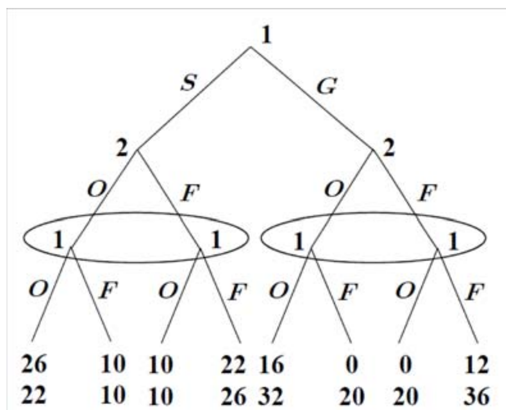
show they will see is determined by the following "battle of the sexes" game:

Player 2

Player 1	O	F
O	16,12	0,0
F	0,0	12,16

(a) Present the *entire* game in extensive form (a game tree).

Answer: Let the choices of player 1 first be S for spliting the \$20 and G for giving it all away. The entire game will have the payoffs from the choice of how to split the money added to the payoffs from the Battle of the Sexes part of the game as follows,



Because the latter is simultaneous, it does not mater which player moves after player 1 as long as the last player cannot distinguish between the choice of the player who moves just before him. ■

(b) Write the (pure) strategy sets for both players.

Answer: Both players can condition their choice in the Battle of the Sexes game on the initial split/give choice of player 1. For player 2, $S_2 = \{OO, OF, FO, FF\}$ where $s_2 = xy$ means that player 2 chooses $x \in \{O, F\}$ after player 1 chose S while player 2 chooses $y \in \{O, F\}$ after player 1 chose G. For player 1, however, even though he chooses first between S or G, he must specify his action for each information set even if he knows it will not happen (e.g., what he will do following

S even when he plans to play G). Hence, he has 8 pure strategies, $S_1 = \{SOO, SOF, SFO, SFF, GOO, GOF, GFO, GFF\}$ where $s_1 = xyz$ means that player 1 first chooses $x \in \{S, G\}$ and then chooses $y \in \{O, F\}$ if he played S and $z \in \{O, F\}$ if he played G. ■

(c) Present the *entire* game in one matrix.

Answer: This will be a 8×4 matrix as follows,

		Player 2			
		OO	OF	FO	FF
Player 1	SOO	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SOF	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SFO	$10, 10$	$10, 10$	$22, 26$	$22, 26$
	SFF	$10, 10$	$10, 10$	$22, 26$	$22, 26$
	GOO	$16, 32$	$0, 20$	$16, 32$	$0, 20$
	GOF	$0, 20$	$12, 36$	$0, 20$	$12, 36$
	GFO	$16, 32$	$0, 20$	$16, 32$	$0, 20$
	GFF	$0, 20$	$12, 36$	$0, 20$	$12, 36$

(d) Find the Nash equilibria of the *entire* game (pure and mixed strategies).

Answer: First note that for player 1, mixing equally between SOO and SFO will strictly dominate the four strategies GOO, GOF, GFO and GFF. Hence, we can consider the reduced 4×4 game,

		Player 2			
		OO	OF	FO	FF
Player 1	SOO	$\underline{\overline{26}}, \underline{\overline{22}}$	$\underline{\overline{26}}, \underline{\overline{22}}$	$10, 10$	$10, 10$
	SOF	$\underline{\overline{26}}, \underline{\overline{22}}$	$\underline{\overline{26}}, \underline{\overline{22}}$	$10, 10$	$10, 10$
	SFO	$10, 10$	$10, 10$	$\underline{\overline{22}}, \underline{\overline{26}}$	$\underline{\overline{22}}, \underline{\overline{26}}$
	SFF	$10, 10$	$10, 10$	$\underline{\overline{22}}, \underline{\overline{26}}$	$\underline{\overline{22}}, \underline{\overline{26}}$

The simple overline-underline method shows that we have eight pure strategy Nash equilibria, four yielding the payoffs $(26, 22)$ and the other four yielding $(22, 26)$. Because of each players indifference between the ways in which the payoffs are reached, there are infinitely many mixed strategies that yield the same payoffs. For example, any profile where player 1 mixes between SOO and SOF and where player 2 mixes between OO and OF will be a Nash equilibrium that yields $(26, 22)$. Similarly, any profile where player 1 mixes between SFO and SFF and where player 2 mixes between FO and FF will be a Nash equilibrium that yields $(22, 26)$. There is, however, one more class of mixed strategy Nash equilibria that are similar to the one found in section 6.2.3. To see this, focus on an even simpler game where we eliminate the duplicate payoffs as follows,

Player 1 \ Player 2	OO	FF
SOO	$\overline{\underline{26}}, \overline{\underline{22}}$	$10, 10$
SFF	$10, 10$	$\overline{\underline{22}}, \overline{\underline{26}}$

which preserve the nature of the game. For player 1 to be indifferent between SOO and SFF it must be that player 2 chooses OO with probability q such that

$$26q + 10(1 - q) = 10q + 22(1 - q)$$

which yields $q = \frac{3}{7}$. Similarly, for player 2 to be indifferent between OO and FF it must be that player 1 chooses SOO with probability p such that

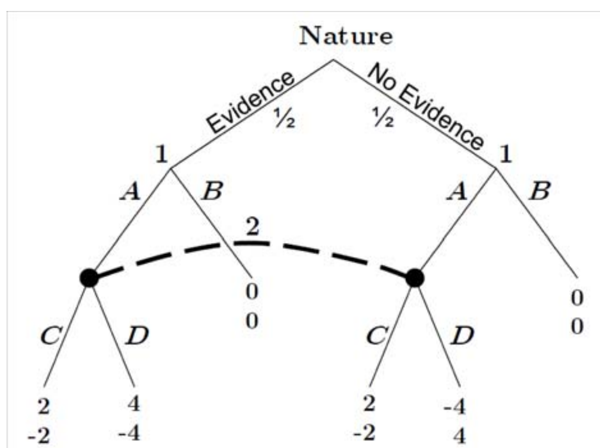
$$22p + 10(1 - p) = 10p + 26(1 - p)$$

which yields $p = \frac{4}{7}$. Hence, we found a mixed strategy Nash equilibrium that results in each player getting an expected payoff of $26 \times \frac{3}{7} + 10 \times \frac{4}{7} = 16\frac{6}{7}$. Notice, however, that player 1 is always indifferent between SOO and SOF, as well as between SFO and SFF so there are infinitely many ways to achieve this kind of mixed strategy, and similarly for player 2 because of his indifference between OO and OF as well as FO and FF. ■

9. **The Dean's Dilemma:** A student stole the DVD from the student lounge. The dean of students (player 1) suspects the student (player 2) and engages in evidence collection. However, evidence collection is a random process, and concrete evidence will be available to the dean only with probability $\frac{1}{2}$. The student knows the evidence generating process, but does not know whether the dean received evidence or not. The game proceeds as follows: The dean realizes if he has evidence or not, and then can choose his action, whether to Accuse the student (A), or Bounce the case (B) and forget it. Once accused, the student has two options: he can either Confess (C) or Deny (D).
   Payoffs are realized as follows: If the dean bounces the case then both players get 0. If the dean accuses the student, and the student confesses, the dean gains 2 and the student loses 2. If the dean accuses the student and the student denies, then payoffs depend on the evidence: If the dean has no evidence then he loses face which is losing 4, while the student gains glory which gives him a payoff 4. If, however, the dean has evidence then he is triumphant and gains 4, while the student is put on probation and loses 4.

   (a) Draw the game-tree that represents the extensive form of this game.

   Answer: Letting the Dean be player 1 and the student player 2,



   ■

   (b) Write down the matrix that represents the normal form of the extensive form you did in (a) above.

Answer: Because player 1 can condition whether or not to accuse on whether or not there is evidence, he has four pure strategies. Let $xy \in \{AA, AB, BA, BB\}$ be the strategy of player 1 where x follows "evidence" and y follows "no evidence." Player 2 does not know whether there is evidence and can only respond by confessing or not:

		Player 2	
		C	D
Player 1	AA	$2, -2$	$0, 0$
	AB	$1, -1$	$2, -2$
	BA	$1, -1$	$-2, 2$
	BB	$0, 0$	$0, 0$

■

(c) Solve for the Nash Equilibria of the game.

Answer: It is easy to see that BB is strictly dominated by AB and BA is strictly dominated by AA. The reduced game is therefore,

		Player 2	
		C	D
Player 1	AA	$2, -2$	$0, 0$
	AB	$1, -1$	$2, -2$

Let player 1 choose AA with probability p and player 2 choose C with probability q. For player 2 to be indifferent it must be that

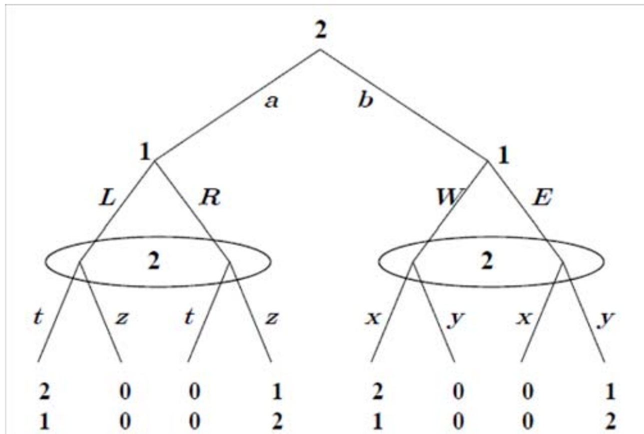
$$p(-2) + (1 - p)(-1) = p(0) + (1 - p)(-2)$$

and the solution is $p = \frac{1}{3}$. Similarly, for player 1 to be indifferent it must be that

$$q(2) + (1 - q)(0) = q(1) + (1 - q)(2)$$

and the solution is $q = \frac{2}{3}$. Hence, $(p, q) = (\frac{1}{3}, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium of this game. As you will see in chapter 15, this is a dynamic game of incomplete information. ■

10. **Perfect and Imperfect Recall:** Consider the game depicted in Figure **??**



Exercise

(a) What are the pure strategies sets for each player?

Answer: T

(b) Show that for any behavioral strategy for player 1, there is a mixed strategy that leads to the same distribution over the terminal nodes regardless of the strategy chosen by player 2.

Answer: T

(c) Show that for any behavioral strategy for player 2, there is a mixed strategy that leads to the same distribution over the terminal nodes regardless of the strategy chosen by player 1.

Answer: T

(d) Now imagine that the game does not have perfect recall so that player 2's two bottom information sets are now one large information set. Can you find an example showing that the claim in (a) above is no longer true?

Answer: T

8

Credibility and Sequential Rationality

1. Find the mixed strategy subgame perfect equilibrium of the Sequential Battle of the Sexes game depicted in Figure **??**

   Answer: The subgame starting with player 1 choosing between O and F is given in the following matrix:

Player 1 \ Player 2	o	f
O	$2,1$	$0,0$
F	$0,0$	$1,2$

   Let player 1 choose O with probability p and player 2 choose o with probability q. For player 2 to be indifferent it must be that

   $$p(1) + (1-p)(0) = p(0) + (1-p)(2)$$

   and the solution is $p = \frac{2}{3}$. Similarly, for player 1 to be indifferent it must be that

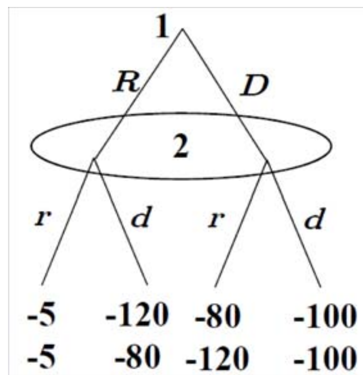
   $$q(2) + (1-q)(0) = q(0) + (1-q)(1)$$

   and the solution is $q = \frac{1}{3}$. Hence, $(p, q) = (\frac{1}{3}, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium of this subgame with expected payoffs of $(v_1, v_2) = (\frac{2}{3}, \frac{2}{3})$. Working backward, player 1 would prefer to choose N over Y. ■

2. **Mutually Assured Destruction (revisited):** Consider the game in section **??**.

   (a) Find the mixed strategy equilibrium of the war stage game and argue that it is unique.

   Answer: The war-game in the text has a weakly dominated Nash equilibrium (D,d) and hence does not have an equilibrium in which any player is mixing. This exercise should have replaced the war-stage game with the following game:



The subgame we called the war-stage game is given in the following matrix:

Player 1 \ Player 2	r	d
R	$-5,-5$	$-120,-80$
D	$-80,-120$	$-100,-100$

Let player 1 choose R with probability p and player 2 choose r with probability q. For player 2 to be indifferent it must be that $\frac{4}{19}(-5)+(1-\frac{4}{19})(-120)=-\frac{1820}{19}$

$$p(-5)+(1-p)(-120)=p(-80)+(1-p)(-100)$$

and the solution is $p=\frac{4}{19}$. By symmetry, for player 1 to be indifferent it must be that $q=\frac{4}{19}$. Hence, $(p,q)=(\frac{4}{19},\frac{4}{19})$ is the unique mixed strategy Nash equilibrium of this subgame with expected payoffs of $(v_1,v_2)=(-95.78,-95.79)$. ■

(b) What is the unique subgame perfect equilibrium that includes the mixed strategy you found above?

Answer: Working backward, player 2 would prefer to choose B over N and player 1 would prefer E over I.

3. **Brothers (revisited):** Find all the subgame prefect equilibria in the "brothers" exercise (exercise 7.8) from the previous chapter.

Answer: In part (d) of exercise 7.8 we found all the Nash equilibria as follows: For player 2, $S_2 = \{OO, OF, FO, FF\}$ where $s_2 = xy$ means that player 2 chooses $x \in \{O, F\}$ after player 1 chose S while player 2 chooses $y \in \{O, F\}$ after player 1 chose G. For player 1,
$S_1 = \{SOO, SOF, SFO, SFF, GOO, GOF, GFO, GFF\}$ where $s_1 = xyz$ means that player 1 first chooses $x \in \{S, G\}$ and then chooses $y \in \{O, F\}$ if he played S and $z \in \{O, F\}$ if he played G. We first noted that for player 1, mixing equally between SOO and SFO will strictly dominate the four strategies GOO, GOF, GFO and GFF. Hence, we can consider the reduced 4×4 game,

		Player 2			
		OO	OF	FO	FF
Player 1	SOO	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SOF	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SFO	$10, 10$	$10, 10$	$22, 26$	$22, 26$
	SFF	$10, 10$	$10, 10$	$22, 26$	$22, 26$

The simple overline-underline method shows that we have eight pure strategy Nash equilibria, four yielding the payoffs $(26, 22)$ and the other four yielding $(22, 26)$.

Now we know that any subgame perfect equilibrium must be a Nash equilibrium, so we can consider the set of Nash equilibria and see which survives backward induction. Because the second stage of the game has players 1 and 2 move simultaneously, the only restriction of subgame perfection is that in each of the simultaneous move games, the players are playing a Nash equilibrium. This implies that the pairs of strategies (SOO, OF) and (SOF, OO)

are not subgame perfect, which is also true for (SFO, FF) and (SFF, FO). Hence, of the eight pure strategy Nash equilibria only four are subgame perfect: (SOO, OO),(SOF, OF),(SFO, FO) and (SFF, FF).

Because of each players indifference between the ways in which the payoffs are reached, there are infinitely many mixed strategies that yield the same payoffs. For example, any profile where player 1 mixes between SOO and SOF and where player 2 mixes between OO and OF will be a Nash equilibrium that yields $(26, 22)$. Similarly, any profile where player 1 mixes between SFO and SFF and where player 2 mixes between FO and FF will be a Nash equilibrium that yields $(22, 26)$. But most of these will not be subgame perfect because in the subgame following G, which is off the equilibrium path, the players are not playing a best response. There is, however, one more class of mixed strategy Nash equilibria that are similar to the one found in section 6.2.3. To see this, focus on an even simpler game where we eliminate the duplicate payoffs as follows,

Player 1		Player 2	
		OO	FF
	SOO	$26, 22$	$10, 10$
	SFF	$10, 10$	$22, 26$

which preserve the nature of the game. For player 1 to be indifferent between SOO and SFF it must be that player 2 chooses OO with probability q such that

$$26q + 10(1 - q) = 10q + 22(1 - q)$$

which yields $q = \frac{3}{7}$. Similarly, for player 2 to be indifferent between OO and FF it must be that player 1 chooses SOO with probability p such that

$$22p + 10(1 - p) = 10p + 26(1 - p)$$

which yields $p = \frac{4}{7}$. Hence, we found a mixed strategy Nash equilibrium that results in each player getting an expected payoff of $26 \times \frac{3}{7} + 10 \times \frac{4}{7} = 16\frac{6}{7}$. Notice, however, that player 1 is always indifferent between SOO and SOF, as well as between SFO and SFF so there are infinitely many ways to achieve this kind of mixed strategy, and similarly for player 2 because of his indifference between OO and OF as well as FO and FF. The mixed strategy subgame perfect equilibria will be those for which in each subgame the players are playing a Nash equilibrium, and hence there will be only 6 such pairs: where they mix after S and play one of three Nash equilibria in the subgame after G, and similarly where they mix after G and play one of the three Nash equilibria in the subgame after S. In all of these player 1's backward induction choice is the play S. ■

4. **The Industry Leader:** Three oligopolists operate in a market with inverse demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$, and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, c, and no fixed cost. The firms choose their quantities dynamically as follows: (1) Firm 1, who is the industry leader, chooses $q_1 \geq 0$; (2) Firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3 respectively.

   (a) How many proper subgames does this dynamic game have? Explain Briefly.

   Answer: There are infinitely many proper subgames because every quantity choice of payer 1 results in a proper subgame. ■

   (b) Is it a game of perfect or imperfect information? Explain Briefly.

   Answer: This is a game of imperfect information because players 2 and 3 make their choice without observing each other's choice first. ■

   (c) What is the subgame perfect equilibrium of this game? Show that it is unique.

   Answer: first we solve for the Nash equilibrium of the simultaneous move stage in which players 2 and 3 make their choices as a function of the choice made first by player 1. Given a choice of q_1 and a belief about q_3, player 2 maximizes

$$\max_{q_2} \ (a - (q_1 + q_2 + q_3) - c)q_2$$

which leads to the first order condition

$$a - q_1 - q_3 - c - 2q_2 = 0$$

yielding the best response function

$$q_2 = \frac{a - q_1 - q_3 - c}{2} \ ,$$

and symmetrically, the best response function of player 3 is

$$q_3 = \frac{a - q_1 - q_2 - c}{2}.$$

Hence, following any choice of q_1 by player 1, the unique Nash equilibrium in the resulting subgame is the solution to the two best response functions, which yields

$$q_2^*(q_1) = q_3^*(q_1) = \frac{a - c - q_1}{3} \ .$$

Moving back to player 1's decision node, he will choose q_1 knowing that q_2 and q_3 be be chosen using the best response function above, and hence player 1 maximizes,

$$\max_{q_1} \ (a - (q_1 + \frac{a - c - q_1}{3} + \frac{a - c - q_1}{3}) - c)q_1$$

which leads to the first order equation

$$\frac{1}{3}(a - c - 2q_1) = 0$$

resulting in a unique solution $q_1 = \frac{a-c}{2}$. Hence, the unique subgame perfect equilibrium dictates that $q_1^* = \frac{a-c}{2}$, and $q_2^*(q_1) = q_3^*(q_1) = \frac{a-c-q_1}{3}$. ■

(d) Find a Nash equilibrium that is not a subgame perfect equilibrium.

Answer: There are infinitely many Nash equilibria of the form "if player 1 plays q_1' then players 2 and 3 play $q_2^*(q_1') = q_3^*(q_1') = \frac{a-c-q_1'}{3}$, and otherwise they play $q_2 = q_3 = a$." In any such Nash equilibrium, players

2 and 3 are playing a Nash equilibrium on the equilibrium path (following q_1') while they are flooding the market and casing the price to be zero off the equilibrium path. One example would be $q_1' = 0$. In this case, following $q_1' = 0$ the remaining two players play the duopoly Nash equilibrium, and player 1 gets zero profits. If player 1 were to choose any positive quantity, his belief is that players 2 and 3 will flood the market and he will earn $-cq_1 < 0$, so he would prefer to choose $q_1' = 0$ given those beliefs. Of course, the threats of players 2 and 3 are not sequentially rational, which is the reason that this Nash equilibrium is not a subgame perfect equilibrium. ■

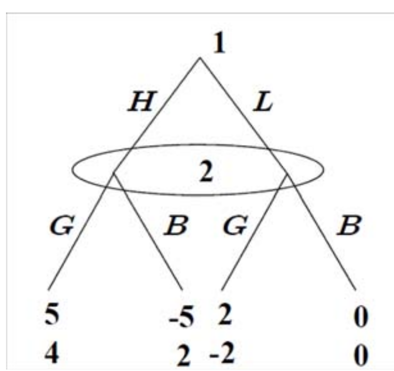
5. **Technology Adoption:** During the adoption of a new technology a CEO (player 1) can design a new task for a division manager. The new task can either be a high level (H) or low level (L). The manager simultaneously chooses to invest in good training (G) or bad training (B). The payoffs from this interaction is given by the following matrix:

Player 2

player 1	G	B
H	$5, 4$	$-5, 2$
L	$2, -2$	$0, 0$

(a) Present the game in extensive form (a game tree) and solve for all the Nash Equilibria and subgame perfect equilibria.

Answer:



It is easy to see in the matrix above that both (H, G) and (L, B) are pure strategy Nash equilibria. To find the mixed strategy Nash equilibrium, let player 1 choose H with probability p and player 2 choose G with probability q. For player 2 to be indifferent it must be that

$$p(4) + (1-p)(-2) = p(2) + (1-p)(0)$$

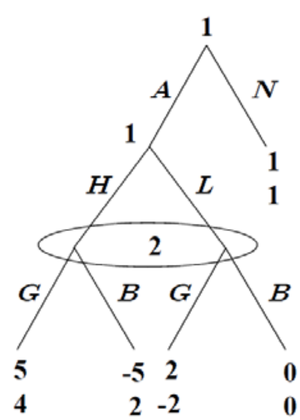
and the solution is $p = \frac{1}{2}$. Similarly, for player 1 to be indifferent it must be that

$$q(5) + (1-q)(-5) = q(2) + (1-q)(0)$$

and the solution is $q = \frac{5}{8}$. Hence, $(p, q) = (\frac{1}{2}, \frac{5}{8})$ is the unique mixed strategy Nash equilibrium of this game with expected payoffs of $(v_1, v_2) = (\frac{5}{4}, 1)$. ■

(b) Now assume that before the game is played the CEO can choose not to adopt this new technology, in which case the payoffs are $(1, 1)$, or to adopt it and then the game above is played. Present the *entire* game in extensive form. How many proper subgames does it have?

Answer:



The game has two proposer subgames. The first is the whole game and the second starts at player 1's second information set. ■

(c) Solve for all the Nash Equilibria and subgame perfect equilibria of the game described in (b) above.

Answer: The game can be represented by the following matrix,

		Player 2	
		G	B
Player 1	AH	$5,4$	$-5,2$
	AL	$2,-2$	$0,0$
	NH	$1,1$	$1,1$
	NL	$1,1$	$1,1$

and it is easy to see that there are three pure strategy Nash equilibria: (AH, G), (NH, B) and (NL, B). From part (a) above we know that player 1 choosing A followed by the mixed strategy Nash equilibrium $(p, q) = (\frac{1}{2}, \frac{5}{8})$ will be a (subgame perfect) Nash equilibrium with expected payoffs of $(v_1, v_2) = (\frac{5}{4}, 1)$. Also, there are infinitely many mixed strategy Nash equilibria in which player 1 is mixing between NH and NL in any arbitrary way and player 2 chooses B. Finally, there is another infinite set of mixed strategy equilibria in which player 1 mixes between AH, NH and NL, and player 2 mixes between G and B. To see this, ignore NL for the moment (as it yields the same payoffs as NH), and let player 1 choose AH with probability p and NH with probability $(1-p)$, and let player 2 choose G with probability q. For player 2 to be indifferent it must be that

$$p(4) + (1-p)(-2) = p(1) + (1-p)(1)$$

and the solution is $p = \frac{1}{2}$. Similarly, for player 1 to be indifferent it must be that

$$q(5) + (1-q)(-5) = q(1) + (1-q)(1)$$

and the solution is $q = \frac{3}{5}$. Hence, $(p, q) = (\frac{1}{2}, \frac{3}{5})$ is a mixed strategy Nash equilibrium of this game with expected payoffs of $(v_1, v_2) = (1, 1)$. Of course, because of the identity between the NH and NL, any pair of the following strategies will also be a mixed strategy Nash equilibrium: player 1 chooses AH with probability $\frac{1}{2}$, chooses NH and NL with probabilities that add up to the remaining $\frac{1}{2}$, and player 2 chooses G

with probability $\frac{3}{5}$.

Turning to subgame perfect equilibria, of the three Nash equilibria only two are subgame perfect: (AH, G) and (NL, B). In the subgame starting with player 1 choosing between H and L, part (a) above showed that the unique (non-degenerate) mixed strategy Nash equilibrium was $(p, q) = (\frac{1}{2}, \frac{5}{8})$ with expected payoffs of $(v_1, v_2) = (\frac{5}{4}, 1)$. If this will be played by the players after player 1 chooses A, then player 1's best reply at the root is indeed to choose A because $\frac{5}{4} > 1$. Hence, as suggested earlier, player 1 choosing A followed by the mixed strategy Nash equilibrium $(p, q) = (\frac{1}{2}, \frac{5}{8})$ is also a subgame perfect Nash equilibrium. ■

6. **Investment in the Future:** Consider two firms that play a Cournot competition game with demand $p = 100 - q$, and costs for each firm given by $c_i(q_i) = 10q_i$. Imagine that before the two firms play the Cournot game, firm 1 can invest in cost reduction. If it invests, the costs of firm 1 will drop to $c_1(q_1) = 5q_1$. The cost of investment is $F > 0$. Firm 2 does not have this investment opportunity.

   (a) Find the value F^* for which the unique subgame perfect equilibrium involves firm 1 investing.

   Answer: If firm 1 does not invest then they are expected to play the Cournot Nash equilibrium where both firms have costs of $10q_i$. Each firm solves,

$$\max_{q_i} \ (100 - (q_i + q_j) - 10)q_i$$

   which leads to the first order condition

$$90 - q_j - 2q_i = 0$$

   yielding the best response function

$$q_i(q_j) = \frac{90 - q_j}{2} \ ,$$

   and the unique Cournot Nash equilibrium is $q_1 = q_2 = 30$ with profits $v_1 = v_2 = 900$. If firm 1 does invest then for firm 1 the problem becomes

$$\max_{q_1} \ (100 - (q_1 + q_2) - 5)q_1$$

which leads to the best response function

$$q_1(q_2) = \frac{95 - q_2}{2} \ .$$

For firm 2 the best response function remains the same as solved earlier with costs $10q_2$, so the unique Cournot Nash equilibrium is now solved using both equations,

$$q_1 = \frac{95 - \frac{90-q_1}{2}}{2} \ ,$$

which yields $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and profits are $v_1 = 1,111\frac{1}{9}$ while $v_2 = 802\frac{7}{9}$. Hence, the increase in profits from the equilibrium with investment for player 1 are $F^* = 1,111\frac{1}{9} - 900 = 211\frac{1}{9}$, which is the most that player 1 would be willing to pay for the investment anticipating that they will play the Cournot Nash equilibrium after any choice of player 1 regarding investment. If $F < F^*$ then the unique subgame perfect equilibrium is that first, player 1 invests, then they players choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and if player 1 did not invest the payers choose $q_1 = q_2 = 30$. (Note that if $F > F^*$ then the unique subgame perfect equilibrium is that first, player 1 does not invest, then they players choose $q_1 = q_2 = 30$, and if player 1 did invest the payers choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$.) ■

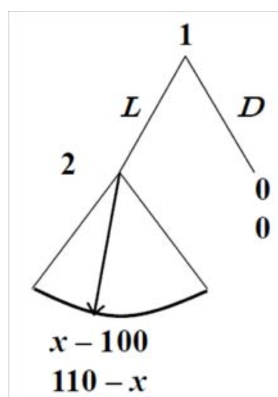
(b) Assume that $F > F^*$. Find a Nash equilibrium of the game that is not subgame perfect.

Answer: We construct a Nash equilibrium in which player 1 will invest despite $F > F^*$. Player 2's strategy will be, play $q_2 = \frac{85}{3}$ if player 1 invests, and $q_2 = 100$ if he does not invest. With this belief, if player 1 does not invest then he expects the price to be 0, and his best response is $q_1 = 0$ leading to profits $v_1 = 0$. If he invests then his best response to $q_2 = \frac{85}{3}$ is $q_1 = \frac{100}{3}$, which together are a Nash equilibrium in the Cournot game after investment. For any $F < 1,111\frac{1}{9}$ this will lead to positive profits, and hence, for $F^* < F < 1,111\frac{1}{9}$ the strategy of player 2 described above, together with player 1 choosing to invest, play $q_1 = \frac{100}{3}$ if he invests and $q_1 = 0$ if he does not is a Nash equilibrium. It is not subgame perfect because in the subgame following no investment, the players are not playing a Nash equilibrium. ■

7. **Debt and Repayment:** A project costing \$100 yields a gross return of \$110. A lender (player 1) is approached by a debtor (player 2) requesting a standard loan contract to complete the project. If the lender chooses *not to offer* a loan, then both parties earn nothing. If the lender chooses to *offer* a loan of \$100, the debtor can realize the projects gains, and is obliged by contract to repay \$105. For simplicity, assume that money is continuous, and that the debtor can choose to return any amount of money $x \leq 110$. Also, ignore the time value of money. Assume first that no legal system is in place that can cause the lender to repay, so that default on the loan (less than full repayment) carries no repercussions for the debtor.

   (a) Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

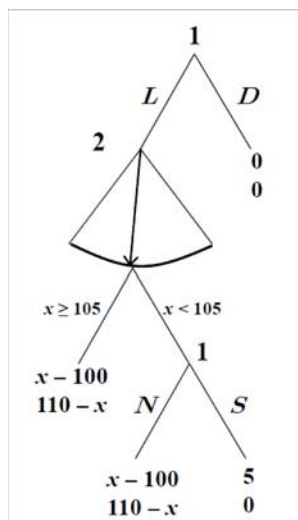
   Answer: Player 1 has two choices first, lend (L) or don't lend (D). After D both players get zero, while after L player 2 chooses a value $x \in [0, 110]$ to repay. The game can be described as follows:



   There is a unique subgame perfect equilibrium. If player 2 is offered the loan then he suffers no penalty from repaying, and his best response is to choose $x = 0$. Anticipating this behavior player 1 should choose D. ■

(b) Now assume that there is a legal system in place that allows the lender to *voluntarily* choose whether to sue or not to sue when the debtor defaults and repays an amount $x < 105$. Furthermore, assume that it is *costless* to use the legal system (it is supplied by the state), and if the lender sues a debtor that defaulted, the lender will get the \$105 repaid in full. After paying the lender, the borrower will pay a fine of \$5 to the court above and beyond the repayment. Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

Answer: The game now distinguished between two conditions: $x \geq 105$ in which case it is like the game in part (a) above, and $x < 105$ in which case player 1 has a new decision node where he can choose to sue (S) or not sue (N).



Starting at the last decision node of player 1, because it is relevant only when $x < 105$, it follows that $x - 100 < 5$ implying that S dominates N. Anticipating this, player 2's best response in the repayment phase is to choose $x = 105$. This is the lowest payment that does not trigger a suit. At the root of the tree player 1 anticipates a payoff of $105 - 100 = 5 > 0$ and hence prefers to choose L. The resulting outcome yields the payoffs

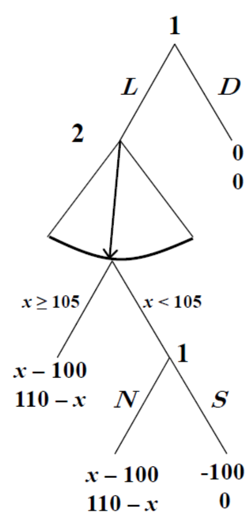
$(5,5)$. This backward induction argument shows that this is the unique subgame perfect equilibrium. ■

(c) Are there Nash equilibria in the game described in (b) above that are not subgame perfect equilibria?

Answer: For player 1, choosing D followed by S is a dominant strategy because it guaranties him a payoff of at least 5 (exactly 5 when $x < 105$ and $x - 100$ when $x \geq 105$.) Given this strategy, player 2's best reply is to choose $x = 105$. Hence, the only Nash equilibrium is also the subgame perfect Nash equilibrium. ■

(d) Now assume that using the legal system is *costly:* if the lender sues, he pays lawyers a *legal fee* of \$105 (this is the lawyers price which is unrelated to the contract above). The rest proceeds the same as before (if the lender sues a debtor that defaulted, the lender will get repaid in full; after paying the lender, the borrower will pay a fine of \$5 above and beyond the repayment.) Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

Answer: The game is now,



Because $x - 100 \geq -100$ it follows that that S is weakly dominated by

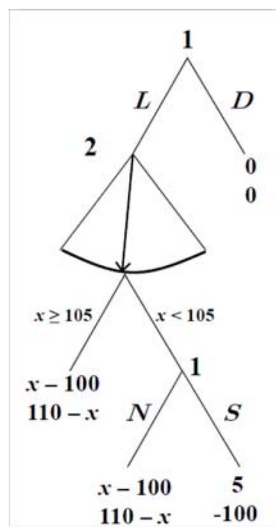
N. Anticipating this, player 2's best response in the repayment phase is to choose $x = 0$. At the root of the tree player 1 anticipates a payoff of $-100 < 0$ and hence prefers to choose D, and the outcome results in payoffs (0,0). This backward induction argument shows that this is the unique subgame perfect equilibrium. ■

(e) Are there Nash equilibria in the game described in (d) above that are not subgame perfect equilibria?

Answer: There are infinitely many. Any choice by player 2 of $x \leq 100$, for which playing D is a best response, will be a Nash equilibrium in which player 2 is not playing a best response. ■

(f) Now assume that a law change is proposed: upon default, if a debtor is sued he has to first repay the lender \$105, and then pay the legal fees of \$105 above and beyond repayment of the loan, and no extra fine is imposed. Should the lender be willing to pay for this law change? If so, how much?

Answer: The game is now as follows:



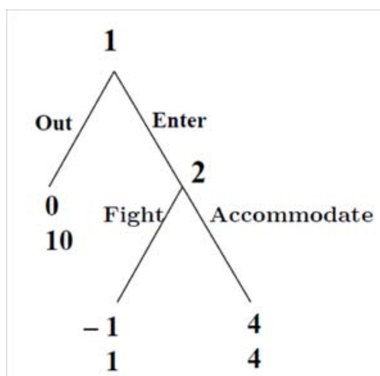
The backward induction argument follows the same logic as in part (b) resulting in the outcome $(5, 5)$. This yields player 1 an extra payoffs of 5 relative to the solution in part (d), implying that he should be willing to pay up to 5 in order to have the law implemented. ■

(g) If you were the "social planner", would you implemented the suggested law?

Answer: Yes because it results in a Pareto superior outcome of $(5,5)$ instead of $(0,0)$. ■

8. **Entry Deterrence 1:** NSG is considering entry into the local phone market in the Bay Area. The incumbent S&P, predicts that a price war will result if NSG enters. If NSG stays out, S&P earns monopoly profits valued at $10 million (net present value, or NPV of profits), while NSG earns zero. If NSG enters, it must incur irreversible entry costs of $2 million. If there is a price war, each firm earns $1 million (NPV). S&P always has the option of accommodating entry (i.e., not starting a price war). In such a case, both firms earn $4 million (NPV). Suppose that the timing is such that NSG first has to choose whether or not to enter the market. Then S&P decides whether to "accommodate entry" or "engage in a price war." What is the subgame perfect equilibrium outcome to this sequential game? (Set up a game tree.)

Answer: Letting NSG be player 1 and S&P be player 2,



Backward induction implies that player 2 will Accommodate, and player 1 will therefore enter. Hence, the unique subgame perfect equilibrium is (Enter,Accommodate). ■

9. **Entry Deterrence 2:** Consider the Cournot duopoly game with demand $p = 100 - (q_1 + q_2)$, and variable costs $c_i(q_i) = 0$ for $i \in \{1, 2\}$. The twist is

that there is now a fixed cost of production $k > 0$ that is the same for both firms.

(a) Assume first that both firms choose their quantities simultaneously. Model this as a normal form game.

Answer: This is a standard Cournot game with two players: $N = \{1, 2\}$, $S_i = \mathbb{R}_+$ (the non-negative real line) and we need to add the fixed costs to the payoff function, $v_i(q_1, q_2) = (100 - q_1 - q_2)q_i - k$ for $i \in \{1, 2\}$.

(b) Write down the firm's best response function for $k = 1000$ and solve for pure strategy Nash equilibrium. Is it unique?

Answer: Because the fixed costs do not affect the first order conditions, from section 5.2.3 we know that the two best response functions ignoring the fixed costs are,

$$q_i(q_j) = \frac{100 - q_j}{2} .$$

With fixed costs, however, each firm will produce only if it has positive profits. For example, using firm 1's best response function, its profits conditional on playing a best response are

$$\begin{aligned} v_1(q_1(q_2), q_2) &= (100 - (\frac{100 - q_2}{2} + q_2))\frac{100 - q_2}{2} - k \\ &= 2500 + \frac{q_2^2}{4} - 50q_2 - k \\ &= 1500 + \frac{q_2^2}{4} - 50q_2 . \end{aligned}$$

where the last inequality follows from $k = 1000$. Now we can compute the value of q_2 for which playing a best response by firm 1 will yield zero profits, which in turn will imply that for higher levels of q_2 firm 1 will incur a loss even when it plays a best response conditional on producing. We have,

$$1500 + \frac{q_2^2}{4} - 50q_2 \geq 0$$

which holds when $q_2 \leq 100 - 20\sqrt{10} \approx 36.75$. A symmetric argument will hold for firm 2, which yields the best response function with a fixed cost of $k = 1000$ to be,

$$q_i(q_j) = \begin{cases} \frac{100-q_j}{2} & \text{if } q_j \leq 100 - 20\sqrt{10} \\ 0 & \text{if } q_j > 100 - 20\sqrt{10} \end{cases} .$$

Using the first portion of the best response function to try and solve for a Nash equilibrium, we obtain that $q_1 = q_2 = 33\frac{1}{3} < 36.75$. Thus, when $k = 1000$, $q_1 = q_2 = 33\frac{1}{3}$ is the unique Nash equilibrium of this game. ■

(c) Now assume that firm 1 is a "Stackelberg leader" in the sense that it moves first and chooses q_1, and then after observing q_1 firm 2 chooses q_2. Also assume that if firm 2 cannot make strictly positive profits then it will not produce at all. Model this as an extensive form game tree as best as you can, and find a subgame perfect equilibrium of this game for $k = 25$. Is it unique?

Answer: similar to the analysis in section 8.3.2 we know that, ignoring fixed costs, firm 2 will choose $q_2(q_1) = \frac{100-q_1}{2}$ as derived above. With $k = 25$ it will not produce for some values of q_1 close to 100. (Similar to the analysis in part (b), q_1 must satisfy $2500 + \frac{q_1^2}{4} - 50q_1 - k > 0$ with $k = 25$. This will be satisfied when $q_1 \leq 90$.) Given firm 2's best response, firm 1 maximizes

$$\max_{q_1} \ (100 - q_1 - \frac{100 - q_1}{2})q_1 - 25 ,$$

which yields the first order condition $50 - q_1 = 0$ or $q_1^* = 50$. Because $q_1^* < 90$ we know that firm 2 will indeed follow $q_2(q_1) = \frac{100-q_1}{2} = 25$, profits for firm 1 are $v_1 = 25 \times 50 - 25 = 1,225$, and for firm 2 are $v_2 = 25 \times 25 - 25 = 600$. By construction, this is the unique subgame perfect equilibrium. ■

(d) How does your answer in (c) change for $k = 725$?

Answer: Now firm 2 will follow $q_2(q_1) = \frac{100-q_1}{2}$ as long as $1775 + \frac{q_1^2}{4} -$

$50q_1 \geq 0$, which holds for $q_1 \leq 100 - 10\sqrt{29} \approx 46.15$. As we saw in part (c), if firm 1 anticipates firm 2 to produce according to $q_2(q_1) = \frac{100-q_1}{2}$ then firm 1 produces $q_1^* = 50$. It turns out that if firm 1 anticipates firm 2 to stay out then it will also produce $q_1^* = 50$ which is the monopolists optimal choice for this market with only fixed costs. However, since $50 > 46.15$ this choice will indeed cause firm 2 to stay out, and the unique subgame perfect equilibrium is now $q_1^* = 50$ and

$$q_2(q_1) = \begin{cases} \frac{100-q_1}{2} & \text{if } q_1 \leq 100 - 10\sqrt{29} \\ 0 & \text{if } q_1 > 100 - 10\sqrt{29} \end{cases},$$

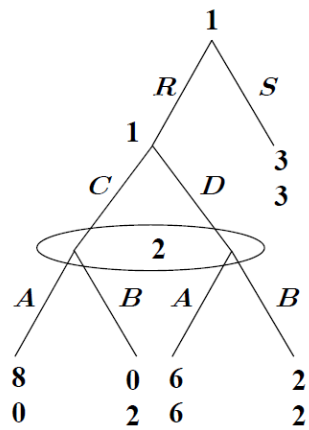
resulting in $q_2^* = 0$. ■

10. **Playing it safe:** Consider the following dynamic game: Player 1 can choose to play it safe (denote this choice by S), in which case both he and player 2 get a **payoff of 3 each**, or he can risk playing a game with player 2 (denote this choice by R). If he chooses R, then they play the following **simultaneous move** game:

		Player 2	
		A	B
player 1	C	$8,0$	$0,2$
	D	$6,6$	$2,2$

(a) Draw a game tree that represents this game. How many proper subgames does it have?

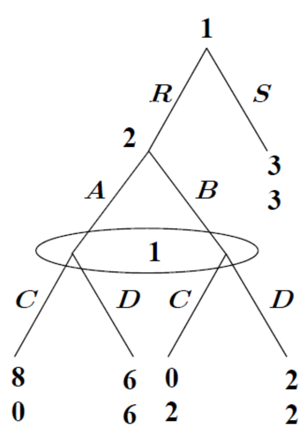
Answer:



The game has two proper subgames: the whole game and the subgame starting at the node where 1 chooses between C and D. ■

(b) Are there other game trees that would work? Explain briefly.

Answer: Yes - it is possible to have player 2 move after 1's initial move, and then have player 1 with an information set as follows:



(c) Construct the matrix representation of the normal form of this dynamic game.

Answer: The game can be represented by the following matrix,

Player 1 \ Player 2	A	B
RC	$8,0$	$0,2$
RD	$6,6$	$2,2$
SC	$3,3$	$3,3$
SD	$3,3$	$3,3$

■

(d) Find all the Nash and subgame perfect equilibria of the dynamic game.

Answer: It is easy to see that there are two pure strategy Nash equilibria: (SC, B) and (SD, B). It follows immediately that there are infinitely many mixed strategy Nash equilibria in which player 1 is mixing between SC and SD in any arbitrary way and player 2 chooses B. It is also easy to see that following a choice of R, there is no pure strategy Nash equilibrium in the resulting subgame. To find the mixed strategy Nash equilibrium in that subgame, let player 1 choose RC with probability p and RD with probability $(1-p)$, and let player 2 choose A with probability q. For player 2 to be indifferent it must be that

$$p(0) + (1-p)(6) = p(2) + (1-p)(2)$$

and the solution is $p = \frac{2}{3}$. Similarly, for player 1 to be indifferent it must be that

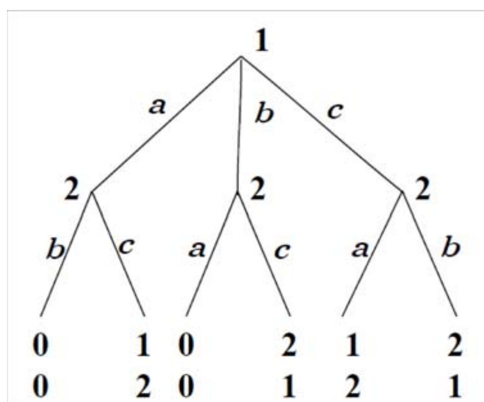
$$q(8) + (1-q)(0) = q(6) + (1-q)(2)$$

and the solution is $q = \frac{1}{2}$. Hence, $(p, q) = (\frac{1}{2}, \frac{3}{5})$ is a mixed strategy Nash equilibrium of the subgame after player 1 chooses R, yielding expected payoffs of $(v_1, v_2) = (4, 2)$. In any subgame perfect equilibrium the players will have to play this mixed strategy equilibrium following R, and because $4 > 3$ player 1 will prefer R over S. Hence, choosing R followed by the mixed strategy computed above is the unique subgame perfect equilibrium. ■

11. **RA Selection with a Twist**: Two staff managers in the ΠΒΦ sorority, the house manager (player 1) and kitchen manager (player 2), are supposed to select a resident assistant (RA) from a pool of three candidates: $\{a, b, c\}$. Player 1 prefers a to b, and b to c. Player 2 prefers b to a, and a to c. The process that is imposed on them is as follows: First, the house manager vetoes one of the candidates, and announces the veto to the central office for staff selection, and to the kitchen manager. Next, the kitchen manager vetoes one of the remaining two candidates and announces it to the central office. Finally, the director of the central office assigns the remaining candidate to be an RA at ΠΒΦ.

   (a) Model this as an extensive form game (using a game tree) where a player's most preferred candidate gives a payoff of 2, the second gives a payoff of 1, and the last gives 0.

   Answer: Since first player 1 effectively removes a candidate, each of the three choices (to veto) of player 1 are followed by two possible veto choices of player 2:



■

   (b) Find the subgame perfect equilibria of this game. Is it unique?

   Answer: If player 1 vetoes a or b then player 2 will veto c, and if player 1 vetoes c then player 2 will veto a. By backward induction, anticipating player 2's behavior player 1 will veto candidate b. This is the unique subgame perfect equilibrium by backward induction resulting in payoffs of $(2, 1)$. ■

(c) Are there Nash Equilibria that are not subgame perfect equilibria.

Answer: Yes. Player 2 can threaten to veto candidate a whenever player 1 vetoes either c or b, and veto candidate c when player 1 vetoes a. Player 1's best reply to this strategy is to veto either a or c. The players will both be playing best responses on the equilibrium path but player 2 is not playing a best response following the choice of player 1 to veto candidate b. Hence, this is a Nash equilibrium that is not subgame perfect.

(d) Now assume that before the two players play the game, player 2 can send an alienating E-mail to one of the candidates, which would result in that candidate withdrawing her application. Would player 2 choose to do this, and if so, with which candidate?

Answer: Player 2 would like to send the email to candidate a. That way, only candidates b and c will be in the pool and both players will veto c, resulting in the payoffs $(1, 2)$ which are better for player 2 than the unique subgame perfect equilibrium payoffs derived in part (a). ■

12. **Agenda Setting:** An agenda-setting game is described as follows. The "issue space" (set of possible policies) is an interval $X = [0, 5]$. An Agenda Setter (player 1) proposes an alternative $x \in X$ against the status quo $q = 4$. After player 1 proposes x, the Legislator (player 2) observes the proposal and selects between the proposal x and the status quo q. Player 1's most preferred policy is 1, and for any final policy $y \in X$, his payoff is given by

$$v_1(y) = 10 - |y - 1|,$$

where $|y - 1|$ denotes the absolute value of $(y - 1)$. Player 2's most preferred policy is 3, and for any final policy $y \in X$, her payoff is given by

$$v_2(y) = 10 - |y - 3|.$$

That is, each player prefers policies that are closer to their most preferred policy.

(a) Write the game down as a normal form game. Is this a game of perfect or imperfect information?

Answer: There are two players, $i \in \{1,2\}$ with strategy sets $S_1 = X = [0,5]$ and $S_2 = \{A, R\}$ where A denotes accepting the proposal $x \in X$ and R means rejecting it and adopting the status quo $q = 4$. The payoffs are given by

$$v_1(s_1, s_2) = \begin{cases} 10 - |s_1 - 1| & \text{if } s_2 = A \\ 7 & \text{if } s_2 = R \end{cases}$$

and

$$v_2(s_1, s_2) = \begin{cases} 10 - |s_1 - 3| & \text{if } s_2 = A \\ 9 & \text{if } s_2 = R \end{cases}$$

(b) Find a subgame perfect equilibrium of this game. Is it unique?

Answer: Player 2 can guarantee himself a payoff of 9 by choosing R, implying that his best response is to choose A if and only if $10-|s_1-3| \geq 9$, which will hold for any $s_1 \in [2,4]$. Player 1 would like to have an alternative adopted that is closest to 1, which implies that his best response to player 2's sequentially rational strategy is to choose $s_1 = 2$. This is the unique subgame perfect equilibrium which results in the payoffs of $(v_1, v_2) = (9,9)$. ■

(c) Find a Nash equilibrium that is not subgame perfect. Is it unique? If yes, explain. If not, show all the Nash equilibria of this game.

Answer: One Nash equilibrium is where player 2 adopts the strategy "I will reject anything except $s_1 = 3$." If player 1 chooses $s_1 = 3$ then his payoff is 8, while any other choice of s_1 is expected to yield player 1 a payoff of 7. Hence, player 1s best response to player 2's proposed strategy is indeed to choose $s_1 = 3$ and the payoffs from this Nash equilibrium are $(v_1, v_2) = (8, 10)$. Since player 2 can guarantee himself a payoff of 9, there are infinitely many Nash equilibria that are not subgame perfect and that follow a similar logic: player 2 adopts the strategy "I will reject anything except $s_1 = x$" for some value $x \in (2, 4)$. Player 1 would strictly prefer the adoption of x over 4, and hence would indeed propose x, and player 2 would accept the proposal. For $x = 4$ both players are indifferent so it would also be supported as a Nash equilibrium. ■

13. **Junk Mail Advertising:** Suppose there is a single good that is owned by a single seller who values it at $c > 0$ (he can consume the good and get a payoff of c). There is a single buyer who has a small transportation cost $k > 0$ to get to and back from the seller's store, and he values the good at $v > c + k$. The buyer first decides whether to make the commute or stay at home, not buy the good and receive a payoff of 0. If the buyers commutes to the store, the seller can then make the buyer a Take-It-Or-Leave-It price offer $p \geq 0$. The buyer can then accept the offer, pay p and get the good, or he can walk out and not buy the good. Assume that c, v and k, are common knowledge.

   (a) As best as you can, draw the extensive form of this game. What is the best response of the buyer at the node where he decides whether to accept or reject the seller's offer?

   Answer: Let the buyer be player 1. Denote by G going to the store and H staying home, and by A accepting or R rejecting the seller's offer. The game can be described as follows:The best response of player 1 after the offer p is to choose A if and only if $v - p - k \geq -k$ or $p \leq v$. ■

   (b) Find the subgame perfect equilibrium of the game and show that it is unique. Is it Pareto Optimal?

   Answer: Given the buyer's best response at the accept/reject node, backward induction implies that the seller's unique best response is to offer $p = v$ because $v > c + k > c$. Anticipating this, the buyer knows that if he chooses G then his payoff will be $-k$, so his unique best

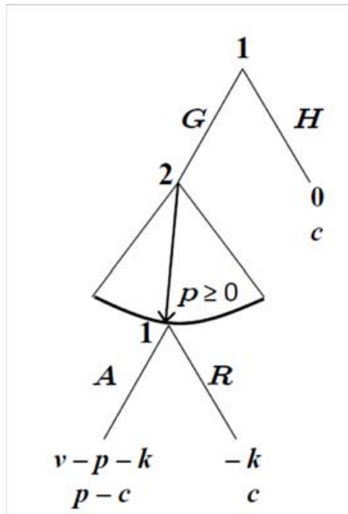


FIGURE 8.1.

response is to choose H and the payoffs from the unique subgame perfect equilibrium are $(v_1, v_2) = (0, c)$. This outcome is not Pareto optimal because if the two players trade at any price p such that $v - k > p > c$ then they would both be better off. ■

(c) Are there Nash equilibria that yield a higher payoff to both players as compared to the subgame perfect equilibrium you found in (b) above?

Answer: There are infinitely many such Nash equilibria. Fix some $p^* \in (c, v-k)$ and let player 1's strategy after the proposal stage be "I will accept any offer $p \leq p^*$ and reject anything else." Given this strategy, and given that $p^* > c$, player 2's best response is to offer p^* and player 1 will therefore wish to choose G because $p^* < v - k$. The resulting payoffs will be $(v_1, v_2) = (v - p^* - k, p^*) > (0, c)$. ■

(d) Now assume that before the game is played, the seller can, at a small cost $\varepsilon < (v - c - k)$, send the buyer a postcard that commits the seller to a certain price at which the buyer can buy the good (e.g., "bring this coupon and get the good at a price p"). Would the seller choose to do so? Justify your answer with an equilibrium analysis.

Answer: The seller would indeed benefit from sending such a postcard.

To see this, let $\delta = v - c - k - \varepsilon > 0$ and imagine that the seller sends he postcard with a price $p^* = v - k - \frac{\delta}{2}$. The buyer who receives this card knows that he can go to the store at a cost of k and pay p^* for the good which would leave the buyer with a payoff of $v_1 = v - k - p^* = \frac{\delta}{2} > 0$, and hence would prefer to go shopping. The seller would receive a payoff of $v_2 = p^* - \varepsilon = c + \frac{\delta}{2} > c$, which is better than no trade. This would work for any price $p^* = v - k - \alpha\delta$ for any $a \in (0, 1)$. ■

14. **Hyperbolic Discounting:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $0 < \delta < 1$ and $0 < \beta < 1$.

    (a) Solve the optimal choice of player 2, the second period self, as a function of his budget K_2, δ and β.

    Answer: Player 2's optimization problem is given by

    $$\max_{x_2} \; v_2(x_2, K - x_2) = \ln(x_2) + \beta\delta \ln(K - x_2),$$

    for which the first order condition is

    $$\frac{dv_2}{dx_2} = \frac{1}{x_2} - \frac{\beta\delta}{K_2 - x_2} = 0 \; ,$$

    which in turn implies that player 2's best response function is,

    $$x_2(K_2) = \frac{K_2}{\beta\delta + 1} \; ,$$

    which leaves $x_3 = K_2 - x_2(K_2) = \frac{\beta\delta K_2}{\beta\delta+1}$ for consumption in the third period. ■

    (b) Solve the optimal choice of player 1, the first period self, as a function of K, δ and β.

    Answer: Player 1 decides how much to allocate between his own consumption and that of player 2 taking into account that $x_2(K_2) = \frac{K_2}{\beta\delta+1}$, hence player 1 solves the following problem,

    $$\max_{x_1} \; v_1\left(x_1, \frac{K - x_1}{\beta\delta + 1}, \frac{\beta\delta(K - x_1)}{\beta\delta + 1}\right) = \ln(x_1) + \beta\delta \ln\left(\frac{K - x_1}{\beta\delta + 1}\right) + \beta\delta^2 \ln\left(\frac{\beta\delta(K - x_1)}{\beta\delta + 1}\right).$$

for which the first order condition is,

$$\frac{dv_1}{dx_1} = \frac{1}{x_1} - \frac{\beta\delta}{K - x_1} - \frac{\beta\delta^2}{K - x_1} = 0 \ ,$$

which in turn implies that player 1's best response function is,

$$x_1(K) = \frac{K}{\beta\delta + \beta\delta^2 + 1} \ .$$

■

15. **Time Inconsistency:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods, with initial budget K and with discount factors $\delta = 1$ and $\beta = \frac{1}{2}$.

    (a) Solve the optimal plan of action of a "naive" player 1 who does not take into account how his future self, player 2, will alter the plan. What is player 1's optimal plan x_1^*, x_2^* and x_3^* as a function of K?

    Answer: A naive player 1 will solve,

$$\begin{aligned} \max_{x_2,x_3} \ v(K - x_2 - x_3, x_2, x_3) &= \ln(K - x_2 - x_3) + \beta\delta\ln(x_2) + \beta\delta^2\ln(x_3) \\ &= \ln(K - x_2 - x_3) + \frac{1}{2}\ln(x_2) + \frac{1}{2}\ln(x_3). \end{aligned}$$

when $\beta = \frac{1}{2}$ and $\delta = 1$. The two fist order conditions are,

$$\frac{\partial v}{\partial x_2} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_2} = 0 \ ,$$

and,

$$\frac{\partial v}{\partial x_3} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_3} = 0 \ .$$

Solving these two equations yields the solution

$$x_2 = x_3 = \frac{K}{4},$$

and using $x_1 = K - x_2 - x_3$ gives,

$$x_1 = \frac{K}{2} \ .$$

■

(b) Let K_2 be the amount left from the solution to part (a) above after player 1 consumes his planned choice of x_1^*. Given K_2, what is the optimal plan of player 2? In what way does it differ from the optimal plan set out by player 1?

Answer: This was solved at the bottom of page 168 in the textbook. After player 1 leave K_2 for player 2, his optimization problem is given by

$$\max_{x_2} \; v_2(x_2, K_2 - x_2) = \ln(x_2) + \frac{1}{2}\ln(K - x_2),$$

for which the first order condition is

$$\frac{dv_2}{dx_2} = \frac{1}{x_2} - \frac{1}{2(K_2 - x_2)} = 0 \ ,$$

which in turn implies that player 2's best response function is,

$$x_2(K_2) = \frac{2K_2}{3} \ ,$$

which leaves $x_3 = K_2 - x_2(K_2) = \frac{K_2}{3}$ for consumption in the third period. From part (a) we know that $K_2 = \frac{K}{2}$ so player 2 will choose $x_2 = \frac{K}{3}$ and $x_3 = \frac{K}{6}$. This is in contrast to what player 1 planned which was $x_2 = x_3 = \frac{K}{4}$ so player 2 is overconsuming relative to what player 1 would have wanted. ■

16. **The Value of Commitment:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $\delta = 1$ and $\beta = \frac{1}{2}$. We solved the optimal consumption plan of a sophisticated player 1.

(a) Imagine that an external entity can enforce any plan of action that player 1 chooses in $t = 1$ and will prevent player 2 from modifying it. What is the plan that player 1 would choose to enforce?

Answer: Player 1 wants to maximize,

$$\begin{aligned}\max_{x_2, x_3} \; v(K - x_2 - x_3, x_2, x_3) &= \ln(K - x_2 - x_3) + \beta\delta\ln(x_2) + \beta\delta^2\ln(x_3) \\ &= \ln(K - x_2 - x_3) + \frac{1}{2}\ln(x_2) + \frac{1}{2}\ln(x_3).\end{aligned}$$

when $\beta = \frac{1}{2}$ and $\delta = 1$. The two fist order conditions are,

$$\frac{\partial v}{\partial x_2} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_2} = 0 \ ,$$

and,

$$\frac{\partial v}{\partial x_3} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_3} = 0 \ .$$

Solving these two equations yields the solution

$$x_2 = x_3 = \frac{K}{4},$$

and using $x_1 = K - x_2 - x_3$ gives,

$$x_1 = \frac{K}{2} \ .$$

Thus, player 1 would choose to enforce $x_1 = \frac{K}{2}$ and $x_2 = x_3 = \frac{K}{4}$. ■

(b) Assume that $K = 90$. Up to how much of his initial budget K will player 1 be willing to pay the external entity in order to enforce the plan you found in part (a)?

Answer: If the external entity does not enforce the plan, then from the analysis on pages 168-169 we know that player 2 will choose $x_2 = \frac{K}{3} = 30$ and $x_3 = \frac{K}{6} = 15$, and player 1 will choose $x_1 = \frac{K}{2} = 45$. The discounted value of the stream of payoffs for player 1 from this outcome is therefore,

$$\ln(45) + \frac{1}{2}\ln(30) + \frac{1}{2}\ln(15) \approx 6.86 \ .$$

If, however, player 1 can have the plan in part (a) above enforced then his discounted value of the stream of payoffs is

$$\ln(45) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) \approx 6.92 \ .$$

We can therefore solve for the amount m of budget $K = 90$ that player 1 would be willing to give up which is found by the following equality,

$$\ln(45 - m) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) = 6.86 \ ,$$

which yields $m \approx 2.63$. Hence, player 1 will be willing to give up to 2.63 of his initial budget $K = 90$ in order to enforce the plan $x_2 = x_3 = \frac{K}{4} = 22.5$. ■

9

Multi-Stage Games

1. Consider the following simultaneous move game that is played twice (the players observe the first period outcome prior to the second period play):

		Player 2		
		L	C	R
	T	10,10	2,12	0,13
player 1	M	12,2	5,5	0,0
	B	13,0	0,0	1,1

   (a) Find all the pure strategy subgame perfect equilibria with no discounting ($\delta = 1$). Be precise in defining history contingent strategies for both players.

   Answer: The simultaneous move game has two pure strategy Nash equilibria: (M, C) and (B, R), which implies that one of these has to be played in the second stage of the game. We know that any unconditional play of these Nash equilibria in each stage is a subgame perfect equilibrium of the multistage game implying four pure strategy Nash equilibria (e.g., player 1 plays M followed by M regardless of what player 2 chose and player 2 plays C followed by C regardless of what player 1 did.)

We now construct other equilibria that are history contingent in which the players will play the "reward" (M, C) in the second period if they followed the first period proposed strategies giving each a payoff of 5, while they will play the "punishment" (B, R) if one of the players deviated from the proposed strategy and both will receive a payoff of 1. Note that the loss from not following the first stage proposed strategies will be $5 - 1 = 4$ in the second period, and because $\delta = 1$ then 4 is also the discounted loss. It is therefore possible to support *any* payoff in the first stage for which the best deviation is no greater then 4 with $\delta = 1$ because the discounted loss from the second stage "punishment" would be greater than the first period gain. The only pair of payoffs from which there is a greater gain than 4 is from $(0, 0)$ because one of the players can deviate to $(5, 5)$. Hence, pick any pure strategy pair (x, y) that is not (M, R) or (B, C). The following is a subgame perfect equilibrium: player 1 plays x in the first stage followed by M if (x, y) was followed and B if it was not. Similarly, player 2 plays y in the first stage followed by C if (x, y) was followed and R if it was not. For $\delta = 1$ this is a subgame perfect equilibrium. ■

(b) For each of the equilibria you found above, find the smallest discount factor that supports it.

Answer: The four subgame perfect equilibria that are just an unconditional sequence of one-stage Nash equilibria are equilibria for any discount factor. The others, however, must guarantee that the discounted loss from punishment is greater than the first period gain for the player who has the most to benefit from the deviation. For the first stage outcomes $(x, y) \in \{(T, L), (T, C), (M, L\}$, the player who gains most can gain 3, and hence the discount factor must satisfy the inequality $3 - \delta 4 \leq 0$ or $\delta \geq \frac{3}{4}$ for these outcomes to be played in the first stage of the subgame perfect equilibrium. For the other two possibilities, $(x, y) \in \{(B, L), (T, R)\}$ the player who gains most can gain only 1, and hence the discount factor must satisfy the inequality $1 - \delta 4 \leq 0$ or

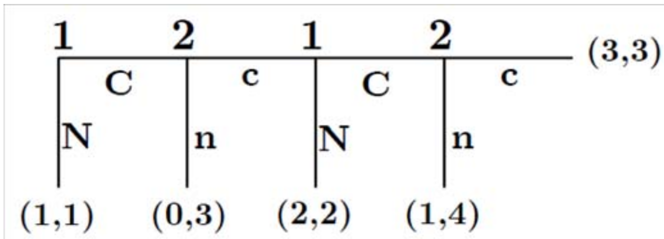


FIGURE 9.1. The Centipede Game

$\delta \geq \frac{1}{4}$ for these outcomes to be played in the first stage of the subgame perfect equilibrium. ■

2. **Centipedes revisited:** Two players are playing two consecutive games. First, they play the centipede game described in Figure 9.1. After the centipede game they play the following coordination game:

Player 1 \ Player 2	a	b
A	$1,1$	$0,0$
B	$0,0$	$3,3$

(a) What are the Nash equilibria of each stage game?

Answer: The first stage game has a unique Nash equilibrium outcome in which player 1 plays N in the first stage and payoffs are $(1,1)$. This can be supported in more than one Mash equilibrium (for example, player 1 plays N always and player 2 does as well, which is the subgame perfect equilibrium, or player 1 player N always and player 2 plays n first and c later – there are more.) The second stage game have three Nash equilibria. The two pure are (A,a) and (B,b) and the mixed one has player 1 (respectively 2) play A (respectively a) with probability $\frac{3}{4}$. ■

(b) How many pure strategies does each player have in the multistage game?

Answer: The players have four pure strategies in the first stage game (two information sets with two actions in each). The second stage strategies can be conditional on the outcomes of the first stage, of which there are 4. (We are defining an outcome is the payoffs of the first stage and not the strategies that players chose to obtain the payoffs. Unlike a matrix game, these will be different here because, as we saw in part **a.** above, there are different combinations of pure strategies that can lead to the same outcome.) Hence, there are $2^4 = 16$ pure strategies for each player in the second stage, which can follow each of the 4 first stage pure strategies, giving every player a total of 64 pure strategies. ■

(c) Find all the pure strategy subgame perfect equilibria with extreme discounting ($\delta = 0$). Be precise in defining history contingent strategies for both players.

Answer: In the second stage the players must play either (A, a) or (B, b) for any history. With extreme discounting we cannot support play in the first stage that is not a Nash equilibrium because there is no second stage "punishment" that can deter first stage deviations. Hence, in the first stage the players must play a subgame perfect equilibrium of the first stage game which is N always for player 1 and n always for player 2. Hence, there are two possible outcomes that can be supported by a subgame prefect equilibrium, $(1, 1)$ followed by $(1, 1)$ or by $(3, 3)$. However, there are $2^4 = 16$ pure strategy subgame perfect equilibria because for each of the 4 outcomes of the first stage the players must specify which of the 2 equilibria (A, a) or (B, b) will be played in the second stage. ■

(d) Now let $\delta = 1$. Find a subgame perfect equilibrium for the two-stage game in which the players receive the payoffs $(2, 2)$ in the first stage-game.

Answer: To get $(2, 2)$ in the first stage player 2 must overcome the temptation to choose n at his first move and get 3 instead. Hence, we can use the following conditional strategies in the second stage: player 1 (respectively 2) plays A (respectively a) if the outcome of the first stage was $(2,2)$ while they play B and b otherwise. In the first stage player 1 will play C followed by N and player 2 will play c followed by n. Player 1 has no reason to deviate in the first stage, and neither does player 2 because the gain of 1 from deviating in the first stage is less than the loss of 2 in the second stage. ■

(e) What is the lowest value of δ for which the subgame perfect equilibrium you found in (d) survives?

Answer: The pain from deviation will deter player 2 if and only if $1 - 2\delta \leq 0$ or $\delta \geq \frac{1}{2}$. ■

(f) For δ greater than the value you found in (e) above, are there other outcomes of the first stage centipede game that can be supported as part of a subgame perfect equilibrium?

Answer: Yes – the exact same idea can be used to support any of the other outcomes because the player who is tempted to deviate will gain 1 in the first period. ■

3. **Campaigning Adds:** Two political candidates are scheduled to campaign in two states, in one in period $t = 1$ and in the other in $t = 2$. In each state they can either choose a positive campaign that promotes their own agenda (P for player 1, p for player 2) or a negative one that attacks their opponent (N for player 1, n for player 2). Residents of the first period state do not mind negative campaigns, which are generally effective, and payoffs in this state are given by the following matrix:

		Player 2	
		p	n
Player 1	P	$2,2$	$0,5$
	N	$5,0$	$3,3$

In the second period state, residents dislike negative campaigns despite their effectiveness and the payoffs are given by the following matrix:

		Player 2	
		p	n
Player 1	P	$6,6$	$1,0$
	N	$0,1$	$2,2$

(a) What are the Nash equilibria of each stage game? Find all the pure strategy subgame perfect equilibria with extreme discounting ($\delta = 0$). Be precise in defining history contingent strategies for both players.

Answer: The first stage game has a unique dominant strategy Nash equilibrium (N, n) while the second stage game has two pure strategy equilibria, (P, p) and (N, n) and a mixed strategy equilibrium in which each player chooses the positive campaign with probability $\frac{1}{5}$. In the second stage the players must play either (A, a) or (B, b) for any history in a pure strategy subgame perfect equilibrium. With extreme discounting we cannot support play in the first stage that is not a Nash equilibrium because there is no second stage "punishment" that can deter first stage deviations. Hence, in the first stage the players must play (N, n). Hence, (N, n) followed by either (P, p) or (N, n) will be the only outcomes that can be supported as subgame perfect equilibria. However, there are $2^4 = 16$ pure strategy subgame perfect equilibria because for each of the 4 outcomes of the first stage the players must specify which of the 2 equilibria (P, p) or (N, n) will be played in the second stage. ■

(b) Now let $\delta = 1$. Find a subgame perfect equilibrium for the two-stage game in which the players choose (P, p) in the first stage-game.

Answer: We can use the conditional second stage strategies in which player 1 (respectively 2) plays P (respectively p) if the choice in the first stage was (P, p) while they play N and n otherwise. In the first stage neither player wants to deviate from (P, p) because the gain of switching actions is 3 (from 2 to 5) while the loss from the punishment in the second stage is 4 (and it is not discounted so it's value remains 4). ■

(c) What is the lowest value of δ for which the subgame perfect equilibrium you found in (b) survives?

Answer: The discounted punishment must be at least as high as the gain from deviation, so the inequality is $3 - \delta 4 \leq 0$, and the solution is $\delta \geq \frac{3}{4}$. ■

(d) Can you find an subgame perfect equilibrium of this game where the players play something other than (P, p) or (N, n) in the first stage?

Answer: The same logic as that for parts **b.** and **c.** follows to support the pairs of actions (P, n) and (N, p) in the first stage. In each of these profiles one player will gain 3 by deviating to his preferred choice, and the loss in the second stage with properly defined contingent strategies is 4, so if $\delta \geq \frac{3}{4}$ the punishment will suffice to support the desired first stage behavior. ■

4. **Online Gaming:** Consider a two-stage game between two firms that produce online games. In the first stage, they play a Cournot competition game (each chooses a quantity q_i) with demand function $p = 100 - q$, and zero marginal production costs ($c_i(q_i) = 0$ for $i = 1, 2$.) In the second stage, after observing the pair (q_1, q_2) and after profits have been distributed, the players play a simultaneous move "access" game where they can either keep their game platforms closed, or each can open it's platform to allow players on the other platform to play online with players on their own platform (O for player 1, o for player 2), or choose to keep their platforms non-compatible (N for player 1, n for player 2), in which case each platform's players can only play with others on their platform. If they choose (N, n) then second stage payoffs are $(0, 0)$. If only one firm chooses to open its platform, it bears a cost of (-10) with no benefit since the other firm did not allow to open access. Finally, if both firms choose (O, o) then each firm gets many more eyeballs

for advertising, and payoffs for each firm are $2,500$. Both players use the same discount factor δ to discount future payoffs.

(a) Find the unique Nash equilibrium in the first stage Cournot Game and all of the pure strategy Nash equilibria of the second stage access game. Find all the pure strategy subgame perfect equilibria with extreme discounting ($\delta = 0$). Be precise in defining history contingent strategies for both players.

Answer: The maximization problem in the Cournot game is

$$\max_{q_i} \; v_i(q_i, q_j) = (100 - q_i - q_j)q_i$$

and the first order condition is $100 - q_j - 2q_i = 0$ resulting in the best response function $q_i = \frac{100-q_j}{2}$, which in turn implies that the unique Nash equilibrium is $q_1 = q_2 = 33\frac{1}{3}$. The second stage game is given by the following matrix:

		Player 2	
		o	n
Player 1	O	$250, 250$	$-10, 0$
	N	$0, -10$	$0, 0$

and it is easy to see that both (O, o) and (N, n) are Nash equilibria. One of these two will have to be played in the second stage of the game in any subgame perfect equilibrium. When $\delta = 0$ the only first stage play that is possible in equilibrium is the unique Cournot equilibrium. Therefore, only two outcomes can result from a subgame perfect equilibrium: choose $q_1 = q_2 = 33\frac{1}{3}$ in the first stage and choose either (O, o) or (N, n) in the second stage.[1]

[1] There are *infinitely many* strategy profiles that will support this outcome and by a subgame perfect equilibrium. For example, have the players each choose $q_i = 33\frac{1}{3}$ in the first stage followed by the following contingent strategy for the second stage: if both q_1 and q_2 are below q^* then choose "open" (O and o) while if either q_1 or q_2 are above q^* then choose "not-open" (N and n). Notice that this is an equilibrium for any value of q^*. If $q^* < 33\frac{1}{3}$ then the second stage equilibrium will be (N, n) while if $q^* \geq 33\frac{1}{3}$ then it will be (O, o).

(b) Now let $\delta = 1$. Find a subgame perfect equilibrium for the two-stage game in which the players choose the monopoly (total profit maximizing) quantities and split them equally (a symmetric equilibrium).

Answer: Monopoly profits in the first stage are given by maximizing $q(100 - q)$ which yields $q = 50$, and an equal split means that $q_1 = q_2 = 25$ with each firm making $(100 - 50)25 = 1250$ in the first stage. However, each firm i is tempted to deviate given that $q_j = 25$. Using the best response derived in part **a.** we know that the best deviation is $q_i' = \frac{100-q_j}{2} = \frac{100-25}{2} = 37.5$ and the deviator's profits would be $(100 - 62.5)37.5 = 1406.3$. Hence, the gain from deviating is $1406.3 - 1250 = 156.3$. Given the two equilibria in the second stage game we can prevent the players from deviating by introducing the following contingent strategies: each player will play "open" if both played $q_i = 33\frac{1}{3}$ in the first stage and they will play "not open" if any other choices were made. With $\delta = 1$ the losses from the punishment outweigh the gains from deviation and hence it is a subgame perfect equilibrium. ■

(c) What is the lowest value of δ for which the subgame perfect equilibrium you found in (b) survives?

Answer: It must be the case that the loss from deviation is at least as painful as the gain, that is, $156.3 - \delta 250 \leq 0$, or, $\delta \geq 0.6252$. ■

(d) Now let $\delta = 0.4$. Can you support a subgame perfect equilibrium for the two-stage game in which the players choose the monopoly quantities and split them equally? If not, what are the highest profits that the firms can make in a symmetric equilibrium?

Answer: From the analysis in part **c.** we know that for $\delta < 0.6252$ we cannot support the split of monopoly profits as a subgame perfect equilibrium. Finding the highest profits that can be supported is a bit tricky. The easy part is starting with the discounted punishment value when $\delta = 0.4$, which is $0.4 \times 250 = 100$. Next we need to find a symmetric pair $(q_1, q_2) = (q^*, q^*)$ for which the extra profits from deviating to the best response to q^* given that the other firm sticks to q^* is exactly equal to 100. First, the profits from sticking to q^* will be $v^* = (100 - 2q^*)q^*$. Next, the best response to q^* is $q' = \frac{100-q^*}{2}$ and the profits from this deviation are $v' = (100 - q^* - \frac{100-q^*}{2})\frac{100-q^*}{2}$, and therefore q^* must solve $v' - v^* = 100$, or

$$(100 - q^* - \frac{100 - q^*}{2})\frac{100 - q^*}{2} - (100 - 2q^*)q^* = 100$$

which yields the solution $q^* = 26\frac{2}{3}$. Hence, for $\delta = 0.4$ the best symmetric equilibrium has both players earning $(100 - 2(26\frac{2}{3}))26\frac{2}{3} = 1244\frac{4}{9}$ in the first period followed by 250 in the second. ■

5. **Campaign Spending:** Two political candidates are destined to play the following two stage game. Assume throughout that there is no discounting ($\delta = 1$). First, they compete in the primaries of their party. Each candidate i can spend $s_i \geq 0$ resources on adds that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$. Each candidate values winning at a payoff of $16 > 0$, and the cost of spending s_i is equal to s_i. After each player observes the resources spent by the other, and a winner in the primaries is selected, they can choose how to interact. Each can choose to be pleasant (P for player 1 and p for player 2) or nasty (N and n respectively). At this stage, both players prefer that they be nice to each other rather than nasty, but if a player is nasty then the other prefers to be nasty too. The payoffs from this stage are given by the matrix where $w > 0$:

		Player 2	
		p	n
Player 1	P	w, w	$-1, 0$
	N	$0, -1$	$0, 0$

   (a) Find the unique Nash equilibrium of the first stage game and the two pure strategy Nash equilibria of the second stage game.

Answer: In the first stage player i maximizes

$$v_i(s_i, s_j) = \frac{s_i}{s_i + s_j} 16 - s_i$$

and the first order condition is

$$\frac{16 s_j}{(s_i + s_j)^2} - 1 = 0$$

which is of course symmetric for both players and represents the best response correspondence. Solving the two FOCs simultaneously yields $s_1 = s_2 = 4$ as the unique Nash equilibrium of the first stage game and each candidate wins with probability $\frac{1}{2}$. The two pure strategy Nash equilibria of the second stage game are (P, p) and (N, n). ■

(b) What are the Pareto optimal outcomes of each stage game?

Answer: In the first stage the symmetric Pareto optimal outcome is for both to choose $s_1 = s_2 = 0$. This way they win with probability $\frac{1}{2}$ each without wasting any resources.[2] It is easy to see that the Pareto optimal outcome of the second stage is (P, p). ■

(c) For which value of w can the players support the path of Pareto optimal outcomes as a subgame perfect equilibrium?

Answer: Because there is no discounting ($\delta = 1$) then the value of the threat of contingent punishment in the second stage is w for each player because the conditional strategies will be "we play (P, p) if we did the right thing in stage 1 and otherwise we play (N, n)." If we wish to support the Pareto optimal action of $s_1 = s_2 = 0$ in the first stage we need to see what the deviation payoff is. A player i who deviates to an infinitesimal value $s_i > 0$ will win for sure and get $16 - s_i$ instead of

[2] Strictly speaking, there is no other Pareto optimal outcome because of the continuous action spaces. If player i chooses $s_i = 0$ then if player j chooses $s_j = \varepsilon > 0$ then this is better for player j than choosing $s_j = 0$, but it is not Pareto optimal because if instead player j chooses $s_j = \frac{\varepsilon}{2}$ then he is better off without making player 1 worse off. This is a technicality in the sense that the "Pareto frontier" includes the point $(s_1, s_2) = (0, 0)$ but any other pair of feasible strategies is Pareto dominated for at least one player.

getting $\frac{1}{2} \times 16 - 0 = 8$, so that the gains from deviating are infinitesimally close to 8. Hence, if $w = 8$ then no player will wish to deviate from the proposed path of play. ■

(d) Assume that $w = 1$. What is the "best" symmetric subgame perfect equilibrium that the players can support?

Answer: The most severe threat is that the players lose $w = 1$ so the gain from deviating cannot be more than 1. We are therefore looking for a symmetric choice in the fist stage, $s_1 = s_2 = s^*$ such that if some player i deviates to the best response to $s_j = s^*$ then his gain in the first period is an expected payoff of 1. First note that if both players choose s^* then each gets a payoff in the first stage of $8 - s^*$ because they win with equal probability. Now consider the first order condition derived in part **a.** above. From it we can derive the best response function of each player to be $s_i(s_j) = 4\sqrt{s_j} - s_j$. This implies that the best response to $s_j = s^*$ is $s_i(s^*) = 4\sqrt{s^*} - s^*$, and if this is what player i deviates to then his expected payoff in the first stage game is

$$\begin{aligned} v_i(s_i(s^*), s^*) &= \frac{4\sqrt{s^*} - s^*}{4\sqrt{s^*} - s^* + s^*} 16 - (4\sqrt{s^*} - s^*) \\ &= 16 - 8\sqrt{s^*} + s^* \end{aligned}$$

The best symmetric equilibrium will be achieved when the gains from deviating are exactly equal to 1, or

$$16 - 8\sqrt{s^*} + s^* - (8 - s^*) = 1,$$

which results in $s^* = \frac{9}{2} - 2\sqrt{2} \approx 1.6716$. ■

(e) What happens to the best symmetric subgame perfect equilibrium that the players can support as w changes? In what way is this related to the role played by a discount factor?

Answer: If w increases then we can have a harsher punishment, and this can allow us to deter more attractive deviations that will happen when we try to implement a smaller value of s^* in the first stage. As w

increases towards 8 we can get closer and closer to the Pareto optimal outcome of $s^* = 0$. This carries the same intuition as a higher discount factor, which makes the punishment more severe for deviations in the first stage. ■

6. **Augmented Competition:** Consider two firms playing a two stage game with discount factor δ. In the first stage they play a Cournot quantity setting game where each firm has costs $c_i(q_i) = 10q_i$ for $i \in \{1, 2\}$ and the demand is given by $p(q) = 100 - q$ where $q = q_1 + q_2$. In the second stage, after the results of the Cournot game are observed, the firms play the following standard setting game:

Player 2

Player 1	a	b
A	$100, 100$	$0, 0$
B	$0, 0$	$300, 300$

(a) Find the unique Nash equilibrium of the first stage game and the two pure strategy Nash equilibria of the second stage game.

Answer: In the first stage game each player i maximizes $(100 - q_i - q_j)q_i - 10q_i$ which yields the best response function $q_i = \frac{90-q_j}{2}$ and the unique Nash equilibrium is $q_i = q_j = 30$. The two pure strategy Nash equilibria in the second stage are (A, a) and (B, b). ■

(b) As far as the two firms are considered, what are the symmetric Pareto optimal outcomes of each stage game?

Answer: In the first stage game it is splitting the monopoly profits and in the second stage it is (B, b) because $300 > 100$. Monopoly profits in the first stage are earned when we maximize $(100 - q_i - q_j)(q_i + q_j) - 10(q_i + q_j)$ which is obtained when $q_i + q_j = 45$. Hence, the symmetric Pareto optimal outcome is $q_1 = q_2 = 22.5$ in the first stage, which yields a payoff of 1012.5 for each player, and (B, b) in the second, which yields a payoff of 300 for each player. ■

(c) For which values of δ can the Pareto optimal outcomes be supported as a subgame perfect equilibrium?

Answer: In the Pareto optimal outcome of the first stage each firm earns a profit of i is tempted to deviate given that $q_j = 22.5$. Using the best response derived in part **a.** we know that the best deviation is $q_i' = \frac{90-q_j}{2} = \frac{90-22.5}{2} = 33.75$ and the deviator's profits would be $(100-22.5-33.75)33.75-10(33.75) = 1139.0625$. Hence, the gain from deviating is $1139.0625 - 1012.5 = 126.5625$. Given the two equilibria in the second stage game we can try and prevent the players from deviating by using contingent strategies: each player will play B (or b) if both played $q_i = 22.5$ in the first stage and they will play A (or a) if any other choices were made. This will cause the deviating player a loss of 200 in the second stage, and for this to deter the best deviation in the first stage it must be that $126.5625 - \delta 200 \leq 0$, and the solution is $\delta \geq 0.63281$. ■

(d) Assume that $\delta = 0.5$. What is the "best" symmetric subgame perfect equilibrium that the players can support?

Answer: Finding the highest profits ("best") that can be supported as a subgame perfect equilibrium is a bit tricky. The easy part is starting with the discounted punishment value when $\delta = 0.5$, which is $0.5 \times 200 = 100$. Next we need to find a symmetric pair $(q_1, q_2) = (q^*, q^*)$ for which the extra profits from deviating to the best response to q^* given that the other firm sticks to q^* is exactly equal to 100. First, the profits from sticking to q^* will be $v^* = (100 - 2q^*)q^* - 10q^*$. Next, the best response to q^* is $q' = \frac{90-q^*}{2}$ and the profits from this deviation are $v' = (100 - q^* - \frac{90-q^*}{2})\frac{90-q^*}{2} - 10(\frac{90-q^*}{2})$, and therefore q^* must solve $v' - v^* = 100$, or

$$(100-q^*-\frac{90-q^*}{2})\frac{90-q^*}{2}-10\left(\frac{90-q^*}{2}\right)-[(100-2q^*)q^*-10q^*] = 100,$$

and the solution is $q^* = 23\frac{1}{3}$. Hence, for $\delta = 0.5$ the best symmetric equilibrium has each player earning $(100-2(23\frac{1}{3}))23\frac{1}{3}-10(23\frac{1}{3}) = 1011\frac{1}{9}$ in the first stage followed by 300 in the second. ■

(e) What happens to the best symmetric subgame perfect equilibrium that the players can support as δ drops towards zero?

Answer: As δ drops towards zero the ability to punish becomes less effective and the best subgame perfect equilibrium quantities in the first stage will grow until they reach the Nash (Cournot) equilibrium of the first stage, $q_1 = q_2 = 30$. ■

10

Repeated Games

1. **Medicare Drug Policy:** In early 2005 there was a discussion of a proposed policy of the US federal administration that supported the use of so called "discount cards" that pharmaceutical firms can offer senior citizens for the purchase of medications. These cards will have a subscription fee, and they will in return offer discounts if prescription drugs are bought through the issuing companies. The federal administration argued that any of the large pharmaceutical companies can enter this market for discount cards, which in turn will promote competition. To ensure this the government has a website with posted prices and posted discounts that go with each card. Some consumer advocates suggest that the companies will just hike up prices and offer a discount over this higher prices, resulting in less welfare for consumers. The administration argued that this does not make too much sense because there is entry and competition. Can you argue, using some formal ideas on tacit collusion, that the way things are set up it is in fact possible, and maybe even easier, for the firms to squeeze more profits at the expense of consumers?

   Answer: By having a central place in which prices are posted the government makes it easy for companies to monitor each other's prices, and this in turn makes it easier to sustain tacit collusion because it companies who deviate from the tacit agreement will be easily detected by the other companies. ■

2. **Grim Trigger**: Consider the infinitely repeated game with discount factor $\delta < 1$ of the following variant of the Prisoner's dilemma:

		Player 2		
		L	C	R
	T	$6, 6$	$-1, 7$	$-2, 8$
player 1	M	$7, -1$	$4, 4$	$-1, 5$
	B	$8, -2$	$5, -1$	$0, 0$

(a) For which values of the discount factor δ can the players support the pair of actions (M, C) played in every period?

Answer: The grim trigger strategy is to revert to playing (B, R) forever yielding a discounted sum of payoffs (and an average payoff) equal to 0. The discounted sum of payoffs from sticking to the pair (M, C) forever is $\frac{4}{1-\delta}$. A player who deviates gets 5 instead of 4 in the period of deviation, but then gets 0 thereafter. Hence a deviation will not be profitable if $\frac{4}{1-\delta} \geq 5$, or $\delta \geq \frac{1}{5}$. ■

(b) For which values of the discount factor δ can the players support the pair of actions (T, L) played in every period? Why is your answer different from part (a) above?

Answer: The discounted sum of payoffs from sticking to the pair (T, L) forever is $\frac{6}{1-\delta}$. A player who deviates gets 8 instead of 6 in the period of deviation, but then gets 0 thereafter using grim trigger. Hence a deviation will not be profitable if $\frac{6}{1-\delta} \geq 8$, or $\delta \geq \frac{1}{4}$. ■

3. **Not so Grim Trigger**: Consider the infinitely repeated Prisoner's Dilemma with discount factor $\delta < 1$ described by the following matrix:

		Player 2	
		m	f
Player 1	M	$4, 4$	$-1, 5$
	F	$5, -1$	$1, 1$

Instead of using "grim trigger" strategies to support a pair of actions (a_1, a_2) other than (F, f) as a subgame perfect equilibrium, assume that the player wish to choose a less draconian punishment called a "length T punishment" strategy. Namely, if there is a deviation from (a_1, a_2) then the players will play (F, f) for T periods, and then resume playing (a_1, a_2). Let δ_T be the critical discount factor so that if $\delta > \delta_T$ then the adequately defined strategies will implement the desired path of play with length T punishment as the threat.

(a) Let $T = 1$. What is the critical value δ_1 to support the pair of actions (M, m) played in every period?

Answer: The proposed one period punishment means that instead of getting 4 for the period after deviation, the players will get 1, and afterwards will resort to getting 4 forever. Hence, the punishment is of size 3 and the discounted value is $\delta 3$. The gain from deviating in one period is getting 5 instead of 4 so this will be deterred if $1 \leq \delta 3$ or $\delta \geq \frac{1}{3}$.[1] ■

(b) Let $T = 2$. What is the critical value δ_T to support the pair of actions (M, m) played in every period?[2]

Answer: The proposed two period punishment means that instead of getting 4 for the two periods after deviation, the players will get 1,

[1] To see this using the whole stream of payoffs, sticking to (M, m) yields $\frac{4}{1-\delta}$ while deviating with the threat of a one period punishment will yield $5 + \delta 1 + \delta^2 \frac{4}{1-\delta}$ and this is not profitable if $\frac{4}{1-\delta} \geq 5 + \delta 1 + \delta^2 \frac{4}{1-\delta}$, which can be rewritten as $4 + \delta 4 + \delta^2 \frac{4}{1-\delta} \geq 5 + \delta 1 + \delta^2 \frac{4}{1-\delta}$, which in turn reduces to $\delta 3 \geq 1$.

[2] Helpful hint: You should encounter an equation of the form $a\delta^3 - (a+1)\delta + 1 = 0$ for which it is easy to see that $\delta = 1$ is a root. In this case, you know that the equation can be written in the form $(\delta - 1)(a\delta^2 + a\delta - 1) = 0$ and solve for the other relevant root of the cubic equation.

and afterwards will resort to getting 4 forever. Hence, the discounted punishment is $(\delta+\delta^2)3$. The gain from deviating in one period is getting 5 instead of 4 so this will be deterred if $1 \leq (\delta+\delta^2)3$, or $\delta \geq \frac{1}{6}\sqrt{3}\sqrt{7}-\frac{1}{2} \approx 0.26376$.[3] ■

(c) Compare the two critical values in parts (a) and (b) above. How do they differ and what is the intuition for this?

Answer: The punishment in part **b.** last for two periods which is more severe than the one period punishment in part **a.** This means that it can be supported with a lower discount factor because the intensity of the punishment is increasing either in the length or when we have less discounting. ■

4. **Trust off-the-equilibrium-path:** Recall the trust game depicted in Figure 10.1. We argued that for $\delta \geq \frac{1}{2}$ the following pair of strategies is a subgame perfect equilibrium. For player 1: "in period 1 I will trust player 2, and as as long as there were no deviations from the pair (T, C) in any period, then I will continue to trust him. Once such a deviation occurs then I will not trust him forever after." For player 2: "in period 1 I will cooperate, and as as long as there were no deviations from the pair (T, C) in any period, then I will continue to do so. Once such a deviation occurs then I will deviate forever after." Show that if instead player 2 uses the strategy "as long as player 1 trusts me I will cooperate" then the path (T, C) played forever is a Nash equilibrium for $\delta \geq \frac{1}{2}$ but is not a subgame perfect equilibrium for any value of δ.

Answer: It is easy to see that this is a Nash equilibrium: the equilibrium path is followed because neither player benefits from deviating as they both believe that a deviation will call for the continuation of grim trigger. To see that it is not subgame perfect consider the subgame that follows after

[3] To see this using the whole stream of payoffs, sticking to (M, m) yields $\frac{4}{1-\delta}$ while deviating with the threat of a two period punishment will yield $5+\delta 1+\delta^2 1+\delta^3\frac{4}{1-\delta}$ and this is not profitable if $\frac{4}{1-\delta} \geq 5+\delta 1+\delta^2 1+\delta^3\frac{4}{1-\delta}$. This can either be solved as a cubic inequality or can be rewritten as $4+\delta 4+\delta^2 4+\delta^3\frac{4}{1-\delta} \geq 5+\delta 1+\delta^2 1+\delta^3\frac{4}{1-\delta}$, which in turn reduces to $(\delta+\delta^2)3 \geq 1$.

a deviation of player 2 from C to D. The strategy of player 1 is to not trust forever which will revert the payoff to 0 in every period, but player 2's strategy is to cooperate as long as player 1 trusts. So, after a deviation by player 2, if player 1 believes that player 2 will indeed cooperate then player 1 should continue to trust. ■

5. **Negative Ad Campaigns (revisited):** Recall the exercise from chapter **??** in which each one of two political parties can choose to buy time on commercial radio shows to broadcast negative ad campaigns against their rival. These choices are made simultaneously. Due to government regulation it is forbidden to buy more than 2 hours of negative campaign time so that each party cannot choose an amount of negative campaigning above 2 hours. Given a pair of choices (a_1, a_2), the payoff of party i is given by the following function: $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$.

   (a) Find the unique pure strategy Nash equilibrium of the one shot game.

   Answer: Each player maximizes $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$ resulting in the first order optimality condition $1 + a_j - 2a_i = 0$, yielding the best response function,

   $$a_i(a_j) = \frac{1 + a_j}{2} \ .$$

   Solving the two best response functions simultaneously,

   $$a_1 = \frac{1 + a_2}{2} \text{ and } a_2 = \frac{1 + a_1}{2}$$

   yields the Nash equilibrium $a_1 = a_2 = 1$, and this is the unique solution to these equations implying that this is the unique equilibrium. Each player obtains a payoff of -1. ■

   (b) If the parties could sign a binding agreement on how much to campaign, what levels would they choose?

   Answer: They would choose $a_1 = a_2 = 0$ and each would obtain a payoff of 0. ■

(c) Now assume that this game is repeated infinitely often, and the above demonstrates the choices and payoffs per period. For which discount factors $\delta \in (0,1)$ can the levels you found in part (b) above be supported as a subgame perfect equilibrium of the infinitely repeated game?

Answer: Consider the grim trigger strategy where the players will revert to playing the one-shot Nash forever after a deviation. The temptation to deviate from 0 is the value a player gains when he chooses the best response to 0, which is $a_i = \frac{1}{2}$, which yields the one shot payoff of $\frac{1}{4}$. Hence, the deviation will not be profitable if $\frac{1}{4} - \delta\frac{1}{1-\delta} \le 0$, or $\delta \in [\frac{1}{5}, 1)$.

(d) Despite the parties ability to coordinate as you have demonstrated in your answer to (c) above, the government is concerned about the parties ability to place up to 2 hours a day of negative campaigning, and it is considering limiting negative campaigning to $\frac{1}{2}$ hour a day so that now $a_i \in [0, \frac{1}{2}]$. Is this a good policy to further limit negative campaigns? Justify your answer with the relevant calculations. What is the intuition for your conclusion?

Answer: If this were just a one shot game then the government's regulation would be beneficial. Instead of choosing $a_1 = a_2 = 1$ they would choose $a_1 = a_2 = \frac{1}{2}$ and receive $-\frac{1}{2}$ each instead of -1. However, for the repeated game this regulation makes the grim trigger threat less severe, and cooperation on spending nothing can only be achieved if $\frac{1}{4} - \delta\frac{0.5}{1-\delta} \le 0$, which holds for $\delta \in [\frac{1}{3}, 1)$. Hence, for $\delta \in [\frac{1}{5}, \frac{1}{3})$ the players will no longer be able to achieve the Pareto optimal outcome using repeated game cooperation, making this regulation a bad idea. ■

6. **Regulating Medications:** Consider a firm (player 1) that produces a unique kind of drug that is used by a consumer (player 2). This drug is regulated by the government so that the price of the drug is $p = 6$. This price is fixed, but the quality of the drug depends on the manufacturing procedure. The "good" (G) manufacturing procedure costs 4 to the firm, and yields a value of 7 to

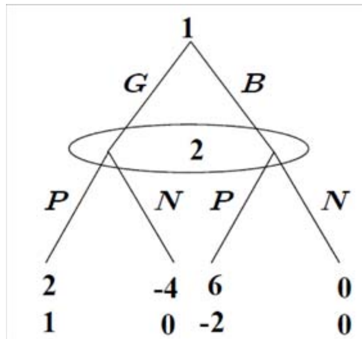


FIGURE 10.1.

the consumer. The "bad" (B) manufacturing procedure costs 0 to the firm, and yields a value of 4 to the consumer. The consumer can choose whether to buy or not at the price p, and this decision must be made before the actual manufacturing procedure is revealed. However, after consumption, the true quality is revealed to the consumer. The choice of manufacturing procedure, and the cost of production, is made before the firm knows whether the consumer will buy or not.

(a) Draw the game tree and the matrix of this game, and find all the Nash equilibria of this game.

Answer: Let player 1 be the firm who can choose G (good) or B (bad), and player 2 is the consumer who can choose P (purchase) or N (not purchase). If, for example, the players choose (G, P) then the firm gets $6-4=2$ and the consumer gets $7-6=1$. In a similar way the complete matrix of this one shot game can be represented as follows:

		Player 2	
		P	N
Player 1	G	$2, 1$	$-4, 0$
	B	$6, -2$	$0, 0$

The extensive form game tree is,■

(b) Now assume that the game described above is repeated twice. (The consumer learns the quality of the product in each period only if he consumes.) Assume that each player tries to maximize the (non-discounted) sum of his stage payoffs. Find *all* the subgame-perfect equilibria of this game.

Answer: It is easy to see that player 1 has a dominant strategy in the stage game: choose B, and player 2's best response is to choose N. This unique Nash equilibrium must be played in the second stage, and by backward induction must also be played in the first stage. hence, it is the unique subgame perfect equilibrium.

(c) Now assume that the game as repeated infinitely many times. Assume that each player tries to maximize the discounted sum of his or her stage payoffs, where the discount rate is $\delta \in (0,1)$. What is the range of discount factors for which the good manufacturing procedure will be used as part of a subgame perfect equilibrium?

Answer: Consider the grim trigger strategies: player 1 chooses G and continues to choose G as long as he chose G in the past and as long as player 2 purchased. Otherwise he chooses B forever after. Player 2 chooses P and continues to choose P as long as he chose P and player 1 chose G. Otherwise he plays N forever after. Player 2 has no incentive to deviate at any stage, but player 1 can gain 4 from switching to B in any period (get 6 instead of 2). He will not have an incentive to deviate if $4 \leq \frac{2}{1-\delta}$, which holds for $\delta \in [\frac{1}{2}, 1)$. ■

(d) Consumer advocates are pushing for a lower price of the drug, say 5. The firm wants to approach the Federal trade Commission and argue that if the regulated price is decreased to 5 then this may have dire consequences for both consumers and the firm. Can you make a formal argument using the parameters above to support the firm? What about the consumers?

Answer: If the price of the drug is lowered to 5 then player 1 has a stronger *relative* temptation to deviate from the grim trigger strategies described in part **c.** above. His gain from deviation is still 4, but the gain from continuing to choose G is only 1 per period and not 2. Hence, he will not have an incentive to deviate if $4 \leq \frac{1}{1-\delta}$, which holds for $\delta \in [\frac{3}{4}, 1)$. Hence, if the firm can argue that $\delta \in [\frac{1}{2}, \frac{3}{4})$ then increasing the price from 4 to 5 will cause the good equilibrium to collapse and no trade will occur. The argument in favor of raising the price can be made if $\delta \in [\frac{3}{4}, 1)$ because then the consumers benefit at the expense of the firm but there is enough surplus to support the good outcome. ■

7. **Diluted Happiness:** Consider a relationship between a bartender and a customer. The bartender serves bourbon to the customer, and chooses $x \in [0, 1]$, which is the proportion of bourbon in the drink served, while $1 - x$ is the proportion of water. The cost of supplying such a drink (standard 4 once glass) is cx where $c > 0$. The Customer, without knowing x, decides on whether or not to buy the drink at the market price p. If he buys the drink, his payoff is $vx - p$, and the bartender's payoff is $p - cx$. Assume that $v > c$, and all payoffs are common knowledge. If the customer does not buy the drink, he gets 0, and the bartender gets $-(cx)$. because the customer has some experience, once the drink is bought and he tastes it, he learns the value of x, but this is only after he pays for the drink.

   (a) Find all the Nash equilibria of this game.

   Answer: The customer has to buy the drink without knowing its content, implying that the bartender has a dominant strategy which is to choose $x = 0$ once the customer pays for the drink. But anticipating that, the customer would not buy the drink. Hence, the unique Nash equilibrium is for the customer not to buy and the bartender to choose $x = 0$ if he does buy. ■

   (b) Now assume that the customer is visiting town for 10 days, and this "bar game" will be played for each of the 10 evenings that the customer is in town. Assume that each player tries to maximize the (non-discounted) sum of his stage payoffs. Find all subgame-perfect equilibria of this game.

   Answer: The game just unravels: in the last period they must play the unique Nash in part **a.** above. But then they will do the same in the penultimate period, and so in until the beginning of the game. The unique subgame perfect equilibrium is therefore for the customer not to buy in any of the 10 periods and for the bartender to choose $x = 0$ in a period where the customer buys. ■

(c) Now assume that the customer is a local, and the players perceive the game as repeated infinitely many times. Assume that each player tries to maximize the discounted sum of his or her stage payoffs, where discount rate is $\delta \in (0, 1)$. What is the range of prices p (expressed in the parameters of the problem) for which there exists a subgame-perfect equilibrium in which everyday the bartender chooses $x = 1$ and the customer buys at the price p?

Answer: For a transaction to occur both have to get a non-negative payoff, implying first that $p \in [c, v]$. We will consider a subgame perfect equilibrium with grim trigger strategies that reverts to no-purchase if anyone ever deviates. Notice that the customer has no incentive to ever deviate if $p \leq v$ because he gains nothing or loses some positive value from not buying. The bartender does benefit in the one shot game from deviating to $x = 0$ and obtaining p instead of $p - c$. Given some value of δ, the bartender will not deviate if $p \leq \frac{p-c}{1-\delta}$, or $p \geq \frac{c}{\delta}$ (which is of course greater than c). Hence, if $\frac{c}{\delta} \leq v$ then for any price $p \in [\frac{c}{\delta}, v]$ there exists a subgame perfect equilibrium in which everyday the bartender chooses $x = 1$ and the customer buys at the price p. If, however, $\frac{c}{\delta} < v$ then no such price exists. ■

(d) For which values of δ (expressed in the parameters of the problem) can such a price range that you found in (5) above exist?

Answer: The condition for such a subgame perfect equilibrium is that $\frac{c}{\delta} \leq v$, which implies that δ must satisfy $\delta \geq \frac{c}{v}$. ■

8. **Tacit Collusion:** There are two firms that have zero marginal cost and no fixed cost that produce some good, each producing $q_i \geq 0$, $i \in \{1, 2\}$. The demand for this good is given by $p = 200 - Q$, where $Q = q_1 + q_2$.

   (a) First consider the case of Cournot competition, where each firm chooses q_i, and that this game is infinitely repeated with a discount factor $\delta < 1$. Solve for the static stage-game Cournot-Nash equilibrium.

   Answer: Each firm solves $\max_{q_i} \ (200 - q_i - q_j)q_i$ so the first order condition is $200 - 2q_i - q_j = 0$ and the best response is $q_i = \frac{200-q_j}{2}$. The unique Nash equilibrium is therefore $q_1 = q_2 = 66\frac{2}{3}$. The profits of each firm would be $4,444\frac{4}{9}$. ■

   (b) For which values of δ can you support the firms **equally splitting** monopoly profits in each period as a subgame perfect equilibrium that uses "trigger strategies"? (i.e., after one deviates from the proposed split, they resort to the static Cournot-Nash equilibrium thereafter). Note: be careful in defining the strategies of the firms.

   Answer: The monopoly profits is obtained from maximizing $(200-Q)Q$ which occurs at $Q = 100$ with combined profits being $10,000$, or $q_i = 50$ and profits are $5,000$ for each firm. If firm j is producing 50, however, then the best deviation for firm i is given by the best response, $q_i = \frac{200-50}{2} = 75$, and firm i's profits in the period when it deviates are $(200-75-50)75 = 5,625$. Consider trigger strategies of the form "start by choosing $q_i = 50$ and continue to choose so as long as both firms follow this path, yet if any firm ever deviates form this path revert to $q_i = 66\frac{2}{3}$ forever after." The deviation will not be worthwhile if

$$5625 + \frac{\delta}{1-\delta}(4444\frac{4}{9}) \leq \frac{5000}{1-\delta}$$

   which holds if $\delta \in [\frac{9}{17}, 1)$. ■

   (c) Now assume that the firms compete à la Bertrand, each choosing a price $p_i \geq 0$, where the lowest priced firm gets all the demand, and in case of a tie they split the market. Solve for the static stage-game Bertrand-Nash equilibrium.

Answer: The static Bertrand-Nash equilibrium is for each form to choose $p_i = 0$ because they have zero marginal costs. Profits will be zero for each firm. ■

(d) For which values of δ can you support the firms splitting monopoly profits in each period as a subgame perfect equilibrium that uses "trigger strategies"? (i.e., after one deviates from the proposed split, they resort to the static Bertrand–Nash equilibrium thereafter). Note: be careful in defining the strategies of the firms!

Answer: The monopoly profits are obtained from choosing $p_i = 100$ with combined profits being $10,000$, and profits are $5,000$ for each firm if they split production equally. If firm j is charges 100, however, then firm i can deviate to some price $p_i = 100 - \varepsilon$ for ε infinitesimally small and firm i's profits in the period when it deviates will be infinitesimally close to $10,000$. Consider trigger strategies of the form "start by choosing $p_i = 100$ and continue to choose so as long as both firms follow this path, yet if any firm ever deviates form this path revert to $p_i = 0$ forever after." The deviation will not be worthwhile if

$$10000 + \frac{\delta}{1-\delta}(0) \leq \frac{5000}{1-\delta}$$

which holds if $\delta \in [\frac{1}{2}, 1)$. ■

(e) Now instead of using trigger strategies, try to support the firms equally splitting monopoly profits as a subgame perfect equilibrium where after a deviation, firms would resort to the static Bertrand competition for **only two periods**. For which values of δ will this work? Why is this different than your answer in (d) above?

Answer: Because we are only punishing for two periods, the deviation will not be worthwhile if

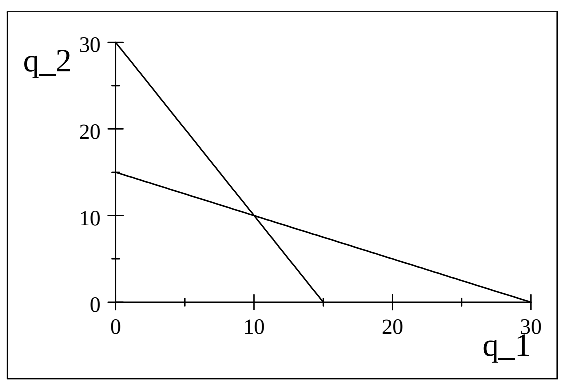
$$10000 + \delta(0) + \delta^2(0) + \delta^3\frac{5000}{1-\delta} \leq \frac{5000}{1-\delta}$$

or $\delta^2 + \delta - 1 \geq 0$, which results in $\delta \geq \frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.618$. The reason we need a larger discount factor is that the punishment is less severe as it lasts for only two periods and not infinitely many. ■

9. **Negative Externalities:** Two firms are located adjacent to one another and each imposes an external cost on the other: the detergent that Firm 1 uses in it's laundry business makes the fish that firm 2 catches in the lake taste funny, and the smoke that firm 2 uses to smoke its caught fish makes the clothes that firm 1 hands out to dry smell funny. As a consequence, each firms profits are increasing it its own production and decreasing in the production of its neighboring firm. In particular, if q_1 and q_2 are the firms' production levels then their per-period (stage game) profits are given by $v_1(q_1, q_2) = (30 - q_2)q_1 - q_1^2$ and $v_2(q_1, q_2) = (30 - q_1)q_2 - q_2^2$.

   (a) Draw the firms' best response functions and find the Nash equilibrium of the stage game. How does this compare to the Pareto optimal stage-game profit levels?

   Answer: Each firm maximizes $v_1(q_i, q_j) = (30 - q_j)q_i - q_i^2$ and the first order condition is $30 - q_j - 2q_i = 0$, resulting in the best response function $q_i = \frac{30-q_j}{2}$ as drawn in the following figure:



   The unique Nash equilibrium is $q_1 = q_2 = 10$ giving each firm a profit of 100. To solve for the Pareto optimal outcome we can maximize the sum of profits,

$$\max_{q_1, q_2} S(q_1, q_2) = (30 - q_2)q_1 - q_1^2 + (30 - q_1)q_2 - q_2^2$$

and the two first order conditions are

$$\frac{\partial S(q_1,q_2)}{\partial q_1} = 30 - q_2 - 2q_1 - q_2 = 0$$
$$\frac{\partial S(q_1,q_2)}{\partial q_2} = 30 - q_1 - 2q_2 - q_1 = 0$$

and solving them together yields $q_1 = q_2 = 7\frac{1}{2}$ and the profits of each firm are $112\frac{1}{2}$. ■

(b) For which levels of discount factors can the firms support the Pareto optimal level of quantities in an infinitely repeated game?

Answer: We consider grim trigger strategies of the form "I will choose $q_i = 7.5$ and continue to do so as long as both chose this value. If anyone ever deviates I will revert to $q_i = 10$ forever." The best deviation from $q_i = 7.5$ given that $q_j = 7.5$ is to choose the best response to 7.5 which is $\frac{30-7.5}{2} = 11.25$, and the profit from deviating is $(30 - 7\frac{1}{2})11\frac{1}{4} - (11\frac{1}{4})^2 = \frac{2025}{16} = 126\frac{9}{16}$. Thus, each player will not want to deviate if

$$126\frac{9}{16} + \delta\frac{100}{1-\delta} \leq \frac{112\frac{1}{2}}{1-\delta}$$

which holds for $\delta \in [\frac{9}{17}, 1)$. ■

10. **Law Merchants (revisited):** Consider the three person game described in section **??**. A subgame perfect equilibrium was constructed with a bond equal to 2, and a wage paid by every player P_2^t to player 3 equal to $w = 0.1$, and it was shown that it is indeed an equilibrium for any discount factor $\delta \geq 0.95$. Show that a similar equilibrium, where players P_1^t trust players P_2^t who post bonds, players P_2^t post bonds and cooperate, and player 3 follows the contract in every period, for any discount factor $0 < \delta < 1$.

Answer: First notice that the bond need not be equal to 2 because player P_2^t only gains 1 from deviating. Hence, any bond of value $1 + \varepsilon > 1$ will deter player P_2^t from choosing to defect instead of cooperate. Second, notice that for any wage to the third party of $1 - \varepsilon < 1$, player P_2^t still get a positive surplus $\varepsilon > 0$ from engaging the services of the third party. Hence, for any value of $\varepsilon \in (0,1)$, posting a bond of $1+\varepsilon$ and paying the third party $1-\varepsilon$ guarantees that player P_2^t will choose to employ the third party and cooperates if trusted, and in turn, P_1^t will choose to trust. We are left to see whether the third party prefers to return the bond as promised or if he would deviate and give up the future stream of all income. By deviating the third party pockets the bon worth $1+\varepsilon$, and gives up the future series of wages $1-\varepsilon$ for all future periods. Hence, he will not deviate if

$$1+\varepsilon \leq \frac{\delta}{1-\delta}(2-\varepsilon),$$

which for $\varepsilon \in (0,1)$ holds for $\delta \in (\frac{1+\varepsilon}{2}, 1)$. Hence, for any $\delta > \frac{1}{2}$ there exists a small enough $\varepsilon > 0$ for which the inequality above holds. ■

11. **Trading Brand Names:** Show that the strategies proposed in Section **??** constitute a subgame perfect equilibrium of the sequence of trust games.

    Answer: Consider any player P_2^t, $t > 1$. Under the proposed strategies, if trust was never abused and the name was bought up till period $t-1$ then (i) by buying the name and cooperating he is guaranteed a payoff of 1, (ii) by buying the name and defecting he receives 2 but cannot sell the name to the next player 2 and hence he gets $2-p^* < 1$, and (iii) by not buying the name he gets 0. Hence, for any t the strategy of P_2^t is a best response. Consider player P_2^1. If he (i) by creating the name and cooperating he is guaranteed a payoff of $1+p^* > 2$, (ii) by not creating the name he gets 0. Hence, the strategy of P_2^1 is a best response. Last, it is easy to see that any player 1 can expect cooperation, and hence trusting is a best response conditional on no one ever defecting and the name being created and transmitted. ■

12. **Folk Theorem (revisited):** Consider the infinitely repeated trust game described in Figure 10.1.

    (a) Draw the convex hull of average payoffs.

    Answer: ■

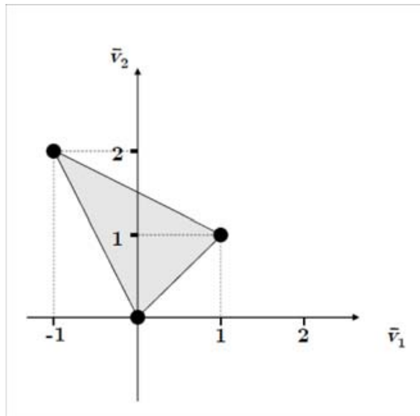


FIGURE 10.2.

(b) Are the average payoffs $(\overline{v}_1, \overline{v}_2) = (-0.4, 1.1)$ in the convex hull of average payoffs? Can they be supported by a pair of strategies that form a subgame perfect equilibrium for a large enough discount factor δ?

Answer: The average payoffs $(\overline{v}_1, \overline{v}_2) = (-0.4, 1.1)$ are in the convex hull of average payoffs. It is easy to see that the point $(-0.4, 0.8)$ is on the line that connects the point $(-1, 2)$ with $(0, 0)$, and the point $(-0.4, 1.7)$ the line that connects the point $(-1, 2)$ with $(1, 1)$. It follows that the point $(-0.4, 1.1)$ is in the interior of the convex hull of payoffs. However, these payoffs cannot be supported by a subgame perfect equilibrium because player 1 is expected to get an average payoff of -0.4, but he can guarantee himself a payoff of 0 by choosing never to trust. ■

(c) Show that there is a pair of subgame perfect equilibrium strategies for the two players that yields average payoffs that approach $(\overline{v}_1, \overline{v}_2) = (\frac{1}{3}, \frac{4}{3})$ as δ approaches 1.

Answer: First note that the point $(\frac{1}{3}, \frac{4}{3})$ the line that connects the point $(-1, 2)$ with $(1, 1)$. That is, it is a weighted average of the two points as follows: $\frac{1}{3}(-1, 2)+\frac{2}{3}(1, 1) = (\frac{1}{3}, \frac{4}{3})$. This suggests that the average payoff we are trying to achieve is a $\frac{1}{3} : \frac{2}{3}$ weighted average between the pairs of actions (T, D) and (T, C). So, consider the the following strategies:

Player 2 will play C twice and then D once, and repeat this pattern (play C in $t = 1, 2, 4, 5, 7, ...$ and play D in $t = 3, 6, 9, ...$). Player 1 will play T every period. If either player deviates from these proposed strategies then both players revert to playing (N, D) forever after. The payoff for player 1 is,

$$\begin{aligned}
\overline{v}_1 &= (1-\delta)(1+\delta+\delta^2(-1)+\delta^3+\cdots) \\
&= (1-\delta)(\frac{1}{1-\delta^3}+\frac{\delta}{1-\delta^3}-\frac{\delta^2}{1-\delta^3}) \\
&= (1-\delta)\frac{1+\delta-\delta^2}{(1-\delta)(1+\delta+\delta^2)} \\
&= \frac{1+\delta-\delta^2}{1+\delta+\delta^2}
\end{aligned}$$

and it follows that

$$\lim_{\delta\to 1}\frac{1+\delta-\delta^2}{1+\delta+\delta^2}=\frac{1}{3}.$$

Similarly,

$$\begin{aligned}
\overline{v}_2 &= (1-\delta)(1+\delta+\delta^2(2)+\delta^3+\cdots) \\
&= (1-\delta)(\frac{1}{1-\delta^3}+\frac{\delta}{1-\delta^3}+\frac{2\delta^2}{1-\delta^3}) \\
&= \frac{1+\delta+2\delta^2}{1+\delta+\delta^2}
\end{aligned}$$

and it follows that

$$\lim_{\delta\to 1}\frac{1+\delta+2\delta^2}{1+\delta+\delta^2}=\frac{4}{3}.$$

Hence, as $\delta \to 1$ the average payoffs from this subgame perfect equilibrium converge to $(\frac{1}{3}, \frac{4}{3})$. ■

11

Strategic Bargaining

1. **Disagreement:** Construct a pair of strategies for the ultimatum game ($T = 1$ bargaining game) that constitute a Nash equilibrium, which together support the outcome that there is no agreement reached by the two players and the payoffs are zero to each. Show that this disagreement outcome can be supported by a Nash equilibrium regardless of the number of bargaining periods.

   Answer: Consider the following strategies: player 1 offers nothing to player 2 ($x = 0$) and player 2 only accepts if he is offered all of the surplus ($x = 1$). In this case both players are indifferent (player 1 is indifferent between any offer and player 2 is indifferent between accepting and rejecting), and both receive zero. It is easy to see that repeating these strategies for any length of the game will still constitute a Nash equilibrium. ■

2. **Hold Up:** Considering an ultimatum game ($T = 1$ bargaining game) where before player 1 makes his offer to player 2, player 2 can invest in the size of the pie. If player 2 chooses a low level of investment (L) then the size of the pie is small, equal to v_L while if player 2 chooses a high level of investment (H) then the size of the pie is large, equal to v_H. The cost to player 2 of choosing

L is c_L, while the cost of choosing H is c_H. Assume that $v_H > v_L > 0$, $c_H > c_L > 0$ and $v_H - c_H > v_L - c_L$.

(a) What is the unique subgame perfect equilibrium of this game? Is it Pareto Optimal?

Answer: Solving this game backward, we know that the ultimatum game has a unique equilibrium in which player 1 will offer nothing to player 2 and player 2 will accept the offer. Working backwards, if player 2 first chooses the low level of investment then his payoff will be $-c_L$, while he will be worse off if he chooses the high level of investment because $-c_H < -c_L$. Hence, the unique subgame perfect equilibrium has player 2 first choose the low level of investment, then player 1 offering to keep all the value v_L to himself, and finally player 2 accepting the offer and getting $-c_L$. ■

(b) Can you find a Nash equilibrium of the game that results in an outcome that is better for both players as compared to the unique subgame perfect equilibrium?

Answer: Consider the following strategy for player 2: first choose the high level of investment, and then accept any offer that gives himself at least $v_H - v_L - \varepsilon$ for ε small. Given this strategy, player 1's best response is to offer to keep $v_L + \varepsilon$ for himself and $v_H - v_L - \varepsilon$ for player 2. Player 2's payoff is then $v_H - v_L - \varepsilon - c_H > -c_L$ for small enough ε, and player 1's payoff is $v_L + \varepsilon > v_L$ so the players are both better off. ■

3. **Even/Odd Symmetry:** In section **??** we analyzed the alternating bargaining game for a finite number of periods when T was odd. Repeat the analysis for T even.

Answer: Consider the case with an even number of rounds $T < \infty$, implying that player 2 has the last mover advantages. The following backward induction argument applies:
- In period T, player 1 accepts any offer, so player 2 offers $x = 0$ and payoffs

are $v_1 = 0$; $v_2 = \delta^{T-1}$
- In period $T-1$ (odd period – player 1 offers), by backward induction player 2 should accept anything resulting in a payoff of $v_2 \geq \delta^{T-1}$. If player 2 is offered x in period $T-1$, then $v_2 = \delta^{T-2}(1-x)$; This implies that in period $T-1$ player 2 will accept any $(1-x) \geq \delta$ and by backward induction player 1 should offer $x = 1-\delta$, which yields player 1 a payoff of $v_1 = (1-\delta)\delta^{T-2}$ and $v_2 = \delta^{T-1}$.
- In period $T-2$ (even period), conditional on the analysis for $T-1$, player 1's best response is to accept any x that gives him $\delta^{T-3}x \geq (1-\delta)\delta^{T-2}$. Player 2's best response to this is to offer the smallest x that satisfies this inequality, and solving it with equality yields player 2's best response: $x = \delta - \delta^2$. This offer followed by 1's acceptance yields $v_1 = \delta^{T-3}x = \delta^{T-2} - \delta^{T-1}$ and $v_2 = \delta^{T-3}(1-x) = \delta^{T-3} - \delta^{T-2} + \delta^{T-1}$.

We can continue with this tedious exercise only to realize that a simple pattern emerges. If we consider the solution for an even period $T-s$ (s being even because T is assumed to be even) then the backward induction argument leads to the sequentially rational offer,

$$x_{T-s} = \delta - \delta^2 + \delta^3 \cdots - \delta^s \ ,$$

while for an odd period $T-s$ (s being odd) then the backward induction argument leads to the sequentially rational offer,

$$x_{T-s} = 1 - \delta + \delta^2 \cdots - \delta^s.$$

We can use this Pattern to solve for the subgame perfect equilibrium offer in the first period, x_1, which by backward induction must be accepted by player 2, and it is equal to

$$\begin{aligned} x_1 &= 1 - \delta + \delta^2 - \delta^3 + \delta^4 \cdots - \delta^{T-1} = \\ &= (1 + \delta^2 + \delta^4 + \cdots + \delta^{T-2}) - (\delta + \delta^3 + \delta^5 + \cdots + \delta^{T-1}) \\ &= \frac{1-\delta^T}{1-\delta^2} - \frac{\delta - \delta^{T+1}}{1-\delta^2} \\ &= \frac{1-\delta^T}{1+\delta} \ , \end{aligned}$$

and this in turn implies that

$$v_1^* = x_1 = \frac{1-\delta^T}{1+\delta} \ , \quad \text{and} \quad v_2^* = (1-x_1) = \frac{\delta+\delta^T}{1+\delta} \ .$$

■

4. **Constant Delay Cost:** Consider a two player alternating bargaining game where instead of the pie shrinking by a discount factor $\delta < 1$, the players each pay a cost $c_i > 0$, $i \in \{1,2\}$ to advance from one period to another. So, if player i receives a share of the pie that gives him a value of x_i in period t then his payoff is $v_i = x_i - (t-1)c_i$. If the game has T periods then a sequence of rejections results in each player receiving $v_i = -(T-1)c_i$.

   (a) Assume that $T = 2$. Find the subgame perfect equilibrium of the game and show in which way it depends on the values of c_1 and c_2.

   Answer: In the last period player 2 makes the offer in an ultimatum game and will offer to keep the whole pie: $x_1 = 0$ and $x_2 = 1$, and player 1 is will accept (he's indifferent). Payoffs would be $v_1 = -c_1$ and $v_2 = 1 - c_2$. Going backwards to period 1, player 1 has to offer at least $x_2 = 1 - c_2$ to player 2 for him to accept, so the unique subgame perfect equilibrium has player 1 offering $1 - c_2$ to player 2, and player 2 accepts anticipating that he will offer and get $x_2 = 1$ in the second period following rejection. Payoffs are $v_1 = c_2$ and $v_2 = 1 - c_2$. Payoffs therefore do not depend on c_1. ■

   (b) Are there Nash equilibria in the two period game that are not subgame perfect?

   Answer: Yes. Just like in the game we studied with a discount factor δ, any split can be supported by a Nash equilibrium. Consider the following strategy by player 2: reject anything but the whole pie in the first period and offer to keep the whole pie in the second. Player 1's best response in the first period is to offer exactly the whole pie to player 2 because that way he is guaranteed 0, while if he believes that player 2 will follow the proposed strategy and he offers anything else then he will get $v_1 = -c_1$. ■

(c) Assume that $T = 3$. Find the subgame perfect equilibrium of the game and show in which way it depends on the values of c_1 and c_2.

Answer: In the last period player 1 makes the offer in an ultimatum game and will offer to keep the whole pie: $x_1 = 1$ and $x_2 = 0$, and player 2 is will accept (he's indifferent). Payoffs would be $v_1 = 1 - 2c_1$ and $v_2 = -2c_2$. Going backwards to period 2, player 2 has to offer at least $x_1 = 1 - 2c_1$ to player 1 for him to accept, so the payoffs starting from the second period are $v_1 = 1 - 3c_1$ and $v_2 = 2c_1 - c_2$ (player 1 gets a piece of the pie equal to $1 - 2c_1$ and because this is the second period he incurs the cost c_1 from the first period.) Finally, in period 1 player 1 must offer player 2 at least $2c_1 - c_2$ so he will offer exactly that, player 2 will accept the offer, and the payoffs will be $v_1 = 1 - 2c_1 + c_2$ and $v_2 = 2c_1 - c_2$. ■

5. **Asymmetric Patience 1:** Consider a 3-period sequential (alternating) bargaining model where two players have to split a pie worth 1 (starting with player 1 making the offer). Now the players have different discount factors, δ_1 and δ_2.

(a) Compute the outcome of the unique subgame perfect equilibrium.

Answer: In the third period player 1 will get the whole pie and hence the payoffs will be $v_1 = \delta_1^2$ and $v_2 = 0$. Moving back to the second period, player 2 will offer player 1 δ_1 and player 1 will accept, so the payoffs are $v_1 = \delta_1^2$ and $v_2 = \delta_2(1-\delta_1)$. Moving back to the first period, player 1 will offer to keep x such that player 2 will receive $v_2 = (1 - x) = \delta_2(1 - \delta_1)$ implying that player 1 gets $v_1 = x = 1 - \delta_2(1 - \delta_1) = 1 - \delta_2 + \delta_1\delta_2$. ■

(b) Show that when $\delta_1 = \delta_2$ then player 1 has an advantage.

Answer: In this case $v_1 = 1 - \delta_2 + \delta_1\delta_2 = 1 - \delta + \delta^2$ and $v_2 = \delta - \delta^2$, implying that

$$v_1 - v_2 = 1 - \delta + \delta^2 - (\delta - \delta^2) = 1 - 2\delta + 2\delta^2 = (1-\delta)^2 + \delta^2 > 0$$

implying that $v_1 > v_2$. ■

(c) What conditions on δ_1 and δ_2 give player 2 an advantage? Why?

Answer: For player 2 to get an advantage it must be that $v_1 < v_2$ which implies using the answer in part **a.** above that $1 - \delta_2 + \delta_1\delta_2 < \delta_2 - \delta_1\delta_2$ or $1 < 2\delta_2(1-\delta_1)$. This condition means that δ_2 has to be significantly greater than δ_1, meaning that player 2 has to be significantly more patient for him to have an advantage. For example, if δ_2 is very close to 1, then δ_1 has to be less than $\frac{1}{2}$ for this condition to hold, and if $\delta_2 < \frac{1}{2}$ then player 2 will never have an advantage. The patience has to overcome the first and last mover advantage that player 1 has in this case. ■

6. **Asymmetric Patience 2:** Consider the analysis of the infinite horizon bargaining model in section 11.3 and assume that the players have different discount factors δ_1 and δ_2. Find the unique subgame perfect equilibrium using the same techniques, and show that as δ_1 and δ_2 become closer in values, the solution you found converges to the solution derived in section 11.3.

Answer: Consider a subgame in which player 1 makes the offer. Player 2 will not accept an offer that gives him less than $\delta_2\underline{v}_2$, implying that

$$\overline{v}_1 \leq 1 - \delta_2\underline{v}_2 \ , \tag{11.1}$$

and player 2 will accept an offer that gives him at least than $\delta_2\overline{v}_2$, implying that

$$\underline{v}_1 \geq 1 - \delta_2\overline{v}_2 \ . \tag{11.2}$$

By symmetry, when player 2 makes the offer we obtain the symmetric inequalities,

$$\overline{v}_2 \leq 1 - \delta_1\underline{v}_1 \ , \tag{11.3}$$

and

$$\underline{v}_2 \geq 1 - \delta_1 \overline{v}_1 \ . \tag{11.4}$$

Subtracting (11.2) from (11.1) yields

$$\overline{v}_1 - \underline{v}_1 \leq \delta_2(\overline{v}_2 - \underline{v}_2) \ , \tag{11.5}$$

and similarly, subtracting (11.4) from (11.3) yields

$$\overline{v}_2 - \underline{v}_2 \leq \delta_1(\overline{v}_1 - \underline{v}_1) \ , \tag{11.6}$$

But (11.5) and (11.6) together imply that

$$\overline{v}_1 - \underline{v}_1 \leq \delta_2(\overline{v}_2 - \underline{v}_2) \leq \delta_2\delta_1(\overline{v}_1 - \underline{v}_1) \ ,$$

and because $\delta_2\delta_1 < 1$ it follows that $\overline{v}_1 = \underline{v}_1 (= v_1)$ and $\overline{v}_2 = \underline{v}_2 (= v_2)$. Revisiting the inequalities above, (11.1) and (11.2) imply that

$$v_1 = 1 - \delta_2 v_2 \ ,$$

and (11.3) and (11.4) imply that

$$v_2 = 1 - \delta_1 v_1 \ ,$$

and from these last two equalities we obtain that in the unique subgame perfect equilibrium, in the first period player 1 receives

$$v_1^* = \frac{1 - \delta_2}{1 - \delta_1\delta_2} \ .$$

and player 2 receives $1 - v_1^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$. Now let $\delta_1 = \delta$, and let δ_2 approach δ. The denominator approaches $1-\delta^2 = (1-\delta)(1+\delta)$ and we get that $v_1^* = \frac{1}{1+\delta}$, which is the solution we obtained is section 11.3 for a symmetric discount factor. ■

7. **Legislative Bargaining (revisited):** Consider a finite T period version of the Baron and Ferejohn legislative bargaining game with an odd number N of players and with a closed rule as described in section 11.4.1.

(a) Find the unique subgame perfect equilibrium for $T = 1$. Also, find a Nash equilibrium that is not subgame perfect.

Answer: If $T = 1$ then following a failed vote (a majority rejects the proposer's proposal) all the players receive a payoff of 0. Hence, like in the Rubinstein game, the proposer will ask for all the surplus and a majority of players will vote in favor. No other outcome can be supported by a subgame perfect equilibrium. There are many Nash equilibria. For example, some player j asks for at least $x_j^* \in [0,1]$ of the surplus while all other players will settle for nothing. Then any player $i \neq j$ will offer j the amount x_j^*, and nothing to the other players, and all the players will vote in favor of the proposal. ■

(b) Find the unique subgame perfect equilibrium for $T = 2$ with a discount factor $0 < \delta \leq 1$. Also, find a Nash equilibrium that is not subgame perfect.

Answer: If the proposal is not accepted in period 1 then period 2 will have the unique subgame perfect equilibrium described in part **a.** above. This implies that in the first period, every player has an expected surplus of $\frac{\delta}{N}$ because they will be the proposer with probability $\frac{1}{N}$ and will get the whole surplus of 1. This means that the player who offers in the first period must offer at least $\frac{\delta}{N}$ to $\frac{N-1}{2}$ other players to form a majority and have the proposal accepted. Hence, the proposing player will keep $1-\frac{N-1}{2}\frac{\delta}{N}$ to himself in the unique subgame perfect equilibrium. Just like in part **a.** above, we can support an arbitrary division of the surplus in a Nash equilibrium by having some players commit to incredible strategies. ■

(c) Compare what the first period's proposer receives in the subgame perfect equilibrium you found in part (b) above to what a first period proposer receives in the two-period two-person Rubinstein-Ståhl bargaining game. What intuitively accounts for the difference?

Answer: In the two-period two-person Rubinstein-Ståhl bargaining game the proposing player 1 gets $1 - \delta$ because player 2 can get the whole pie in the second period. Notice that the difference between the payoff in the Baron-Ferejohn model and the Rubinstein-Ståhl model is,

$$1 - \frac{N-1}{2}\frac{\delta}{N} - (1-\delta) = \frac{\delta(N+1)}{2N} > \frac{\delta}{2}\ .$$

As we can see, the first proposer has a lot more surplus in the Baron-Ferejohn model. This is because the responder is not one player who plays an ultimatum game in the second period, but a group of player from which a majority needs to be selected. This lets the proposer pit the responders against each other and capture more surplus. ■

(d) Compare the subgame perfect equilibrium you found in part (b) above to the solution of the infinite horizon model in section **??**. What intuitively accounts for the similarity?

Answer: The share received by the first proposer is the same as what we derived in equation (11.8). The intuition is that the same forces are at work: the larger the discount factor the more the proposer needs to give away, and the more people there are, the more he has to give away. Still, he gets to keep at least $\frac{1}{2}$ because of the competitive nature of the situation in which the responders are put. ■