Automata Theory - Midterm (Solutions)

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1 Problems

1. Consider the $\epsilon - NFA$ defined below:

	ϵ	a	b	c
$\rightarrow p$	ϕ	$\{p\}$	$\{q\}$	$\{r\}$
q	$\{p\}$	$\{q\}$	$\{r\}$	ϕ
$*r$	$\{q\}$	$\{r\}$	ϕ	$\{p\}$

   (a) Compute the ϵ-closure of each state. (3 points)
   Solution:

$$\begin{aligned} \epsilon - closure(p) &= \{p\} \\ \epsilon - closure(q) &= \{p, q\} \\ \epsilon - closure(r) &= \{p, q, r\} \end{aligned}$$

   □

   (b) Convert the automaton to a DFA. (4 points)
   Solution: □

	a	b	c
$\rightarrow \{p\}$	$\{p\}$	$\{p, q\}$	$\{p, q, r\}$
$\{p, q\}$	$\{p, q\}$	$\{p, q, r\}$	$\{p, q, r\}$
$*\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$

2. Let $\Sigma = \{a, b, c\}$. Write a regular expression for the language consisting of the set of strings containing at least one a and at least one b. (4 points)

   Solution: Observe that the simplest approach is to consider those strings in which the first a precedes the first b separately from those where the opposite occurs. The regular expression is:
   $c^*a(a+c)^*b(a+b+c)^* + c^*b(b+c)^*a(a+b+c)^*$. □

3. Let $\Sigma = \{0, 1\}$. Which of the following languages is regular? Provide an explanation in each case. (6 points)

   (a) $L = \{0^n 1^m \mid n \leq m,\ n, m \geq 0\}$
   Proof:

i. Player 1 picks the language L to be proved nonregular, where $L = \{0^n1^m | n \leq m, n, m \geq 0\}$.
ii. Player 2 picks n.
iii. Player 1 picks $w = 0^n1^{n+1}$.
iv. Player 2 breaks w into xyz, in which $y \neq \epsilon$ and $|xy| \leq n$.
v. Player 1 wins. Since $|xy| \leq n$ and xy comes at the front of w, we know that x and y consist of only 0's. Thus, $y = 0^k$ for $0 < k \leq n$, since $y \neq \epsilon$. The Pumping Lemma tells us that xy^kz is in L if L is regular. If we choose $k = 2$, the resulting string is $w' = 0^{n+2}1^{n+1}$. Clearly w' is not in L. Therefore, we have contradicted our assumption that L is regular.

□

(b) $L = \{0^n1^m |\ n \geq m,\ n, m \geq 0\}$

Proof:

i. Player 1 picks the language L to be proved nonregular, where $L = \{0^n1^m | n \geq m, n, m \geq 0\}$.
ii. Player 2 picks n.
iii. Player 1 picks $w = 0^n1^n$.
iv. Player 2 breaks w into xyz, in which $y \neq \epsilon$ and $|xy| \leq n$.
v. Player 1 wins. We know that $|xy| \leq n$ and $y \neq \epsilon$. Since xy comes at the front of w, we know that x and y consist of only 0's, and that y must contain at least one 0. The Pumping Lemma tells us that xz is in L if L is regular, however, xz has n 1's, since all of the 1's of w are in z. However, xz also has fewer than n 0's, because we have lost the 0's of y. Since $y \neq \epsilon$, we know that there can be no more than $n - 1$ 0's among x and z. We have assumed L to be a regular language, but have proved that xz is not in L. Therefore, we have contradicted our assumption that L is regular.

□

(c) $L = \{0^n1^m |\ n, m \geq 0\}$

Solution: Observe that the following regular expression 0^*1^* corresponds to L. Since we can write a regular expression for L, we know that L is regular. □

4. Let $\Sigma = \{0, 1\}$. Let L be the language that consists of strings having either 01 repeated one or more times *or* 010 repeated one or more times. Is L regular? Explain. (4 points)

   Solution: Observe that L can be written as the following regular expression $((0 + 1)^*01(0 + 1)^*01(0 + 1)^*) + ((0 + 1)^*010(0 + 1)^*010(0 + 1)^*)$. Since we are able to write L as a regular expression, we know that L is regular. (Note each pattern must occur twice in order to be repeated once!) □

5. Assume that a regular language L is provided to you as a DFA $\mathbf{A} =< Q, \Sigma, \delta, q_0, F >$. How would you check whether L is infinite? (5 points).
   Hint: Pumping Lemma.

   Proof: Let n be the Pumping Lemma constant. Test all strings of length between n and $2 \cdot n - 1$ for membership in L. If we find even one string, then L is infinite. The reason is that the Pumping Lemma applies to such a string, and it can be "pumped" to show an infinite sequence of strings are in L.
   Suppose, however, that there are no strings in L whose length is in the range n to $2 \cdot n - 1$. We claim that there are no strings in L of length $2 \cdot n$ or more, and thus there are only a finite number of strings in L.
   Suppose w is a string in L of length at least $2 \cdot n$, and w is as short as any string in L that has length at least $2 \cdot n$. Then the Pumping Lemma applies to w, and we can write $w = xyz$, where xz is also in L. How long could xz be? It can't be as long as $2 \cdot n$, because it is shorter than w, and w is as short as any string in L of length $2 \cdot n$ or more. Secondly, $|z| \geq n$ and hence $|xz| \geq n$. Thus, xz is of length between n and $2 \cdot n - 1$, which is a contradiction, since we assumed that there were no strings in L with a length in that range. □

6. Let $\Sigma = \{0, 1\}$. We showed in class that the language $L = \{0^n1^n |\ n \geq 0\}$ is not regular. Argue using closure properties of regularity, that $L' = \{0^i1^j |\ i \neq j\}$ is not regular. (4 points)

Proof: Start out by complementing the language L'; the resulting language is the language consisting of all strings of 0's and 1's that are not in 0^*1^*, plus the strings in L. Now, if we intersect the complement of L' with 0^*1^*, the result is precisely the language L. Since complementation and intersection with a regular set preserve regularity, if the given language were regular, then so would be L. We already know that L is not regular, therefore, we can conclude that the given language L' is not regular. $\square$