Automata Theory - Scrimmage I (Solutions)

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1 Problems

1. Prove using Mathematical Induction:

$$\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$$

Proof: Base case $P(1)$:

$$\begin{aligned} LHS &= \sum_{i=1}^{1} i^3 \\ &= 1^3 \\ &= 1 \\ RHS &= (\sum_{i=1}^{1} i)^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

Thus, $LHS = RHS$ and $P(1)$ is true.
Let us assume that $P(k)$ is true, i.e.

$$\sum_{i=1}^{k} i^3 = (\sum_{i=1}^{k} i)^2$$

We need to show that $P(k+1)$ is true.

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^3 \\ &= 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 \\ &= \sum_{i=1}^{k} i^3 + (k+1)^3 \end{aligned}$$

$$
\begin{aligned}
&= (\sum_{i=1}^{k} i)^2 + (k+1)^3 \quad (using \;\; the \;\; inductive \;\; hypothesis) \\
&= (\frac{k \cdot (k+1)}{2})^2 + (k+1)^3 \\
RHS &= (\sum_{i=1}^{k+1} i)^2 \\
&= (1 + 2 + 3 + \ldots + k + (k+1))^2 \\
&= (\sum_{i=1}^{k} i + (k+1))^2 \\
&= (\frac{k \cdot (k+1)}{2} + (k+1))^2 \\
&= (\frac{k \cdot (k+1)}{2})^2 + k \cdot (k+1)^2 + (k+1)^2 \\
&= (\frac{k \cdot (k+1)}{2})^2 + (k+1)^3
\end{aligned}
$$

$LHS{=}RHS$. Thus, we have shown that $P(k) \rightarrow P(k{+}1)$; applying the principle of mathematical induction, we conclude that the conjecture is true. □

2. Let $\Sigma = \{0, 1\}$. Draw a DFA for the language L containing strings having the pattern 010 in them.

   Solution: □

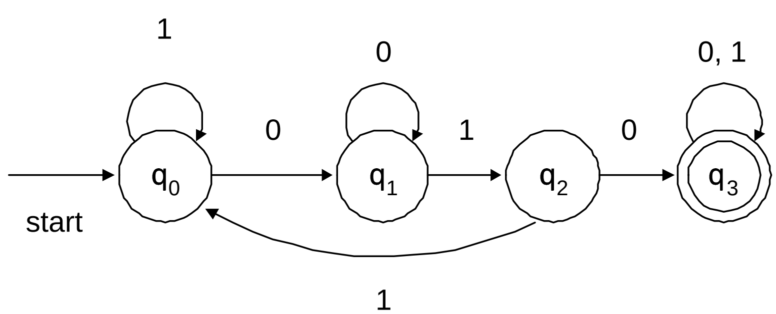

Figure 1: DFA

3. Let $\Sigma = \{0, 1\}$. Draw a NFA for the language L consisting of strings in which the final digit has appeared before.

Solution: □

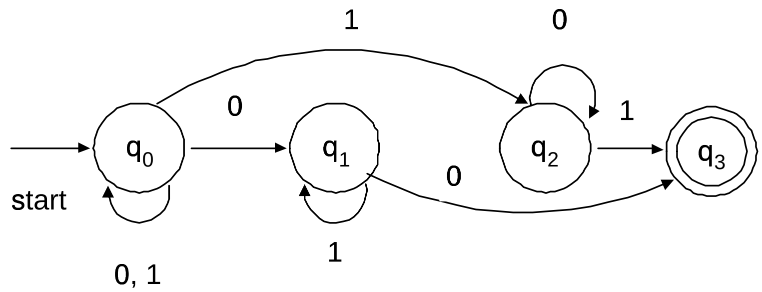

Figure 2: NFA

4. Repeat the above problem for strings in which the final digit has not appeared before.

   Solution: Observe that the language L consists only of $\{\epsilon, 0, 1, 01, 10\}$. Note that in case of NFAs, switching accepting and non-accepting states *does not* necessarily result in an NFA accepting the complement language. The switching technique works only for DFAs. Secondly, the string ϵ may be part of L or $\bar{L}$, depending upon the definition of L. In this case, the definition did not specify whether $\epsilon \in L$; we chose to make ϵ part of $\bar{L}$.

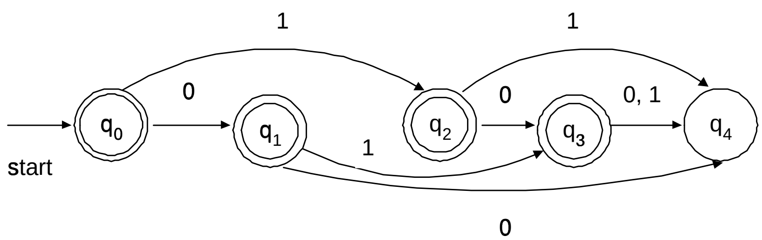

Figure 3: NFA

□

5. Let $\Sigma = \{0, 1\}$. Let $L \subseteq \Sigma^*$ represent the language of those strings that do not contain the pattern 101. Argue that L is regular.

   Solution: Let L' be the language that consists of only those strings containing the pattern 101. It is clear that $L' = L((0+1)^*101(0+1)^*)$. Since L' is regular, it follows that L is regular, since L is the complement of L' and regular languages are closed under complementation. □

6. Argue that $(R^*)^* = R^*$, for any regular expression R.

   Proof: Observe that by the concretization theorem (Theorem 3.13 in [HMU01]), in order to show that $(R^*)^* = R^*$, all we really need to show is $(a^*)^* = a^*$, where a is the concrete symbol, replacing the regular expression R. It is not hard to see that both $(a^*)^*$ and a^* represent all possible strings of a's and are therefore identical. □

7. Is $L = \{0^n10^n | \, n \geq 1\}$ regular? Explain.

   Proof: Let n be the pumping-lemma constant. Choose $w = 0^n10^n$, then write $w = xyz$. We know that $|xy| \leq n$, and $|y| \geq 1$ thus, y consists of 0's only. Thus the z part of xyz contains the 1 and n zeros after the 1. As per the Pumping Lemma, xz must be in L. But x contains strictly less than n zeros and cannot be in L, as per the definition of L. It follows that L is not regular. □

8. Let $\Sigma = \{0, 1, 2\}$ and $\tau = \{a, b\}$. Consider the homomorphism h defined by $h(0) = a$, $h(1) = ab$ and $h(2) = ba$. Let L be the unit string language $\{ababa\}$. What is $h^{-1}(L)$?

   Solution: Observe that each b must come from either 1 or 2. However, if the first b comes from 2 and the second comes from 1, then they will both need the a between them as part of $h(2)$ and $h(1)$, respectively. Therefore, $h^{-1}(L)$ consists of the strings $\{110, 102, 022\}$. (Clearly $h(110) = h(102) = h(022) = ababa$. Make sure that no other string in Σ^* is mapped by $h()$ onto $ababa$!) □

References

[HMU01] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *"Introduction to Automata Theory, Language, and Computation"*. Addison–Wesley, 2nd edition edition, 2001.