An Introduction to First-Order Logic

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Completeness, Compactness and Inexpressibility

Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof
2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability

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   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability

Outline

Soundness and Completeness

Theorem

Soundness: If $\Delta \vdash \phi$, then $\Delta \models \phi$.

Theorem

Completeness (Gödel's traditional form): If $\Delta \models \phi$, then $\Delta \vdash \phi$.

Theorem

Completeness (Gödel's altenate form): If Δ is consistent, then it has a model.

Soundness and Completeness (contd.)

Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model M that satisfies all the expressions in Δ, also satisfies ϕ and hence falsifies $\neg\phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg\phi\}$. It follows that $\Delta \cup \{\neg\phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \vdash \phi$. □

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Outline

Proof Sketch of Completeness Theorem

Proof.

http://www.maths.bris.ac.uk/~rp3959/firstordcomp.pdf □

Outline

Validity

Theorem

VALIDITY *is Recursively enumerable.*

Proof.

Follows instantaneously from the completeness theorem. □

Outline

Compactness

Theorem

If all finite subsets of a set of sentences Δ are satisfiable, then so is Δ.

Proof.

Assume that Δ is unsatisfiable, but all finite subsets of Δ are satisfiable. As per the completeness theorem, there is a proof of a contradiction from Δ, say $\Delta \vdash \phi \wedge \neg\phi$. However, this proof has finite length! Therefore, it can involve only a finite subset of Δ! □

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Model Size

Theorem

If a sentence has a model, it has a countable model.

Proof.

The model M constructed in the proof of the completeness theorem is countable, since the corresponding vocabulary is countable. □

Outline

Query

Do all sentences have infinite models?

Theorem

If a sentence ϕ has finite models of arbitrary large cardinality, then it has an infinite model.

Proof.

Consider the sentence $\psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \wedge_{1 \leq i < j \leq k} \neg(x_i = x_j)$. ψ_k cannot be satisfied with a model having less than k elements.

Assume that ϕ has arbitrarily large models, but no infinite models. Let $\Delta = \phi \cup \{\psi_k : k = 2, 3, \ldots\}$. If Δ has a model M, M can neither be finite nor infinite. Thus, Δ does not have a model. . By the compactness theorem, a finite subset $D \subset \Delta$ does not have a model. ϕ must be in D. Let k denote the largest integer, such that $\psi_k \in D$. But there is a large enough model that satisfies both ϕ (hypothesis) and ψ_k! □

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Reachability

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Given a directed graph G and two nodes x and y in G, is there a directed path from x to y in G?

Theorem

There is no first-order expression ϕ with two free variables x and y, such that ϕ-Graphs expresses REACHABILITY.

Proof.

Assume that there exists such a ϕ. Consider the sentence, $\psi' = \psi_0 \land \psi_1 \land \psi_2$, where,

$$\begin{aligned}
\psi_0 &= (\forall x)(\forall y)\phi \\
\psi_1 &= (\forall x)(\exists y)G(x,y) \land (\forall x)(\forall y)(\forall z)((G(x,y) \land G(x,z)) \rightarrow (y = z)) \\
\psi_2 &= (\forall x)(\exists y)G(y,x) \land (\forall x)(\forall y)(\forall z)((G(y,x) \land G(z,x)) \rightarrow (y = z))
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Arbitrarily large models are possible for ψ', but no infinite models!

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