Introduction to Second-Order Logic

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Expressibility and Complexity

Outline

1. Second Order Logic
   - Definition
   - Examples
2. Existential second-order expressions over graph theory's vocabulary
   - The general case
   - The Horn case

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Rudiments

Definition

An expression of existential second-order logic over a vocabulary $\Sigma = (\Phi, \Pi, r)$ is of the form $\exists P \phi$, where ϕ is a first-order expression over $\Sigma' = ((\Phi, \Pi \cup \{P\}, r)$. $P \notin \Pi$ is a new relational symbol with arity $r(P)$.

Semantics

A model M appropriate to Σ, satisfies $\exists P \phi$, if there exists a relation $P^M \subseteq (U^M)^{r(P)}$, such that M augmented with P^M, together comprise a model (that is appropriate to Σ' and satisfies ϕ.

Fact

In Second-order logic, variables can be quantified over both terms **and** *relations.*

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Examples

Even Numbers

$\phi = \exists P(\forall x)((P(x) \vee P(x+1)) \wedge \neg(P(x) \wedge P(x+1)))$.

The above sentence is satisfied by the traditional model of Number theory, by setting $P^N = \{\text{even numbers}\}$.

Subgraph

$\phi = \exists P(\forall x)(\forall y)(P(x,y) \rightarrow G(x,y))$.

The above sentence can be satisfied by the model for graph theory, in which P represents the subgraph relation.

Unreachability

$\phi(x,y) = \exists P((\forall u)(\forall v)(\forall w)((P(u,u)) \wedge (G(u,v) \rightarrow P(u,v)) \wedge ((P(u,v) \wedge P(v,w)) \rightarrow P(u,w)) \wedge \neg P(x,y)))$.

The first conjunct of the sentence specifies that P is the reflexive and transitive closure of G, while the second conjunct specifies that x and y should not be related under P.

Examples

Even Numbers

$\phi = \exists P(\forall x)((P(x) \lor P(x+1)) \land \neg(P(x) \land P(x+1)))$.

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Examples (contd.)

Hamilton Path

$$
\begin{aligned}
\phi &= \exists P(\psi_1 \wedge \psi_2 \wedge \psi_3), \text{ where,} \\
\psi_1 &= (\forall x)(\forall y)(P(x,y) \vee P(y,x) \vee (x = y)) \\
\psi_2 &= (\forall x)(\forall y)(\forall z)((\neg P(x,x)) \wedge ((P(x,y) \wedge P(y,z)) \rightarrow P(x,z)) \\
\psi_3 &= (\forall x)(\forall y)(((P(x,y) \wedge (\forall z)(\neg P(x,z) \vee \neg P(z,y))) \rightarrow G(x,y))
\end{aligned}
$$

ϕ_1 specifies that either there is a path from x to y or a path from y to x or $x = y$. This establishes an ordering on the vertices.
ϕ_2 specifies that P is transitive but not reflexive.

ϕ_3 specifies that if there is a path from x to y and there is no vertex z, such that there is a path from x to z and a path from z to y, then it must be the case that the path from x to y is an edge of G.

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Complexity of existential second-order expression

Theorem

For any existential second-order expression $\exists P\phi$, the problem $\exists P\phi$-GRAPHS is in NP.

Proof.

Let $\mathbf{G} = \langle V, E \rangle$ have n nodes. The NDTM can guess a relation $P^M \subseteq V^{r(P)}$; note that $|V^{r(P)}| = n^{r(P)}$, which is polynomial. It then tests whether M satisfies the first-order expression ϕ, using the recursive decomposition technique developed before. □

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Theorem

For any Horn existential second-order expression $\exists P\phi$, the problem $\exists P\phi$-GRAPHS is in $\mathbf{P}$.

Proof.

Let $\exists P\phi = \exists P(\forall x_1)(\forall x_2)\ldots(\forall x_k)\eta$, where η is a Horn system with h clauses and $r(P) = r$. Assume that the instance of the graph problem has n vertices $\{v_1, v_2, \ldots, v_n\}$; thus the computational problem is asking whether there is a subset of $\{1, 2, \ldots, n\}^r$, such that ϕ is satisfied.

Idea 1: Since η must hold for all values of the x_is, generate sub-expressions corresponding to each unique substitution pattern. The total number of expressions generated is $h \cdot n^k$.

Idea 2: Each atomic expression is either $G(v_i, v_j)$ or $(v_i = v_j)$ or $P(v_{i_1}, v_{i_2}, \ldots, v_{i_r})$. The first two can be easily disposed of. The last can be solved as a Boolean Horn expression! □

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