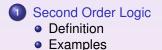
Introduction to Second-Order Logic

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Expressibility and Complexity

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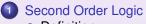


Existential second-order expressions over graph theory's vocabulary

- The general case
- The Horn case

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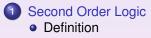
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Examples

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Rudiments

An expression of existential second-order logic over a vocabulary $\Sigma = (\Phi, \Pi, r)$ is of the form $\exists P\phi$, where ϕ is a first-order expression over $\Sigma' = ((\Phi, \Pi \cup \{P\}, r). P \notin \Pi$ is a new relational symbol with arity r(P).

Definition

Examples

Semantics

A model *M* appropriate to Σ , satisfies $\exists P\phi$, if there exists a relation $P^M \subseteq (U^M)^{r(P)}$, such that *M* augmented with P^M , together comprise a model (that is appropriate to Σ' and satisfies ϕ .

Fact

In Second-order logic, variables can be quantified over both terms and relations.

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Examples

Even Numbers

$$\phi = \exists P(\forall x)((P(x) \lor P(x+1)) \land \neg (P(x) \land P(x+1))).$$

The above sentence is satisfied by the traditional model of Number theory, by setting $P^{N} = \{\text{even numbers}\}.$

Subgraph

 $\phi = \exists P(\forall x)(\forall y)(P(x,y) \to G(x,y)).$

The above sentence can be satisfied by the model for graph theory, in which *P* represents the subgraph relation.

Jnreachability

 $\begin{array}{l} \phi(x,y) = \exists P((\forall u)(\forall v)(\forall w)((P(u,u)) \land (G(u,v) \rightarrow P(u,v)) \land ((P(u,v) \land P(v,w)) \rightarrow P(u,w)) \land \neg P(x,y))). \end{array}$

The first conjunct of the sentence specifies that P is the reflexive and transitive closure of G, while the second conjunct specifies that x and y should not be related under P.

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Examples (contd.)

Hamilton Path

$$\phi = \exists P(\psi_1 \land \psi_2 \land \psi_3), \text{ where,}$$

$$\psi_1 \quad = \quad (\forall x)(\forall y)(P(x,y) \lor P(y,x) \lor (x=y))$$

$$\psi_2 = (\forall x)(\forall y)(\forall z)((\neg P(x,x)) \land ((P(x,y) \land P(y,z)) \to P(x,z))$$

$$\psi_3 \quad = \quad (\forall x)(\forall y)(((P(x,y) \land (\forall z)(\neg P(x,z) \lor \neg P(z,y))) \to G(x,y))$$

 ϕ_1 specifies that either there is a path from *x* to *y* or a path from *y* to *x* or *x* = *y*. This establishes an ordering on the vertices.

 ϕ_2 specifies that *P* is transitive but not reflexive.

 ϕ_3 specifies that if there is a path from x to y and there is no vertex z, such that there is a path from

x to z and a path from z to y, then it must be the case that the path from x to y is an edge of G.

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Complexity of existential second-order expression

Theorem

For any existential second-order expression $\exists P\phi$, the problem $\exists P\phi$ -GRAPHS is in NP.

Proof.

Let $\mathbf{G} = \langle V, E \rangle$ have *n* nodes. The NDTM can guess a relation $P^M \subseteq V^{r(P)}$; note that $|V^{r(P)}| = n^{r(P)}$, which is polynomial. It then tests whether *M* satisfies the first-order expression ϕ , using the recursive decomposition technique developed before.

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Theorem

For any Horn existential second-order expression $\exists P\phi$, the problem $\exists P\phi$ -GRAPHS is in P.

Proof.

Let $\exists P\phi = \exists P(\forall x_1)(\forall x_2) \dots (\forall x_k)\eta$, where η is a Horn system with h clauses and r(P) = r. Assume that the instance of the graph problem has n vertices $\{v_1, v_2, \dots, v_n\}$; thus the computational problem is asking whether there is a subset of $\{1, 2, \dots, n\}^r$, such that ϕ is satisfied.

Idea 1: Since η must hold for all values of the x_i s, generate sub-expressions corresponding to each unique substitution pattern. The total number of expressions generated is $h \cdot n^k$.

Idea 2: Each atomic expression is either $G(v_i, v_j)$ or $(v_i = v_j)$ or $P(v_{i_1}, v_{i_2}, \ldots, v_{i_r})$. The first two

can be easily disposed of. The last can be solved as a Boolean Horn expression!

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