An Introduction to First-Order Logic

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Axioms, Proofs and Theoremhood

Outline

1. Axioms and Proofs
   - Notion of truth
   - First-order theorems
   - Theoremhood and Validity

2. Model-specific theorems
   - Definition of model-specific theorems
   - Three fundamental techniques
   - The Soundness Theorem

Outline

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Axioms and Proofs

Goal

A systematic procedure to reveal truth, where truth coincides with the notion of validity.

Logical Axioms (Fundamental valid expressions)

(i) Any expression whose Boolean form is a tautology.
(ii) Any expression deemed valid by the rules of equality.
(iii) Any expression deemed valid by the rules of quantification.

The above set of logical axioms is denoted by Λ.

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Definition

Let $S = (\phi_1, \phi_2, \ldots, \phi_n)$ denote a set of first-order expressions, such that for each ϕ_i, $1 \leq i \leq n$ in the sequence, either $\phi_i \in \Lambda$ or there are two expressions of the form ψ, $\psi \rightarrow \phi_i$, among the expressions $\phi_1, \phi_2, \ldots, \phi_{i-1}$. Then, S is a proof of expression ϕ_n. ϕ_n, in turn, is called a first-order theorem and this is denoted by $\vdash \phi_n$.

Proof of Symmetry $(x = y) \rightarrow (y = x)$

(i) $\phi_1 = [(x = y) \wedge (x = x)] \rightarrow [(x = x) \rightarrow (y = x)]$, properties of equality

(ii) $\phi_2 = (x = x)$, properties of equality

(iii) $\phi_3 = [(x = x)] \rightarrow [(((x = y) \wedge (x = x)) \rightarrow ((x = x) \rightarrow (y = x))) \rightarrow (x = y) \rightarrow (y = x)]$, boolean tautology

(iv) $\phi_4 = [((x = y) \wedge (x = x)) \rightarrow ((x = x) \rightarrow (y = x))] \rightarrow [(x = y) \rightarrow (y = x)]$, from ϕ_2 and ϕ_3, using Modus Ponens.

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Two fundamental questions

Theoremhood

Given a formula ϕ, is it a theorem?

Theorem

Theoremhood is recursively enumerable.

Validity

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Fact

$\models \phi \Leftrightarrow \vdash \phi$

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Axiomatic Method

If $M \models \phi_0$ and $M \models \phi \quad \leftrightarrow \quad \models \phi_0 \rightarrow \phi$, then $M \models \phi$.

Example

A group is a set S and a binary operator $\circ$, such that

(i) $(\forall x)(\forall y)(\forall z)((x \circ y) \circ z) = (x \circ (y \circ z))$.
(ii) $(\forall x)(x \circ 1) = x$.
(iii) $(\forall x)(\exists y)(x \circ y = 1)$.

The above axioms are called the non-logical axioms of the theory.

Definition

An expression ϕ is a valid consequence of a set of expressions Δ, written $\Delta \models \phi$, if every model that satisfies Δ also satisfies ϕ.

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Defining model-specific theorems

Definition

Let Δ denote a set of expressions. Let $S = (\phi_1, \phi_2, \ldots, \phi_n)$ denote finite sequence of first-order expressions, such that for each ϕ_i, $i \leq i \leq n$, one of the following holds:

(a) $\phi_i \in \Lambda$,

(b) $\phi_i \in \Delta$,

(c) there are two expressions of the form ψ, $\psi \rightarrow \phi_i$, among the expressions $\phi_1, \phi_2, \ldots, \phi_{i-1}$.

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Derivation of model-specific theorems

Theorem

The Deduction Technique: Suppose that $\Delta \cup \phi \vdash \psi$*; then* $\Delta \vdash (\phi \rightarrow \psi)$.

Theorem

The Contradiction Technique: If $\Delta \cup \{\neg\phi\}$ *is inconsistent, then* $\Delta \vdash \phi$.

Theorem

Justified Generalization: Suppose that $\Delta \vdash \phi$ *and* x *is not free in any expression of* Δ*. Then* $\Delta \vdash (\forall x)\phi$.

Example of applying the derivation theorems

Example

Show that $(\forall x)\phi \rightarrow (\exists x)\phi$ is a theorem, i.e., show that $\vdash (\forall x)\phi \rightarrow (\exists x)\phi$.

Proof.

(i) $\phi_1 = (\forall x)\phi$, hypothesis.

(ii) $\phi_2 = (\forall x)\phi \rightarrow \phi$, logical axiom arising from properties of quantifiers.

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Soundness of the proof system

Theorem

Soundness: *If $\Delta \vdash \phi$, then $\Delta \models \phi$.*

Proof.

Let $S = (\phi_1, \phi_2, \ldots, \phi_n)$, $\phi_n = \phi$, denote a proof of ϕ from Δ. We will show that $\Delta \models \phi_i$, for each $i = 1, 2, \ldots, n$. If ϕ_i is a logical or non-logical axiom, then clearly $\Delta \models \phi_i$. Assume that ϕ_i is obtained using Modus Ponens from ϕ_j and $\phi_j \rightarrow \phi_i, j < i$. By the inductive hypothesis, $\Delta \models \phi_j$ and $\Delta \models \phi_j \rightarrow \phi_i$. Thus, any model that satisfied Δ also satisfies ϕ_j and $\phi_j \rightarrow \phi_i$. It follows that $\Delta \models \phi_i$. □

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Soundness: *If $\Delta \vdash \phi$, then $\Delta \models \phi$.*

Proof.

Let $S = (\phi_1, \phi_2, \ldots, \phi_n)$, $\phi_n = \phi$, denote a proof of ϕ from Δ. We will show that $\Delta \models \phi_i$, for each $i = 1, 2, \ldots, n$. If ϕ_i is a logical or non-logical axiom, then clearly $\Delta \models \phi_i$. Assume that ϕ_i is obtained using Modus Ponens from ϕ_j and $\phi_j \rightarrow \phi_i, j < i$. By the inductive hypothesis, $\Delta \models \phi_j$ and $\Delta \models \phi_j \rightarrow \phi_i$. Thus, any model that satisfied Δ also satisfies ϕ_j and $\phi_j \rightarrow \phi_i$. It follows that $\Delta \models \phi_i$. □