Computational Complexity - Homework II

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1 Instructions

1. The homework is due on March 5, in class.
2. Each question is worth 3 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. Assume that you are given a formula ϕ in CNF over the variables $X = \{x_1, x_2, \ldots, x_n\}$. Let $C_1 = (x_1 \vee P)$ and $C_2 = (\bar{x_1} \vee Q)$ denote two clauses in ϕ, where P and Q are arbitrary disjuncts over X. Let $\phi' = \phi \wedge (P \vee Q)$. The above process is called a resolution step on variable x_1.
   (i) Show that resolution steps are solution preserving.
   (ii) We can continue in this fashion resolving on x_1 and other variables, deriving a new formula at each time till no new clauses can be added. Call the final formula ϕ^*. Show that ϕ is satisfiable if and only if ϕ^* does not contain the empty clause ($\Box$). Note that $\Box$ is the resolvent of the clauses (x_i), $(\bar{x_i})$, for any $i = 1, 2, \ldots, n$.
2. Let Σ_{EG} denote an enhanced vocabulary for graph theory. Σ_{EG} has one function symbol $\langle + \rangle$. It has three relational symbols, viz., the binary relation $=$, a binary relation $<$, and a ternary relation, called G_w. Typical expressions in the new vocabulary are: $G_w(x, y, 5)$, $G_w(y, z, -5)$, $(\forall z)(\exists y)G_w(z, y, \infty)$ and so on. Write a sentence in Σ_{EG} which describes graphs containing negative cost cycles.
3. Let P_1 denote the statement: If Δ is consistent, then Δ has a model. Let P_2 denote the statement: If $\Delta \models \phi$, then $\Delta \vdash \phi$. Argue that P_1 and P_2 are equivalent.
4. Show that for any first-order expression ϕ over the vocabulary Σ_G, the property $\phi-$GRAPHS can be tested in logarithmic space.
5. Gödel's Incompleteness theorem is based on the fact that if number theory was axiomatizable, then it would be decidable. Curiously enough, group theory is axiomatizable, but still undecidable. How would you explain this discrepancy?