Computational Complexity - Homework II (Solutions)

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1 Problems

1. Assume that you are given a formula ϕ in CNF over the variables $X = \{x_1, x_2, \ldots, x_n\}$. Let $C_1 = (x_1 \vee P)$ and $C_2 = (\bar{x_1} \vee Q)$ denote two clauses in ϕ, where P and Q are arbitrary disjuncts over X. Let $\phi' = \phi \wedge (P \vee Q)$. The above process is called a resolution step on variable x_1.

   (i) Show that resolution steps are solution preserving.

   (ii) We can continue in this fashion resolving on x_1 and other variables, deriving a new formula at each time till no new clauses can be added. Call the final formula ϕ^*. Show that ϕ is satisfiable if and only if ϕ^* does not contain the empty clause ($\Box$). Note that $\Box$ is the resolvent of the clauses (x_i), $(\bar{x_i})$, for any $i = 1, 2, \ldots, n$.

   Solution:

   (i) We need to show that ϕ is satisfiable if and only if ϕ' is.
   Clearly, if ϕ' is satisfiable, then so is ϕ, since the clauses defining ϕ are a subset of the clauses defining ϕ'.

   Assume that ϕ is satisfiable and let $\vec{\mathbf{x}}$ denote an assignment satisfying ϕ. Since all the clauses in ϕ are satisfied, it must be the case that at least one of the literals in $(P \vee Q)$ is set to **true**. To see this, observe that if all the variables in $(P \vee Q)$ are set to **false** under $\vec{\mathbf{x}}$, then regardless of the value assigned to x_1, either C_1 is falsified or C_2 is! It follows that ϕ' is satisfiable.

   (ii) Using induction on the number of resolution steps, it is clear that ϕ is satisfiable if and only if ϕ^* is. Thus, if ϕ^* contains $\Box$, then ϕ^* is unsatisfiable and hence so if ϕ. Now assume that ϕ is satisfiable; by the solution preserving nature of resolution, each derived formula including ϕ^* is satisfiable. Assume that ϕ^* contains $\Box$. This means that in the step preceding the derivation of $\Box$, the intermediate formula, say ϕ' had a pair of clauses $(x_i)(\bar{x_i})$ for some $i = 1, 2, \ldots, n$. However, this implies that ϕ^* is unsatisfiable and hence so is ϕ, contradicting the hypothesis. It follows that if ϕ is satisfiable, then ϕ^* cannot contain $\Box$.

   $\Box$

2. Let Σ_{EG} denote an enhanced vocabulary for graph theory. As with Σ_G, Σ_{EG} does not have a function symbol. It has a single ternary relation, called G_w, in addition to the binary relation $=$. Typical expressions in the new vocabulary are: $G_w(x, y, 5)$, $G_w(y, z, -5)$, $(\forall z)(\exists y)G_w(z, y, \infty)$ and so on. Write a sentence in Σ_{EG} which describes graphs containing negative cost cycles.

   Solution: Consider the following formula:

   $$\Phi = (\exists P)(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4),$$

   where,

   (i) $\phi_1 = (\exists c)(\exists c_1)(G(x, y, c) \rightarrow P(x, y, c_1) \wedge (c_1 \leq c))$ represents the fact that if there is an edge from x to y of length c, then there exists a path from x to y of length at most c,

(ii) $\phi_2 = (\exists x)(\exists c)(P(x,x,c) \wedge (c < 0))$ represents the fact that if there exists some vertex x, such that there is a negative cost path from x to x,

(iii) $\phi_3 = (\forall u)(\forall v)(\forall w)(\exists c_1)(\exists c_2)(\exists c_3)(P(u,v,c_1) \wedge P(v,w,c_2) \rightarrow P(u,v,c_3) \wedge (c_3 \leq c_1 + c_2))$ represents the fact that if there is a path of length c_1 from u to v and a path of length c_2 from v to w, then there exists a path of length at most $(c_1 + c_2)$ from u to w.

(iv) $\phi_4 = (\forall x)(\forall y)(\exists w)(\exists c)(\exists c_1)(\exists c_2)(P(x,y,c) \rightarrow (G_w(x,y,c) \vee (P(x,w,c_1) \wedge P(w,y,c_2) \wedge (c = c_1 + c_2))))$, represents the fact that if there is a path of length c from x to y, then either there is an edge in the graph of length c or there exists a vertex w, such that there is a path of length c_1 from x to w and a path of length c_2 from w to y and $c = c_1 + c_2$.

Technically speaking, we do have all constants in $\mathcal{Z}$ in the vocabulary! □

3. Let P_1 denote the statement: If Δ is consistent, then Δ has a model. Let P_2 denote the statement: If $\Delta \models \phi$, then $\Delta \vdash \phi$. Argue that P_1 and P_2 are equivalent.

Solution:

(i) $P_1 \rightarrow P_2$ - Assume that for all sets of sentences Δ, whenever Δ is consistent, it has a model. Now consider a specific pair (Δ_1, ϕ_1) such that $\Delta_1 \models \phi_1$. Thus, any model satisfying all the expressions in Δ_1 also satisfies ϕ_1 and therefore falsifies $\neg\phi_1$. It follows that no model can satisfy $\Delta_1 \cup \{\neg\phi_1\}$ and hence this set is inconsistent. We apply Theorem 5.4 of [Pap94] to conclude that $\Delta_1 \vdash \phi_1$.

(ii) $P_2 \rightarrow P_1$ - Assume that for all sets of sentences Δ, and all sentences ϕ, if $\Delta \models \phi$ then $\Delta \vdash \phi$. Now consider a specific set of sentences Δ_1, such that Δ_1 is consistent. We need to show that Δ_1 has a model. Assume Δ_1 does not have a model. It follows that for any expression ϕ, $\Delta_1 \models \phi$. However, as per P_2, this implies that $\Delta_1 \vdash \phi$, for any expression ϕ. The only conclusion from the previous statement is that Δ_1 is inconsistent, which contradicts the hypothesis!

□

4. Show that for any first-order expression ϕ over the vocabulary Σ_G, the property $\phi-$GRAPHS can be tested in logarithmic space.

Solution: We simply use induction on the length of the expression ϕ. If the expression is atomic, i.e., of the form $G(x,y)$ or $G(y,x)$ it can be trivially checked in logarithmic space. We now consider the following cases:

(i) $\phi = \neg\psi$ - By induction ψ can be decided in logarithmic space. The answer to ψ can be inverted to get the answer for ϕ.

(ii) $\phi = \psi_1 \vee \psi_2$ - By induction, ψ_1 can be decided in logarithmic space. If the answer to ψ_1 is **true**, then so is the answer to ϕ. Otherwise, the answer to ϕ is the answer to ψ_2, which can be inductively determined in logarithmic space.

(iii) $\phi = \psi_1 \wedge \psi_2$ - The answer to ψ_1 can be determined inductively in logarithmic space. If this answer is **false**, so is the answer to ϕ. Otherwise, the answer to ϕ is the answer to ψ_2 which can be inductively determined in logarithmic space.

(iv) $\phi = (\forall x)\psi$ - Substitute each node of G in the expression ψ and inductively determine the corresponding answer in logarithmic space. If any answer is **false**, the answer to ϕ is **false**!

□

5. Gödel's incompleteness theorem is based on the fact that if number theory was axiomatizable, then it would be decidable. Curiously enough, group theory is axiomatizable, but still undecidable. How would you explain this discrepancy?

Solution: The axiomatization of number theory employed by Gödel, refers specifically to the model $\mathcal{N}$ of non-negative integers. Consequently, any recursive axiomatization would imply the existence of a recursive procedure to decide the truth of a statement. In case of group theory, there are an infinite number of groups and hence axiomatization does not imply the existence of an algorithm to determine truth.

□

References

[Pap94] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley, New York, 1994.