Asymptotics

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Outline

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(i) Which function grows faster: $100x^2$ or $\frac{1}{10^6}x^3$?

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Example

(i) Which function grows faster: $100x^2$ or $\frac{1}{10^6}x^3$?

(ii) Which function grows faster: $x^2 - 10$ or $x + 10$?

Order of Magnitude (contd.)

Definition

Let f and g be functions mapping non-negative reals to non-negative reals. Then $f = O(g)$, if there exist constants c and n_0 such that for all $n \geq n_0$, $f(x) \leq c \cdot g(x)$.

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The above rule is called L'Hospital's rule.

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(i) Show that $x = o(x^2)$.

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(i) Show that $x = o(x^2)$.

(ii) Show that $x = o(x \log x)$.

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(ii) Show that $x = o(x \log x)$.

(iii) Show that $\log x = o(x)$.