Counting

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Outline

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Multiplication Principle

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If there are n_1 possible outcomes for a first event and n_2 possible outcomes for a second event, then there are $n_1 \cdot n_2$ possible outcomes for the sequence of the two events.

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Solution

There are 10 ways to select the first digit,

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Solution

There are 10 ways to select the first digit, 10 ways to select the second digit, 10 ways to select the third digit and 10 ways to select the fourth digit. As per the multiplication principle, there are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ ways to construct a 4 digit number.

Examples of the Multiplication Principle

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How many 4 digit numbers are there, if no digit can be repeated?

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Let A and B denote two sets. How many elements does $A \times B$ have?
Solution: Using the multiplication principle, $|A| \times |B|$.

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Solution: $10 + 5 = 15$.

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Let A and B be two disjoint sets. What is $|A \cup B|$?
Solution: By the addition principle, $|A| + |B|$.

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$$|A - B| = |A| - |A \cap B|$$

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Solution: The key observation is that $(A - B)$ and $A \cap B$ are disjoint. Further, the union of $(A - B)$ and $A \cap B$ is A! Therefore, by the addition principle, $|A| = |A - B| + |A \cap B|$.

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Solution: Every even number ends in 0, 2, 4, 6 or 8.

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How many 3 digit numbers between 100 and 999 (inclusive) are even?
Solution: Every even number ends in 0, 2, 4, 6 or 8. Use multiplication principle to compute the number of even numbers that end in 0, that end in 2 and so on $(9 \cdot 10 \cdot 1)$.

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How many 3 digit numbers between 100 and 999 (inclusive) are even?
Solution: Every even number ends in 0, 2, 4, 6 or 8. Use multiplication principle to compute the number of even numbers that end in 0, that end in 2 and so on $(9 \cdot 10 \cdot 1)$. Use the addition principle to get the total number of even numbers (450).

One more example

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How many 4 digit numbers are there in which at least one digit is repeated?

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How many 4 digit numbers are there in which at least one digit is repeated?
Solution: Find the total number of 4 digit numbers and subtract the 4 digit numbers with no repetitions!

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Let $n_1, n_2, \ldots, n_{51}$ denote the chosen numbers.

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Let $n_1, n_2, \ldots, n_{51}$ denote the chosen numbers.
Therefore, each $n_i = 2^{k_i} \cdot b_i$, where b_i is some odd number, such that $1 \leq b_i \leq 99$.

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Example (contd.)

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But there are exactly 50 odd numbers between 1 and 99.

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In other words, we must have $n_i = 2^{k_i} \cdot b_i$ and $n_j = 2^{k_j} \cdot b_i$.

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Therefore, $b_i = b_j$, for some pair (n_i, n_j) (pigeonhole principle).
In other words, we must have $n_i = 2^{k_i} \cdot b_i$ and $n_j = 2^{k_j} \cdot b_i$.
Depending on whether $k_i \geq k_j$ or vice versa, one of n_i and n_j must divide the other.

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Permutations (contd.)

Example

Compute $P(7, 3)$, $P(n, 0)$, $P(n, 1)$, and $P(n, n)$.

Permutations (contd.)

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Compute $P(7,3)$, $P(n,0)$, $P(n,1)$, and $P(n,n)$.
Solution: 210,

Permutations (contd.)

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Solution: 210, 1, n, and $n!$.

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Example

How many 3 letter words can be formed using the letters in the word “compiler”?

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In how many ways can a president and vice-president be chosen from a group of 20 people?

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Example

How many 3 letter words can be formed using the letters in the word "compiler"?
Solution: $P(8, 3)$.

Example

In how many ways can a president and vice-president be chosen from a group of 20 people?
Solution: $P(20, 2)$.

One more exampe

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A library has 4 books on programming, 7 on algorithms and 3 on complexity. In how many ways can the books be ordered on a shelf?

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Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.

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Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.
Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects.

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Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.
Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.

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Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$.
Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.
Corresponding to each such arrangement, the programming books can be permuted in $P(4, 4) = 4!$ ways, the algorithms books can be permuted in $P(7, 7) = 7!$ ways and the complexity books can be permuted in $P(3, 3) = 3!$ ways.

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Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways.
Corresponding to each such arrangement, the programming books can be permuted in $P(4, 4) = 4!$ ways, the algorithms books can be permuted in $P(7, 7) = 7!$ ways and the complexity books can be permuted in $P(3, 3) = 3!$ ways. Using the multiplication principle, the total number of arrangements is $3! \cdot 4! \cdot 7! \cdot 3!$.

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Compute $C(7, 3)$, $C(n, 0)$, $C(n, 1)$ and $C(n, n)$.

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Solution: 35, 1, n, 1.

Combinations (examples)

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2. it must contain exactly one freshman.

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3. it can contain at most one freshman. Solution: $C(34,8) + C(19,1) \cdot C(34,7)$.
4. it contains at least one freshman. Solution: $C(53,8) - C(34,8)$.

Outline

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(i) $(a+b)^1 =$

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We want a general formula that permits us to write down the terms of $(a+b)^n$ without actual multiplication.

Outline

Pascal's Triangle

The coefficient table

Consider the following table:

Pascal's Triangle

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Row 0: $C(0, 0)$

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Row 0:		$C(0,0)$	
Row 1:	$C(1,0)$		$C(1,1)$

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Row 0:			$C(0,0)$			
Row 1:		$C(1,0)$		$C(1,1)$		
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Row 3:		$C(3,0)$		$C(3,1)$		$C(3,2)$		$C(3,3)$	
$\vdots$									
Row n:	$C(n,0)$		$C(n,1)$	$\ldots$	$\ldots$		$C(n,n-1)$		$C(n,n)$

Pascal's triangle (contd.)

The Value Table

Writing down the values of the terms gives the following table:

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Row 0:			1			
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Row 2:	1		2		1	

Pascal's triangle (contd.)

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Pascal's triangle (contd.)

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Pascal's formula

Theorem

$C(n, k) = C(n-1, k-1) + C(n-1, k), 1 \leq k \leq n-1.$

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Proof.

Observe that,

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&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k)!}[k + (n-k)]
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&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= C(n, k)
\end{aligned}$$

□

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A second Proof

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Note

The above proof is called a combinatorial proof and is always preferred on account of its elegance.

Outline

The Theorem

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$$(a+b)^n = \sum_{i=0}^{n} C(n,\, i)\ a^{n-i} \cdot b^i, \ \ \forall n \geq 0.$$

Outline

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Expand $(x - 3)^4$.

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Solution:

$$\begin{aligned}(x-3)^4 &= C(4,0)x^4\cdot(-3)^0 + C(4,1)x^3\cdot(-3)^1 + C(4,2)x^2\cdot(-3)^2 \\ &\quad + C(4,3)x^1\cdot(-3)^3 + C(4,4)x^0\cdot(-3)^4\end{aligned}$$

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One more example

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Show that

$$\sum_{i=0}^{n} C(n, i) = 2^n$$

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Proof using the binomial theorem

As per the binomial theorem,

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Substituting $x = 1$, we get,

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$$\begin{aligned}(1+1)^n &= C(n, 0) \cdot (1) + C(n, 1) \cdot (1) + \ldots C(n, n) \cdot (1) \\ \Rightarrow \sum_{i=0}^{n} C(n, i) &= 2^n\end{aligned}$$