Discrete Probability - Fundamentals

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Outline

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1. Motivation
2. Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events

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Note

Discrete Probability combines aspects of combinatorics, logic and inference.

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Any subset of the sample space S is called an event.

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Some more examples

Example

A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b, g),\ (b, b),\ (g, b),\ (g, g)\}$ and that all outcomes are equally likely.

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A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b, g),\ (b, b),\ (g, b),\ (g, g)\}$ and that all outcomes are equally likely.

Exercise

Assume that an urn contains 7 *black balls and* 5 *white balls. Two balls are chosen from this urn, one after the other, without replacement and at random. What is the probability that both balls are black?*

Solution

Let E denote the event that the first ball is black and F denote the event that the second ball is black. Clearly, we are interested in $P(EF)$. Observe that $P(E) = \frac{7}{12}$ and $P(F \mid E) = \frac{6}{11}$. Now, $P(F \mid E) = \frac{P(EF)}{P(E)}$, and hence,

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Two events E and F on a sample space S are said to be pairwise independent, if the occurrence of one does not affect the occurrence of the other.

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Mutual independence

Definition

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ denote a set of n events defined on a sample space S. If for each k-subset of $\mathcal{A}$ ($2 \leq k \leq n$), we have that the probability of the intersection of the events is equal to the product of the event probabilities, then the events of $\mathcal{A}$ are said to be mutually independent.

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Consider the experiment of tossing two fair coins. Let A_1 denote the event that the first coin turns up heads. Let A_2 denote the event that the second coin turns up heads. Let A_3 denote the event that the two coins turn up differently. Are A_1, A_2 and A_3 pairwise independent?

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Bayes' Formula

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Thus, the probability of an event E is the weighted average of the conditional probability of E, given that event F has occurred and the conditional probability of E, given that event F has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.

One Final Example

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Consider two urns.

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Consider two urns. Urn 1 has 2 white balls and 7 black balls.

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}}$

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.