Recurrence Relations

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

18 January, 2011

Outline

1. Recurrences

Outline

Sample Recurrences

Examples

Sample Recurrences

Examples

(i)

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

Sample Recurrences

Examples

(i)

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1),\ n \geq 2. \end{aligned}$$

(ii)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3,\ n \geq 2. \end{aligned}$$

Sample Recurrences

Examples

(i)

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1),\ n \geq 2. \end{aligned}$$

(ii)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3,\ n \geq 2. \end{aligned}$$

(iii)

$$\begin{aligned} F(1) &= 1 \\ F(2) &= 1 \\ F(n) &= F(n-1) + F(n-2),\ n \geq 3 \end{aligned}$$

Solving recurrences

Three methods

Solving recurrences

Three methods

(i) Expand-Guess-Verify (EGV).

Solving recurrences

Three methods

(i) Expand-Guess-Verify (EGV).

(ii) Formula.

Solving recurrences

Three methods

(i) Expand-Guess-Verify (EGV).
(ii) Formula.
(iii) Recursion Tree.

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

EGV

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 2 \\
S(n) &= 2 \cdot S(n-1), \quad n \geq 2.
\end{aligned}
$$

(i) Expand: $S(1) = 2$,

EGV

Example

Consider the recurrence:

$$S(1) = 2$$
$$S(n) = 2 \cdot S(n-1), \quad n \geq 2.$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$,

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8$,

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify:

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.
(ii) Guess: $S(n) = 2^n$
(iii) Verify: Using Induction!

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$LHS = 2$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \ \ n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.
(ii) Guess: $S(n) = 2^n$
(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \end{aligned}$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots.$

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.

EGV

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 2 \\
S(n) &= 2 \cdot S(n-1), \quad n \geq 2.
\end{aligned}
$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$
\begin{aligned}
LHS &= 2 \\
RHS &= 2^1 \\
&= 2
\end{aligned}
$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$.

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$.

EGV

Example

Consider the recurrence:

$$
\begin{aligned}
S(1) &= 2 \\
S(n) &= 2 \cdot S(n-1), \quad n \geq 2.
\end{aligned}
$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$
\begin{aligned}
LHS &= 2 \\
RHS &= 2^1 \\
&= 2
\end{aligned}
$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \;\; n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$S(k+1) \quad =$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$S(k+1) = 2 \cdot S(k), \quad \text{by definition}$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \end{aligned}$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \\ &= 2^{k+1}! \end{aligned}$$

EGV

Example

Consider the recurrence:

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \quad n \geq 2. \end{aligned}$$

(i) Expand: $S(1) = 2$, $S(2) = 2 \cdot 2 = 4$, $S(3) = 2 \cdot S(2) = 8, \ldots$.

(ii) Guess: $S(n) = 2^n$

(iii) Verify: Using Induction!
BASIS: $n = 1$

$$\begin{aligned} LHS &= 2 \\ RHS &= 2^1 \\ &= 2 \end{aligned}$$

Since LHS=RHS, the basis is proven.
INDUCTIVE STEP: Assume that $S(k) = 2^k$. We need to show that $S(k+1) = 2^{k+1}$. Observe that,

$$\begin{aligned} S(k+1) &= 2 \cdot S(k), \quad \text{by definition} \\ &= 2 \cdot 2^k, \quad \text{by inductive hypothesis} \\ &= 2^{k+1}! \end{aligned}$$

Applying the first principle of mathematical induction, we conclude that $S(n) = 2^n$.

EGV (contd.)

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1)+3, \ n \geq 2. \end{aligned}$$

EGV (contd.)

Example

Solve the recurrence:

$$
\begin{aligned}
T(1) &= 1 \\
T(n) &= T(n-1) + 3, \; n \geq 2.
\end{aligned}
$$

(i) Expand: $T(1) = 1$,

EGV (contd.)

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1)+3, \; n \geq 2. \end{aligned}$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$,

EGV (contd.)

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3,\ n \geq 2. \end{aligned}$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$, $T(3) = 3 + T(n-2) = 7$,

EGV (contd.)

Example

Solve the recurrence:

$$T(1) = 1$$
$$T(n) = T(n-1) + 3, \; n \geq 2.$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$, $T(3) = 3 + T(n-2) = 7, \ldots$

EGV (contd.)

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + 3, \; n \geq 2. \end{aligned}$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$, $T(3) = 3 + T(n-2) = 7, \ldots$
(ii) Guess: $T(n) = 3 \cdot n - 2$.

EGV (contd.)

Example

Solve the recurrence:

$$T(1) = 1$$
$$T(n) = T(n-1) + 3, \; n \geq 2.$$

(i) Expand: $T(1) = 1$, $T(2) = T(1) + 3 = 4$, $T(3) = 3 + T(n-2) = 7, \ldots$

(ii) Guess: $T(n) = 3 \cdot n - 2$.

(iii) Verify: Somebody from class!

Formula approach

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear, *because the* $S()$ *terms occur only in the first power.*

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power. It is called* first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power.*
It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*.*

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power.*
It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*. The recurrence is called homogeneous, if* $g(n) = 0$*, for all* n*.*

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear, *because the* $S()$ *terms occur only in the first power.*
It is called first-order, *if* $S(n)$ *depends only on* $S(n-1)$.
For example, $S(n) = c \cdot S(n-1) + g(n)$. *The recurrence is called homogeneous, if* $g(n) = 0$, *for all n.*

Formula for Linear first-order recurrence

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power.*
It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*. The recurrence is called homogeneous, if* $g(n) = 0$*, for all* n*.*

Formula for Linear first-order recurrence

$$S(1) = k_0$$

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power.*
It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*. The recurrence is called homogeneous, if* $g(n) = 0$*, for all* n*.*

Formula for Linear first-order recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

Formula approach

Definition

A general linear recurrence has the form:

$$S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$$

Note

The above formula is called linear*, because the* $S()$ *terms occur only in the first power.*
It is called first-order*, if* $S(n)$ *depends only on* $S(n-1)$*.*
For example, $S(n) = c \cdot S(n-1) + g(n)$*. The recurrence is called homogeneous, if* $g(n) = 0$*, for all n.*

Formula for Linear first-order recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

$\Rightarrow S(n) = c^{n-1} \cdot k_0 + \sum_{i=2}^{n} c^{n-i} \cdot g(i)$.

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \ n \geq 2. \end{aligned}$$

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1),\ n \geq 2. \end{aligned}$$

As per the formula, $k_0 =$

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 2$, $g(n) =$

Linear first-order recurrence

Example

$$S(1) = 2$$
$$S(n) = 2 \cdot S(n-1), \ n \geq 2.$$

As per the formula, $k_0 = 2$, $g(n) = 0$ and $c =$

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 2$, $g(n) = 0$ and $c = 2$. Thus,

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 2$, $g(n) = 0$ and $c = 2$. Thus,

$$S(n) =$$

Linear first-order recurrence

Example

$$S(1) = 2$$
$$S(n) = 2 \cdot S(n-1), \; n \geq 2.$$

As per the formula, $k_0 = 2$, $g(n) = 0$ and $c = 2$. Thus,

$$S(n) = 2^{n-1} \cdot 2 + \sum_{i=2}^{n} 2^{n-i} \cdot 0$$

Linear first-order recurrence

Example

$$\begin{aligned} S(1) &= 2 \\ S(n) &= 2 \cdot S(n-1), \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 2$, $g(n) = 0$ and $c = 2$. Thus,

$$\begin{aligned} S(n) &= 2^{n-1} \cdot 2 + \sum_{i=2}^{n} 2^{n-i} \cdot 0 \\ &= 2^n \end{aligned}$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \;\; n \geq 2. \end{aligned}$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \;\; n \geq 2. \end{aligned}$$

As per the formula, $k_0 =$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) =$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c =$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \;\; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \ n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

$$S(n) =$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \;\; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

$$S(n) = 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \ n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

$$\begin{aligned} S(n) &= 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3 \\ &= 2^{n+1} + 3 \sum_{i=2}^{n} 2^{n-i} \end{aligned}$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \; n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

$$\begin{aligned} S(n) &= 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3 \\ &= 2^{n+1} + 3 \sum_{i=2}^{n} 2^{n-i} \\ &= 2^{n+1} + 3 \cdot [2^{n-2} + 2^{n-3} + \ldots + 2^0] \end{aligned}$$

Another Example

Example

Solve the recurrence:

$$\begin{aligned} S(1) &= 4 \\ S(n) &= 2 \cdot S(n-1) + 3, \ \ n \geq 2. \end{aligned}$$

As per the formula, $k_0 = 4$, $g(n) = 3$ and $c = 2$. Thus,

$$\begin{aligned} S(n) &= 2^{n-1} \cdot 4 + \sum_{i=2}^{n} 2^{n-i} \cdot 3 \\ &= 2^{n+1} + 3 \sum_{i=2}^{n} 2^{n-i} \\ &= 2^{n+1} + 3 \cdot [2^{n-2} + 2^{n-3} + \ldots + 2^0] \\ &= 2^{n+1} + 3 \cdot [2^{n-1} - 1] \end{aligned}$$

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

   (a) If $r_1 \neq r_2$, solve

$$\begin{aligned} p + q &= S(1) \\ p \cdot r_1 + q \cdot r_2 &= S(2) \end{aligned}$$

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

   (a) If $r_1 \neq r_2$, solve

$$\begin{aligned} p + q &= S(1) \\ p \cdot r_1 + q \cdot r_2 &= S(2) \end{aligned}$$

   Then, $S(n) = p \cdot r_1^{n-1} + q \cdot r_2^{n-1}$

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

(a) If $r_1 \neq r_2$, solve

$$\begin{aligned} p + q &= S(1) \\ p \cdot r_1 + q \cdot r_2 &= S(2) \end{aligned}$$

Then, $S(n) = p \cdot r_1^{n-1} + q \cdot r_2^{n-1}$

(b) If $r_1 = r_2 = r$, solve

$$\begin{aligned} p &= S(1) \\ (p + q) \cdot r &= S(2) \end{aligned}$$

Second Order homogeneous Linear Recurrence with constant coefficients

Formula

(i) Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.

(ii) Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.

(a) If $r_1 \neq r_2$, solve

$$\begin{aligned} p + q &= S(1) \\ p \cdot r_1 + q \cdot r_2 &= S(2) \end{aligned}$$

Then, $S(n) = p \cdot r_1^{n-1} + q \cdot r_2^{n-1}$

(b) If $r_1 = r_2 = r$, solve

$$\begin{aligned} p &= S(1) \\ (p + q) \cdot r &= S(2) \end{aligned}$$

Then, $S(n) = p \cdot r^{n-1} + q \cdot (n-1) \cdot r^{n-1}$

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \; n \geq 3 \end{aligned}$$

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \; n \geq 3 \end{aligned}$$

Solution:

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \; n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6, \; c_2 = -5.$

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation:

Examples of second order recurrences

Example

Solve the recurrence relation

$$
\begin{aligned}
T(1) &= 5 \\
T(2) &= 13 \\
T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \; n \geq 3
\end{aligned}
$$

Solution:

(i) $c_1 = 6, \; c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2), \ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6, \ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.
Solution is: $r_1 = 1$, $r_2 = 5$.

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$. Solution is: $r_1 = 1$, $r_2 = 5$.

(ii) Solve the equations:

$$\begin{aligned} p + q &= T(1) = 5 \\ p \cdot 1 + q \cdot 5 &= T(2) = 13 \end{aligned}$$

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$. Solution is: $r_1 = 1$, $r_2 = 5$.

(ii) Solve the equations:

$$\begin{aligned} p + q &= T(1) = 5 \\ p \cdot 1 + q \cdot 5 &= T(2) = 13 \end{aligned}$$

We get $p = 3$ and $q = 2$.

Examples of second order recurrences

Example

Solve the recurrence relation

$$\begin{aligned} T(1) &= 5 \\ T(2) &= 13 \\ T(n) &= 6 \cdot T(n-1) - 5 \cdot T(n-2),\ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 6,\ c_2 = -5$. Characteristic equation: $t^2 - 6 \cdot t + 5 = 0$.
Solution is: $r_1 = 1$, $r_2 = 5$.

(ii) Solve the equations:

$$\begin{aligned} p + q &= T(1) = 5 \\ p \cdot 1 + q \cdot 5 &= T(2) = 13 \end{aligned}$$

We get $p = 3$ and $q = 2$.

(iii) Accordingly, the solution is $T(n) = 3 \cdot 1^{n-1} + 2 \cdot 5^{n-1} = 3 + 2 \cdot 5^{n-1}$.

□

One More Example

Example

Solve the recurrence relation:

$$
\begin{aligned}
S(1) &= 1 \\
S(2) &= 12 \\
S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \; n \geq 3
\end{aligned}
$$

Solution:

One More Example

Example

Solve the recurrence relation:

$$
\begin{aligned}
S(1) &= 1 \\
S(2) &= 12 \\
S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \ \ n \geq 3
\end{aligned}
$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$.

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \;\; n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$.

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \ n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

One More Example

Example

Solve the recurrence relation:

$$
\begin{aligned}
S(1) &= 1 \\
S(2) &= 12 \\
S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \;\; n \geq 3
\end{aligned}
$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

(ii) Solve the equations:

$$
\begin{aligned}
p &= 1 \\
p \cdot 4 + q \cdot 4 &= 12
\end{aligned}
$$

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \; n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

(ii) Solve the equations:

$$\begin{aligned} p &= 1 \\ p \cdot 4 + q \cdot 4 &= 12 \end{aligned}$$

We get $p = 1$ and $q = 2$.

One More Example

Example

Solve the recurrence relation:

$$\begin{aligned} S(1) &= 1 \\ S(2) &= 12 \\ S(n) &= 8 \cdot S(n-1) - 16 \cdot S(n-2), \; n \geq 3 \end{aligned}$$

Solution:

(i) $c_1 = 8$, $c_2 = -16$. Characteristic equation: $t^2 - 8t + 16 = 0$. Solution is $r_1 = r_2 = 4$.

(ii) Solve the equations:

$$\begin{aligned} p &= 1 \\ p \cdot 4 + q \cdot 4 &= 12 \end{aligned}$$

We get $p = 1$ and $q = 2$.

(iii) Accordingly, the solution is $S(n) = 4^{n-1} + 2 \cdot (n-1) \cdot 4^{n-1} = (2n-1) \cdot 4^{n-1}$.

□

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m \end{aligned}$$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n), \; n \geq 2, \; n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$.

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m \end{aligned}$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$S(1) = k_0$$
$$S(n) = c \cdot S(\frac{n}{2}) + g(n), \; n \geq 2, \; n = 2^m$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$C(1) = 1$$
$$C(n) = 1 + C(\frac{n}{2}), \; n \geq 2, \; n = 2^m$$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m \end{aligned}$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m \end{aligned}$$

Note that $k_0 =$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c =$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m \end{aligned}$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$\begin{aligned} C(1) &= 1 \\ C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m \end{aligned}$$

Note that $k_0 = 1$, $c = 1$ and $g(i) =$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1,\ \forall i$.

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n), \; n \geq 2, \; n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}), \; n \geq 2, \; n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1, \;\; \forall i$. As per the formula,

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1,\ \forall i$. As per the formula,

$$
C(n) = 1^{\log n} \cdot 1 + \sum_{i=1}^{\log n} 1^{\log n - i} \cdot (1)
$$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1, \ \forall i$. As per the formula,

$$
\begin{aligned}
C(n) &= 1^{\log n} \cdot 1 + \sum_{i=1}^{\log n} 1^{\log n - i} \cdot (1) \\
&= 1 + (\log n) \cdot 1
\end{aligned}
$$

Divide and Conquer Recurrence

Formula for Divide and Conquer Recurrence

$$
\begin{aligned}
S(1) &= k_0 \\
S(n) &= c \cdot S(\frac{n}{2}) + g(n),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

$\Rightarrow S(n) = c^{\log n} \cdot k_0 + \sum_{i=1}^{\log n} c^{logn-i} \cdot g(2^i)$. (All logarithms are to base 2).

Example

Solve the recurrence:

$$
\begin{aligned}
C(1) &= 1 \\
C(n) &= 1 + C(\frac{n}{2}),\ n \geq 2,\ n = 2^m
\end{aligned}
$$

Note that $k_0 = 1$, $c = 1$ and $g(i) = 1,\ \forall i$. As per the formula,

$$
\begin{aligned}
C(n) &= 1^{\log n} \cdot 1 + \sum_{i=1}^{\log n} 1^{\log n - i} \cdot (1) \\
&= 1 + (\log n) \cdot 1 \\
&= 1 + \log n
\end{aligned}
$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$.

Example

Solve the recurrence:

$$T(1) = 3$$
$$T(n) = 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \ \ n \geq 2, \ \ n = 2^m$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \ \ \forall i$. As per the formula,

Example

Solve the recurrence:

$$
\begin{aligned}
T(1) &= 3 \\
T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \; n \geq 2, \; n = 2^m
\end{aligned}
$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \; \forall i$. As per the formula,

$$
T(n) = 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i)
$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \ \ n \geq 2, \ \ n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \ \ \forall i$. As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n) \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n) \\ &= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \ \ n \geq 2, \ \ n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \ \ \forall i$. As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n) \\ &= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\ &= 3 \cdot n + n \cdot 2 \cdot \log n, \end{aligned}$$

Example

Solve the recurrence:

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \;\; n \geq 2, \;\; n = 2^m \end{aligned}$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \;\; \forall i$. As per the formula,

$$\begin{aligned} T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\ &= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= 3 \cdot n + 2^{\log n + 1} \cdot (\log n) \\ &= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\ &= 3 \cdot n + n \cdot 2 \cdot \log n, \;\; \text{since } (a^{\log_a n} = n, \;\; a \neq 0) \end{aligned}$$

Example

Solve the recurrence:

$$
\begin{aligned}
T(1) &= 3 \\
T(n) &= 2 \cdot T(\frac{n}{2}) + 2 \cdot n, \ \ n \geq 2, \ \ n = 2^m
\end{aligned}
$$

Note that $k_0 = 3$, $c = 2$ and $g(i) = 2 \cdot i, \ \ \forall i$. As per the formula,

$$
\begin{aligned}
T(n) &= 2^{\log n} \cdot 3 + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2 \cdot (2^i) \\
&= 3 \cdot 2^{\log n} + \sum_{i=1}^{\log n} 2^{\log n + 1} \\
&= 3 \cdot n + 2^{\log n + 1} \cdot (\log n) \\
&= 3 \cdot n + 2^{\log n} \cdot 2 \cdot \log n \\
&= 3 \cdot n + n \cdot 2 \cdot \log n, \ \ \text{since } (a^{\log_a n} = n, \ \ a \neq 0) \\
&= 3 \cdot n + 2 \cdot n \cdot \log n
\end{aligned}
$$

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants.

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

(i) If $f(n) = O(n^{\log_b a - \epsilon})$, for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

(i) If $f(n) = O(n^{\log_b a - \epsilon})$, for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

(ii) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

(i) If $f(n) = O(n^{\log_b a - \epsilon})$, for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

(ii) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.

(iii) If $f(n) = \Omega(n^{\log_b a + \epsilon})$, for some constant $\epsilon > 0$, and if $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$, for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

(i) If $f(n) = O(n^{\log_b a - \epsilon})$, for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
(ii) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
(iii) If $f(n) = \Omega(n^{\log_b a + \epsilon})$, for some constant $\epsilon > 0$, and if $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$, for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

Note

Proof is via induction. Outside scope of class.

The Master Method

Theorem

Let $a \geq 1$ and $b \geq 1$ be constants. Let $f(n)$ be a function and let $T(n)$ be defined recursively as follows:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

(i) If $f(n) = O(n^{\log_b a - \epsilon})$, for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

(ii) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.

(iii) If $f(n) = \Omega(n^{\log_b a + \epsilon})$, for some constant $\epsilon > 0$, and if $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$, for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

Note

Proof is via induction. Outside scope of class. The master theorem does not cover all cases!

Some examples

Example

(i) $T(n) = 9 \cdot T(\frac{n}{3}) + n.$

Some examples

Example

(i) $T(n) = 9 \cdot T(\frac{n}{3}) + n.$

(ii) $T(n) = T(\frac{2n}{3}) + 1.$

Some examples

Example

(i) $T(n) = 9 \cdot T(\frac{n}{3}) + n$.

(ii) $T(n) = T(\frac{2n}{3}) + 1$.

(iii) $T(n) = 3 \cdot T(\frac{n}{4}) + n \cdot \log n$.

Some examples

Example

(i) $T(n) = 9 \cdot T(\frac{n}{3}) + n$.

(ii) $T(n) = T(\frac{2n}{3}) + 1$.

(iii) $T(n) = 3 \cdot T(\frac{n}{4}) + n \cdot \log n$.

(iv) $T(n) = 2 \cdot T(\frac{n}{2}) + n$.

Some examples

Example

(i) $T(n) = 9 \cdot T(\frac{n}{3}) + n$.

(ii) $T(n) = T(\frac{2n}{3}) + 1$.

(iii) $T(n) = 3 \cdot T(\frac{n}{4}) + n \cdot \log n$.

(iv) $T(n) = 2 \cdot T(\frac{n}{2}) + n$.

(v) $T(n) = 2 \cdot T(\sqrt{n}) + \log n$.