Design of Algorithms - Homework I

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Instructions

1. The homework is due on February 8, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. Consider the following algorithm for sorting an array of n numbers.

```
Function ARRAY-SORT(A, n)
for (i = 1 to n) do
    for (j = n downto i + 1) do
        if (A[j] < A[j − 1]) then
            SWAP(A[j], A[j − 1]).
        end if
    end for
end for
```

Algorithm 2.1: Sorting Algorithm

Argue the correctness of the algorithm using loop invariants and analyze its running time.

2. (a) Show that $\log n! = \Theta(n \cdot \log n)$.
   (b) Show that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$, where $f(n)$ and $g(n)$ are non-negative functions.

3. An $m \times n$ array $\mathbf{A}$ of integers is said to be a **Monge Array**, is for all i, j, k, and l, such that $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$, we have,
$$A[i,j] + A[k,l] \leq A[i,l] + A[k,j]$$
   (a) Prove that an array $\mathbf{A}$ is Monge, if and only if for all $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots n-1$, we have,
$$A[i,j] + A[i+1,j+1] \leq A[i,j+1] + A[i+1,j]$$
.
   (b) Let $f(i)$ be the index of the column containing the leftmost minimum element of row i. Prove that $f(1) \leq f(2) \leq \ldots \leq f(m)$, for any $m \times n$ Monge array.

4. Show that for any integer $n \geq 0$,

$$\sum_{k=0}^{n} C(n,k) \cdot k = n \cdot 2^{n-1}.$$

5. Let X be a non-negative random variable and suppose that $E[X]$ and $\sigma = \sqrt{Var(X)}$ are well-defined.

   (a) Show that $Pr[X \geq t] \leq \frac{E[X]}{t}$, for all $t > 0$.

   (b) Show that $Pr[|X - E[X]| \geq t \cdot \sigma] \leq \frac{1}{t^2}$, for any $t > 0$.