Balls and Bins (Advanced)

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- Recap
- 2 The Poisson Approximation
 - Some theorems and lemmas

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 - Chain Hashing
 - Bit String Hashing
 - Bloom Filters
 - Breaking Symmetry

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The experiment of throwing m balls into n bins, each bin being chosen independently and uniformly at random. Several questions regarding the above random process were examined, such as expected maximum load, expected number of balls in a bin, expected number of empty bins, and expected number of bins with r balls. We also examined the Poisson random variable and its applications to Balls and Bins questions.

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The distribution of $(Y_1^{(m)}, \dots, Y_n^{(m)})$ conditioned on $\sum_i Y_i^{(m)} = k$ is the same as $(X_1^{(k)}, \dots, X_n^{(k)})$.

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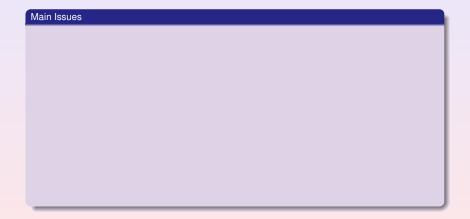
Let Δ be an event whose probability is either monotonically increasing or decreasing in the number of balls. If Δ has probability p in the Poisson case, then it has probability at most $2 \cdot p$ in the exact case.

Lemma

When n balls are thrown independently into n bins, the maximum load is at least $\frac{\ln n}{\ln \ln n}$ with probability at least $(1-\frac{1}{n})$, for sufficiently large n.

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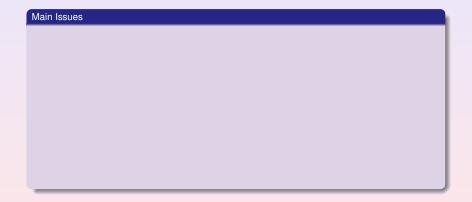
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• If we choose $b=2\cdot \log_2 m$, the probability of a false positive is: $1-\left(1-\frac{1}{m^2}\right)^m<\frac{1}{m}$.

If our dictionary has 2^{16} words, using 32 bits when hashing, leads to an error probability of at most $\frac{1}{65,536}$.

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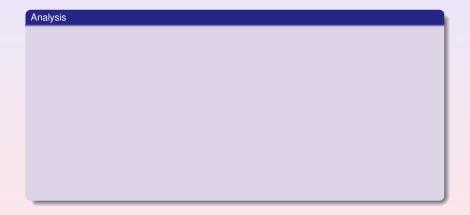
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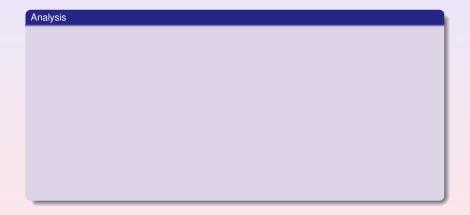
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- How to check if $x \in S$? Check all locations $A[h_i(x)]$, $1 \le i \le k$.

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- Set $A[h_i(s)]$ to 1, for each $1 \le i \le k$ and each $s \in S$.
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• Optimizing for k, we get $k = \ln 2$ and $f \approx (0.6185)^{\frac{m}{n}}$.

Outline

- 1 Recap
- 2 The Poisson Approximation
 - Some theorems and lemmas
- Applications to Hashing
 - Chain Hashing
 - Bit String Hashing
 - Bloom Filters
 - Breaking Symmetry

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Can also be used for the leader election problem.