Martingales

Naga Venkata Nitesh Tadepalli[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

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Outline

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A sequence of random variables $Z_0, Z_1, \cdots$ is called martingale, when it is a martingale with respect to itself. That is, $\mathbf{E}[|Z_n|] < \infty$, and $\mathbf{E}[Z_{n+1} \mid Z_0, \ldots, Z_n] = Z_n$.

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2. *The indexing of the martingale sequence does not need to start at 0.*

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Thus, $Z_1, Z_2, \ldots, Z_n$ is a martingale with respect to the sequence $X_1, X_2, \ldots, X_n$.

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The sequence is a martingale regardless of the amount bet on each game, even if these amounts are dependent upon previous results.

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Note

$\boldsymbol{E}[V \mid W] = \boldsymbol{E}[\boldsymbol{E}[V \mid U, W] \mid W]$

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Consider that we want to estimate the value of Y, whose value is a function of the values of the random variables $X_1, \ldots, X_n$. The sequence $Z_0, Z_1, \ldots, Z_n$ represents a sequence of refined estimates of the value of Y, gradually using more information on the values of the random variables $X_1, X_2, \ldots, X_n$. Then

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1. $Z_0 = \mathbf{E}[Y]$
2. $Z_i = \mathbf{E}[Y \mid X_1, \ldots X_i]$
3. $Z_n = Y$, if Y is fully determined by $X_1, \ldots, X_n$

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The sequence $Z_0, Z_1, \ldots, Z_m$ is a Doob martingale that represents the conditional expectations of $F(G)$ as we reveal whether each edge is in the graph, one edge at a time.

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The sequence $Z_0, Z_1, \ldots, Z_m$ is a Doob martingale that represents the conditional expectations of $F(G)$ as we reveal whether each edge is in the graph, one edge at a time.
This process of revealing edges gives a martingale called the edge exposure martingale.

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gives a Doob martingale that is commonly called the vertex exposure martingale.

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Repeating this argument yields

$$\mathbf{E}[Z_n] = \mathbf{E}[Z_0]$$

Stopping Times

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Since $Z_0, Z_1, \ldots, Z_n$ is a martingale with respect to $X_0, X_1, \ldots, X_n$, it follows that

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Note

The gambler could decide to keep playing until his winnings total at least a hundred dollars. The following notion proves quite powerful.

Stopping Times

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A nonnegative, integer-valued random variable T is a stopping time for the sequence $\{Z_n, n \geq 0\}$ if the event $T = n$ depends only on the value of the random variables $Z_0, Z_1, \ldots, Z_n$.

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Note

The subtle problem with the stopping times like, the first T such that $Z_T > B$ where B is a fixed constant greater than 0, is that it may not be finite, so the gambler may never finish playing.

Outline

Martingale Stopping Theorem

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Theorem

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If $Z_0, Z_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and if T is a stopping time for $X_1, X_2, \ldots$, then

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whenever one of the following holds:

1. The Z_i are bounded, so there is a constant c such that, for all i, $|Z_i| \leq c$;
2. T is bounded;
3. $\mathbf{E}[T] < \infty$, and there is a constant c such that $\mathbf{E}[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i] < c$.

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$$\text{gives } q = \frac{l_1}{l_1 + l_2}$$

Wald's Equation

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Let $X_1, X_2, \ldots$ be a nonnegative, independent, identically distributed random variables with distribution X. Let T be a stopping time for this sequence. If T and X have bounded expectation, then

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The sequence $Z_1, Z_2, \ldots$ is a martingale with respect to $X_1, X_2 \ldots$, and $\mathbf{E}[Z_1] = 0$. Now, $\mathbf{E}[T] < \infty$ and

$$\mathbf{E}[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i] = \mathbf{E}[|X_{i+1} - E[X]|] \leq 2\mathbf{E}[X].$$

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Wald's Equation

Definition

Let $Z_0, Z_1, \ldots$ be a sequence of independent random variables. A non-negative, integer-valued random variable T is a stopping time for the sequence if the event $T = n$ is independent of $Z_{n+1}, Z_{n+2,\ldots}$

Example

Consider a gambling game in which a player first rolls one standard die. If the outcome of the roll is X then she rolls X new standard dice and her gain Z is the sum of the outcome of the X dice. What is the outcome of this game?

Solution

For $1 \leq i \leq X$, let Y_i be the outcome of the i^{th} die in the second round. Then

$$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{i=1}^{X} Y_i\right] = \mathbf{E}[X] \cdot \mathbf{E}[Y_i] = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

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Wald's equation can be used in the analysis of Las Vegas algorithms. If N is the number of trials until a correct answer is found and X_i is the running time for the two subroutines on the i^{th} trial, then according to Wald's equation

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$$\mathbf{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbf{E}[N] \cdot \mathbf{E}[X].$$

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At each time slot, if the server's buffer is not empty then with probability $1/n$ it attempts to send the first packet in its buffer.

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At each time slot, if the server's buffer is not empty then with probability $1/n$ it attempts to send the first packet in its buffer. What is the expected number of time slots used until each server successfully sends at least one packet?

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Let N be the number of packets successfully sent until each server has successfully sent at least one packet.

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Solution

Let N be the number of packets successfully sent until each server has successfully sent at least one packet. Let t_i be the time slot in which the i^{th} successfully transmitted packet is sent, starting from time $t_0 = 0$, and let $r_i = t_i - t_{i-1}$ and

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$$T = \sum_{i=1}^{N} r_i$$

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Given that a packet was successfully sent at a given time slot, the sender of that packet is uniformly distributed among the n servers, independent of previous steps.

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&= \mathbf{E}[N] \cdot \mathbf{E}[r_i] \\
&= \frac{n \cdot H_n}{p} \\
&\approx e \cdot n \cdot \ln n
\end{aligned}$$