Random Variables - Expectation

K. Subramani[1]

[1]Lane Department of Computer Science and Electrical Engineering
West Virginia University

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Outline

1. Recap

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2. Random Variables

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3. Expectation

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1. Recap
2. Random Variables
3. Expectation
4. Expectation of a function of a random variable
5. Linearity of Expectation
6. Conditional Expectation

Recap

Main points

Random experiment,

Recap

Main points

Random experiment, sample spaces,

Recap

Main points

Random experiment, sample spaces, events,

Recap

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Random experiment, sample spaces, events, combining events,

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Random experiment, sample spaces, events, combining events, conditional probability,

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Random experiment, sample spaces, events, combining events, conditional probability, independence.

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In case of certain random experiments, we are not so much interested in the actual outcome, but in some function of the outcome, e.g., in the experiment of tossing two dice, we could be interested in knowing whether or not the the sum of the upturned faces is 7. We may not care whether the actual outcome is $(1, 6)$, $(6, 1)$, or $\ldots$.

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$$P\{X = 1\} \quad =$$

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Let X denote the random variable that is defined as the sum of two fair dice. What are the values that X can take?

$$\begin{aligned} P\{X = 1\} &= 0 \\ P\{X = 2\} &= \frac{1}{36} \end{aligned}$$

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Let X denote the random variable that is defined as the sum of two fair dice. What are the values that X can take?

$$
\begin{aligned}
P\{X = 1\} &= 0 \\
P\{X = 2\} &= \frac{1}{36} \\
&\vdots \\
P\{X = 12\} &= \frac{1}{36}
\end{aligned}
$$

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P\{Y = 0\} &= \frac{1}{4} \\
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Consider the experiment of tossing two fair coins; let Y denote the random variable that counts the number of heads. What values can Y take?

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Definition

A random variable that can take on only a countable number of possible values is said to be *discrete*.

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Definition

A random variable that can take on only a countable number of possible values is said to be *discrete*. For a discrete random variable X, the probability mass function (pmf) $p(a)$ is defined as:

$$p(a) = P\{X = a\}.$$

The Bernoulli Random Variable

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$$\begin{aligned} p(1) &= P\{X = 1\} = p \\ p(0) &= P\{X = 0\} = 1 - p \end{aligned}$$

where $0 \leq p \leq 1$ is the probability that the experiment results in a success.

The Binomial Random Variable

Motivation

Consider an experiment which consists of n independent Bernoulli trials, with the probability of success in each trial being p. If X is the random variable that counts the number of successes in the n trials, then X is said to be a Binomial Random Variable.

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$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1 - p)^{n-i}, \; i = 0, 1, 2, \ldots n$$

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Consider the experiment of tossing four fair coins. What is the probability that you will get two heads and two tails?

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Example (contd.)

Solution

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Let the event of heads turning up denote a "success." Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

$$\begin{aligned} p(2) &= C(4,2) \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2 \\ &= \frac{3}{8} \end{aligned}$$

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$$p(i) = P\{X = i\} = (1-p)^{i-1} \cdot p, \ i = 1, 2, \ldots$$

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&= \sum_{i=0}^{n} i \cdot C(n,i) \cdot p^{i} \cdot (1-p)^{n-i} \\
&= \sum_{i=0}^{n} i \cdot \frac{n!}{i!(n-i)!} \cdot p^{i} \cdot (1-p)^{n-i}
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&= \sum_{i=1}^{n} i \cdot \frac{n!}{i!(n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\
&= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \cdot p^i \cdot (1-p)^{n-i}
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&= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \cdot p^i \cdot (1-p)^{n-i} \\
&= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}
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\end{aligned}$$

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Expectation of a Binomial Random Variable (contd.)

Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$

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&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem} \\
&= n \cdot p \cdot 1
\end{aligned}
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&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem} \\
&= n \cdot p \cdot 1 \\
&= n \cdot p
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Let X denote a Geometric Random Variable with parameters n and p. What is $E[X]$?
Solution:

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Expectation of a Geometric Random Variable

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Let X denote a Geometric Random Variable with parameters n and p. What is $E[X]$?
Solution:

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition} \\
&= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p
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Expectation of a Geometric Random Variable

Example

Let X denote a Geometric Random Variable with parameters n and p. What is $E[X]$?
Solution:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition} \\
&= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p \\
&= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p, \text{ where } q = 1-p \\
&= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} \\
&= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq}[q^i]
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$$E[X] = p \cdot \frac{d}{dq}[\sum_{i=1}^{\infty} q^i]$$

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Consider yet another variation to the initial game: The die is tossed ten times. For each toss that turns up an even number, *A* gets 5 dollars. For tosses turning up an odd number, *A* loses 4 dollars. How much money can *A* expect to make from this game?

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Let X be a random variable, with the following pmf:

$$p(0) = 0.3, \; p(1) = 0.5, \; p(2) = 0.2$$

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Compute $E[X^2]$.

Expectation of functions of random variables (contd.)

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Let $Y = X^2$.

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Accordingly,

$$E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$$

Expectation of functions - The Direct Approach

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Theorem

If X is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

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Linearity of Expectation

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Lemma

Let X and Y denote two random variables. Then, $E[X+Y] = E[X] + E[Y]$.

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Lemma

Let X and Y denote two random variables. Then, $E[X + Y] = E[X] + E[Y]$.

Lemma

Let X denote a random variable and let c denote a constant. Then, $E[c \cdot X] = c \cdot E[X]$.

Linearity of Expectation (contd.)

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Note

Note that linearity of expectation holds even when the random variables are **not** *independent.*

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Example

What is the expected value of the sum of the upturned faces, when two fair dice are tossed?

Another Application

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Compute the expected value of the Binomial random variable.

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Let X and Y denote two random variables. The conditional expectation of X, given that $Y = y$, is defined as follows:

$$E[X \mid Y = y] = \sum_{x} x \cdot Pr(X = x \mid Y = y).$$

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Conditional Expectation (contd.)

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\begin{aligned}
\sum_y Pr(Y=y)\cdot E[X \mid Y=y] &= \sum_y Pr(Y=y)\cdot \sum_x x\cdot Pr(X=x \mid Y=y) \\
&= \sum_x \sum_y x\cdot Pr(X=x \mid Y=y)\cdot Pr(Y=y) \\
&= \sum_x x\cdot \sum_y Pr(X=x \mid Y=y)\cdot Pr(Y=y) \\
&= \sum_x x\cdot Pr(X=x)
\end{aligned}
$$

Conditional Expectation (contd.)

Proof.

Observe that,

$$\begin{aligned}
\sum_y Pr(Y=y)\cdot E[X \mid Y=y] &= \sum_y Pr(Y=y)\cdot \sum_x x\cdot Pr(X=x \mid Y=y) \\
&= \sum_x \sum_y x\cdot Pr(X=x \mid Y=y)\cdot Pr(Y=y) \\
&= \sum_x x\cdot \sum_y Pr(X=x \mid Y=y)\cdot Pr(Y=y) \\
&= \sum_x x\cdot Pr(X=x) \\
&= E[X]
\end{aligned}$$

□

Conditional Expectation (contd.)

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Definition

The expression $E[X \mid Y]$ is a random variable and takes on the values $E[X \mid Y = y]$, when $Y = y$.

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Theorem

Let X and Y denote any two random variables.

Conditional Expectation (contd.)

Definition

The expression $E[X \mid Y]$ is a random variable and takes on the values $E[X \mid Y = y]$, when $Y = y$.

Theorem

Let X and Y denote any two random variables. Then,

$$E[X] = E[E[X \mid Y]]$$