The Lovasz Local Lemma

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Outline

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Also if $P(E_i) < 1$ for all i, then

$$P(\cap_{i=1}^{n} \overline{E}_i) = \prod_{i=1}^{n} P(\overline{E}_i) > 0$$

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There exists an element of the sample space that is not included in any bad event.

Lovasz Local Lemma

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The Lovasz local lemma generalizes the argument to the case where n events are not mutually independent but the dependency is limited.

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The dependency between events can be represented in terms of a dependency graph.

Lovasz Local Lemma

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A dependency graph for a set of events $E_1, E_2, \ldots E_n$ is a graph $G = (V, E)$ such that $V = \{1 \ldots n\}$ and for $i = 1 \ldots n$, event E_i is mutually independent of the events $\{E_j \mid (i, j) \notin E\}$

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3. $4 \cdot d \cdot p \leq 1$

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Proof

Assume $S \subset \{1 \ldots n\}$
By induction on $s = 0, \ldots n - 1$ we prove that if $|S| \leq s$, then for all $k \notin S$,

$$P(E_k \mid \cap_{j \in S} \overline{E}_j) \leq 2 \cdot p$$

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Assume $S \subset \{1 \dots n\}$
By induction on $s = 0, \dots n-1$ we prove that if $|S| \leq s$, then for all $k \notin S$,

$$P(E_k \mid \cap_{j \in S} \overline{E}_j) \leq 2 \cdot p$$

Also, for this to be well defined when S is not empty

$$P(E_k \mid \cap_{j \in S} \overline{E}_j) \geq 0$$

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Using induction hypothesis,

$$\geq \prod_{i=1}^{s} (1 - 2 \cdot p) \geq 0$$

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Let $S_1 = \{j \in S \,|(k,j) \in E\}$ and $S_2 = S - S_1$.
If $S_2 = S$, then E_k is mutually independent of the events $\overline{E}_i, i \in S$

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$$P\left(E_k \mid \cap_{j \in S} \overline{E}_j\right) = P(E_k) \leq p$$

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$$P\left(E_k \mid \cap_{j \in S}\, \overline{E}_j\right) = P(E_k) \leq p$$

Let $|S_2| \leq s$
Let F_s be $F_s = \cap_{j \in S} \overline{E}_j$
Also $F_s = F_{s_1} \cap F_{s_2}$

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Let $|S_2| \leq s$
Let F_s be $F_s = \cap_{j \in S} \overline{E}_j$
Also $F_s = F_{s_1} \cap F_{s_2}$
Applying Conditional Probability,

$$P(E_k \mid F_s) = \frac{P(E_k \cap F_s)}{P(F_s)}$$

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$$P(F_s) = P(F_{s_1} \cap F_{s_2})$$

$$= P(F_{s_1} \mid F_{s_2}) \cdot P(F_{s_2})$$

Hence,

$$P(E_k \mid F_s) = \frac{P(E_k \cap F_{s_1} \mid F_{s_2})}{P(F_{s_1} | F_{s_2})}$$

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$|S_2| \leq |S| = s$, applying induction hypothesis to,

$$P(E_i \mid F_{s_2}) = P(E_i \mid \cap_{j \in S_2} \overline{E}_j)$$

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Establishing a lower bound on the denominator using $|S_1| \leq d$

$$P(F_{s_1} | F_{s_2}) = P(\cap_{i \in s_1} \overline{E}_i \mid \cap_{j \in s_2} \overline{E}_j)$$

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$$\geq \frac{1}{2}$$

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Using the upper bound and lower bound,

$$P(E_k \mid F_s) = \frac{P(E_k \cap F_{s_1} \mid F_{s_2})}{P(F_{s_1} \mid F_{s_2})}$$

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If any path in F_i shares edges with no more than k paths in F_j , where $i \neq j$ and $\frac{8 \cdot n \cdot k}{m} \leq 1$, then there is a way to choose n edge-disjoint paths connecting the n pairs.

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Let $E_{i,j}$ be the event that the paths chosen by i and j share at least one edge.
Path in F_i shares edges with no more than k paths in F_j. Hence,

$$p = P(E_{i,j}) \leq \frac{k}{m}$$

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All the conditions of Lovasz local lemma are satisfied and hence

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Hence there is a choice of paths such that the n paths are edge disjoint.

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Let there be a probability space defined by some variables that are randomly assigned. Let E_i for $i = 1 \ldots m$ be an event that the i^{th} clause is not satisfied by the random assignment.

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$$P(\cap_{i=1}^{m} \overline{E}_i) \geq 0$$