# Introduction to Markov Chains

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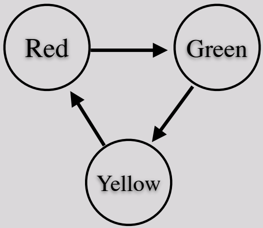
# Outline

# Graphical Representation

## State Diagram

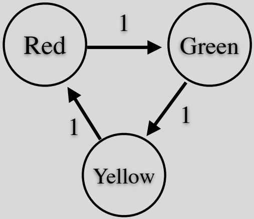
- Directed graph
- Each vertex is a state
- Each edge is a transition
- Move from state to state

## Stop Light



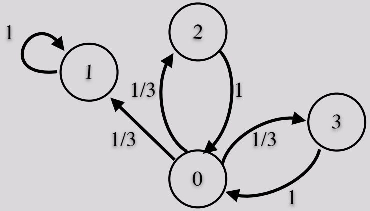
# Graphical Representation

## Stop Light Revisited



# Graphical Representation

## Example




## Questions

- What is the sum of the transition probabilities from a state? 1
- Where have we seen this before? Miner problem
- Which distribution did we use to solve the problem? Geometric
- What property of the Geometric distribution did we use? “Memoryless”

# Markov Chain Definition

## Definitions

A **stochastic process** $X = \{X(t) : t \in T\}$ is a collection of random variables where $t$ represents time and $X(t)$ is the state of the process at time $t$. We can also write $X(t) = X_t$. If $X_t$ assumes values from a countably infinite set we say $X$ is a **discrete space** process. If $X_t$ assumes values from a finite set, we say $X$ is a **finite** process. If $T$ is a countably infinite set we say $X$ is a **discrete time** process.

## Definition

A discrete time stochastic process $X_0, X_1, X_2, \cdots$ is a **Markov chain** if

$$P(X_t = a_t | X_{t-1} = a_{t-1}, X_{t-2} = a_{t-2}, \cdots, X_0 = a_0) = P(X_t = a_t \mid X_{t-1} = a_{t-1})$$

Does this mean that $X_t$ is independent of $X_0, X_1, \cdots, X_{t-2}$? No.

# Transition Matrix Representation

## A Transition Matrix

$$\mathbf{P} = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\ P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

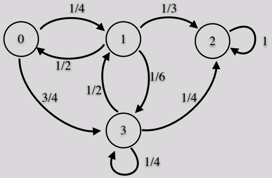
where $P_{i,j} = P(X_t = j \mid X_{t-1} = i)$. In other words, $P_{i,j}$ is the transition probability of state $i$ to state $j$.

## Transition Matrix

What can we say about the sum of the elements of row $i$? $\sum_{j \geq 0} P_{i,j} = 1$

# Graphical and Matrix

## Graphical




## Matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

## Question

What is the probability that, starting in state 0, we are in state 3 in exactly 3 steps? What are the paths? Total is $\frac{41}{192}$

# Transition Matrix

## Transition probability to future states

$p_i(t) = \sum_{j \geq 0} p_j(t-1) \cdot P_{j,i}$ where $p_i(t)$ denotes the probability of being in state $i$ at time $t$. Explain what this is saying.

## *m*-step Transition

For any $m \geq 0$ we can define the $m$-step transition probability by

$$\begin{aligned} P_{i,j}^{m} &= P(X_{t+m} = j \mid X_t = i) \\ &= \sum_{k \geq 0} P_{i,k} \cdot P_{k,j}^{m-1} \end{aligned} \tag{1}$$

# Matrix

## Matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

## Question

What is the probability that, starting in state 0, we are in state 3 in exactly 3 steps?

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{13}{48} & \frac{11}{48} \\ 0 & \frac{5}{24} & \frac{9}{24} & \frac{10}{24} \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{23}{48} & \frac{7}{48} \end{bmatrix}$$

$$\mathbf{P}^3 = \begin{bmatrix} \frac{3}{16} & \frac{7}{48} & \frac{29}{64} & \frac{41}{192} \\ \frac{5}{48} & \frac{5}{24} & \frac{79}{144} & \frac{5}{36} \\ 0 & 0 & 1 & 0 \\ \frac{1}{16} & \frac{13}{96} & \frac{107}{192} & \frac{47}{192} \end{bmatrix}$$

# Matrix

## Matrix

$$\mathbf{P}^3 = \begin{bmatrix} \frac{3}{16} & \frac{7}{48} & \frac{29}{64} & \frac{41}{192} \\ \frac{5}{48} & \frac{5}{24} & \frac{79}{144} & \frac{5}{36} \\ 0 & 0 & 1 & 0 \\ \frac{1}{16} & \frac{13}{96} & \frac{107}{192} & \frac{47}{192} \end{bmatrix}$$

## Question

What is the probability that we get to state 3 in exactly 3 steps starting from a state chosen uniformly at random? $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \cdot \mathbf{P}^3 = ?$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \cdot \mathbf{P}^3 = (\frac{17}{192}, \frac{47}{384}, \frac{737}{1152}, \frac{43}{288})$

# 2-SAT

**Example**

Are there values for $x_1, x_2,$ and $x_3$ which cause this formula to evaluate to true?

$$(x_1, x_2)(x_2, \bar{x_3})(x_1, \bar{x_2})(x_3, x_1)$$

Yes.

$$x_1 = T, x_2 = T, x_3 = T$$

How about this one?

$$(x_1, x_2)(x_1, \bar{x_2})(\bar{x_1}, x_2)(\bar{x_1}, \bar{x_2})$$

No.

# 2-SAT

## Definition

The satisfiability (SAT) problem is a Boolean formula made of the conjunction(AND) of clauses where each clause is made of the disjunction(OR) of Boolean variables or the negation of a Boolean variable. The problem 2-SAT deals with each clause containing exactly two literals.

## Question

Let $n$ be the number of variables.
What is the order of the number of clauses? $O(n^2)$
Why?

# 2-SAT

## A solution

- Do something random! Assign values picked uniformly at random to each of the variables.
- If the formula isn't satisfiable, pick a clause that is wrong and flip a variable.
- Keep going for some number of times

# 2-SAT Algorithm

Start with an arbitrary truth assignment
**for** up to $2 \cdot m \cdot n^2$ times, terminating if all clauses are satisfied **do**
    Choose an arbitrary clause that is not satisfied
    Choose uniformly at random one of the literals in the clause and switch the value of its variable
**end for**
**if** a valid truth assignment was found **then**
    **return** the valid truth assignment
**else**
    **return** return that the formula is unsatisfiable
**end if**

**Algorithm 3.1:** Randomized 2-SAT Algorithm

## Questions:

- What could go wrong?
- Will the algorithm return a false positive?
- Will the algorithm return a false negative?
- How long do we expect it to run? We need to figure this out.

# 2-SAT Analysis: Intution

## A man walks into a bar . . . ouch

Consider a man at a bar, who has had a few too many root beers. Each time he stumbles he might go forward or he might go backward.

- How might we model this? A chain of states!
- What are the states? How far he is from the bar.
- What are the transitions? Step to step.
- What are the probabilites? $\frac{1}{2}$ to move forward and $\frac{1}{2}$ to move backward.
- Should he be able to move backward into the bar? No.

# 2-SAT Analysis

**Lemma**

Assume that 2-SAT formula with $n$ variables has a satisfying assignment and that the 2-SAT algorithm is allowed to run until it finds a satisfying assignment. Then the expected number of steps until the algorithm finds the assignment is at most $n^2$.

# Lemma

## Proof

Assume the formula is satisfiable and let $S$ be a solution. Let $A_i$ represent the variable assignment after the $i$th step and let $X_i$ be the number of variables in $A_i$ that have the same value in $S$. When $X_i = n$ what happens in the algorithm? It terminates with satisfying values.(In the worst case)

So how does $X_i$ change over time?

Look at $X_0$. What does this mean? Zero variables are assigned satisfying values.

$$P(X_{i+1} = 1 \mid X_i = 0) = 1$$

Now what about $X_i$ for $0 \leq i \leq n-1$? Two directions to consider, increasing the number of matches and decreasing the number of matches.

$$\begin{aligned} P(X_{i+1} = j+1 \quad &\mid \quad X_i = j) \geq 1/2 \\ P(X_{i+1} = j-1 \quad &\mid \quad X_i = j) \leq 1/2 \end{aligned}$$

Why? If the clause isn't right, then both of the literals are wrong.

# Lemma Cont.

## Proof Cont.

Is $X_0, X_1, X_2, \cdots$ a Markov chain?
Not necessarily since the probability can change depending on previous states.
So consider the Markov chain:

$$\begin{aligned} Y_0 &= X_0 \\ P(Y_{i+1} = 1 \mid Y_i = 0) &= 1 \\ P(Y_{i+1} = j+1 \mid Y_i = j) &= 1/2 \\ P(Y_{i+1} = j-1 \mid Y_i = j) &= 1/2 \end{aligned}$$

The expected time to reach $n$ is larger for $Y$ than for $X$. Let $h_j$ be the expected number of steps to reach $n$ when starting from $j$. So $h_n = 0$ and $h_0 = h_1 + 1$ What about $h_j$?
Let $Z_j$ be a random variable representing the number of steps to reach $n$ from state $j$.
Now if $1 \leq j \leq n-1$ then half the time the next state will be $j-1$ and $Z_j = 1 + Z_{j-1}$.
Likewise, half of the time the next state will be $j+1$ and $Z_j = 1 + Z_{j+1}$.

# Lemma Cont.

## Proof Cont.

So

$$\mathbf{E}[Z_j] = \mathbf{E}[1/2 \cdot (1 + Z_{j-1}) + 1/2 \cdot (1 + Z_{j+1})] = h_j$$

Therefore

$$\begin{aligned} h_j &= \frac{h_{j-1}+1}{2} + \frac{h_{j+1}+1}{2} \\ &= \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} + 1 \end{aligned}$$

And so we have

$$\begin{aligned} h_n &= 0 \\ h_0 &= h_1 + 1 \\ h_j &= \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} + 1 \end{aligned}$$

We can use induction to show $h_{j+1} = h_j - 2 \cdot j - 1$ and hence $h_0 = h_1 + 1 = h_2 + 1 + 3 = \cdots = \sum_{i=0}^{n-1} 2 \cdot i + 1 = n^2$.

# Theorem

## Theorem

The 2-SAT algorithm always returns a correct answer if the formula is unsatisfiable. If the formula is satisfiable, then with probability at least $1 - 2^{-m}$ then algorithm returns a satisfying assignment.

## Proof

Suppose the formula is satisfiable. Divide the execution of the algorithm into segments of $2 \cdot n^2$ steps each. We want to know that if it is given that no satisfying assignment was found in the first $i - 1$ segments then what is the conditional probability that the algorithm did not find a satisfying assignment in the $i^{\text{th}}$ segment? What is the expected time to find a satisfying assignment? $n^2$ Let $Z$ be the number of steps from the start of segment $i$ until the algorithm finds a solution. We have $P(Z > 2 \cdot n^2)$ What bound to use? Markov's and therefore $P(Z > 2 \cdot n^2) \leq \frac{n^2}{2 \cdot n^2} = \frac{1}{2}$. So after $m$ iterations what is the probability that the algorithm fails to find the satisfying assignment? $(1/2)^m$

# 3-SAT

## Definition

The problem 3-SAT deals with each clause containing exactly three literals.

## Example

Are there values for $x_1, x_2, x_3$ and $x_4$ which cause this formula to evaluate to true?

$$(x_1, \bar{x_2}, x_4)(x_2, \bar{x_3}, \bar{x_4})(x_3, \bar{x_4}, x_1)$$

Yes.

## Questions

Is 3-SAT a more difficult problem than 2-SAT? Yep. 3-SAT is NP-Complete.
How many possible truth assignments are there? $2^n$
What is different about 3-SAT that makes it more difficult?

# 3-SAT Algorithm

```
Start with an arbitrary truth assignment
for up to m times, terminating if all clauses are satisfied do
    Choose an arbitrary clause that is not satisfied
    Choose uniformly at random one of the literals in the clause and switch the value of its variable
end for
if a valid truth assignment was found then
    return  the valid truth assignment
else
    return  return that the formula is unsatisfiable
end if
```

**Algorithm 3.2:** 3-SAT Algorithm

# Bounds

## Expected time

Similar to the 2-SAT algorithm so start the same way.
Let $S$ be the satisfying assignment and let the assignment after $i$ steps be $A_i$ and let $X_i$ be the number of variables of $A_i$ which match $S$. So for $1 \leq j \leq n-1$,

$$\begin{aligned} P(X_{i+1} = j+1 &\quad | \quad X_i = j) \geq 1/3 \\ P(X_{i+1} = j-1 &\quad | \quad X_i = j) \leq 2/3 \end{aligned}$$

Using another Markov chain $Y_0, Y_1, \cdots$ such that $Y_0 = X_0$ and

$$\begin{aligned} P(Y_{i+1} &\quad | \quad Y_i = 0) = 1 \\ P(Y_{i+1} = j+1 &\quad | \quad Y_i = j) = 1/3 \\ P(Y_{i+1} = j-1 &\quad | \quad Y_i = j) = 2/3 \end{aligned}$$

The chain is more likely to go down than up.

# Bounds cont.

## Expected time cont.

Again we let $h_j$ be the expected number of steps to reach $n$ when starting from $j$.

$$\begin{aligned} h_n &= 0 \\ h_0 &= h_1 + 1 \\ h_j &= \frac{2h_{j-1}}{3} + \frac{h_{j+1}}{3} + 1, 1 \leq j \leq n-1 \end{aligned}$$

Solution: $h_j = 2^{n+2} - 2^{j+2} - 3(n-j)$ hence $\Theta(2^n)$ steps.
Not an improvement!

# 3-SAT Algorithm Modified

**for** up to $m$ times, terminating if all clauses are satisfied **do**
    Choose an arbitrary clause that is not satisfied
    **for** up to $3 \cdot n$ times, terminating if all clauses are satisfied **do**
        Start with an arbitrary truth assignment
        Choose uniformly at random one of the literals in the clause and switch the value of its variable
    **end for**
**end for**
**if** a valid truth assignment was found **then**
    **return** the valid truth assignment
**else**
    **return** return that the formula is unsatisfiable
**end if**

**Algorithm 3.3:** 3-SAT Algorithm Modified

## Question

- What is different about this algorithm? Each time choosing an arbitrary truth assignment
- Why do this? Over time we expect to get fewer assignments right, so reset.
- What kind of distribution is the arbitrary assignment? Uniform
- What is the distribution of the number of correct literals? Binomial

# Bounds

## Analysis

Let $q_j$ be the lower bound on the probability that this algorithm reaches $S$ when it starts with exactly $j$ variables that do not agree with $S$.
What is the probability of moving down? $(2/3)$
How many different moves are there $C(j + 2 \cdot k, k)$
What is the probability of $k$ moves down and $j + k$ moves up in a sequence of $j + 2 \cdot k$ moves?
Hence, $C(j + 2 \cdot k, k) \cdot (2/3)^k \cdot (1/3)^{j+k}$
Therefore $q_j \geq \max_{k=0,\cdots,j} C(j + 2 \cdot k, k) \cdot (2/3)^k \cdot (1/3)^{j+k}$ for $j + 2 \cdot k \leq 3 \cdot n$. So for $j = k$ we get $q_j \geq C(3 \cdot j, j) \cdot (2/3)^j \cdot (1/3)^{2 \cdot j}$

# Bounds cont.

## Analysis Cont.

Using the following form of Stirling's Formula

$$\sqrt{2\pi m}\cdot\left(\frac{m}{\mathrm{e}}^{m}\right)\leq m!\leq 2\cdot\sqrt{2\cdot\pi\cdot m}\cdot\left(\frac{m}{\mathrm{e}}^{m}\right)$$

When $j>0$ we get

$$\begin{aligned}
C(3\cdot j,j) &= \frac{3\cdot j!}{j!2\cdot j!}\\
&\geq \frac{\sqrt{2\cdot\pi\cdot 3\cdot j}}{4\cdot\sqrt{2\cdot\pi\cdot j}\cdot\sqrt{2\cdot\pi\cdot 2\cdot j}}\cdot\left(\frac{3\cdot j}{\mathrm{e}}\right)^{3\cdot j}\cdot\left(\frac{\mathrm{e}}{2\cdot j}\right)^{2\cdot j}\cdot\left(\frac{\mathrm{e}}{j}\right)^{j}\\
&= \frac{\sqrt{3}}{8\cdot\sqrt{\pi\cdot j}}\cdot\left(\frac{27}{4}\right)^{j}\\
&= \frac{c}{\sqrt{j}}\cdot\left(\frac{27}{4}\right)^{j}
\end{aligned}$$

Where $c=\sqrt{3/8}\cdot\sqrt{\pi}$

# Bounds cont.

## Analysis Cont.

Hence, for $j > 0$,

$$
\begin{aligned}
q_j &\geq C(3 \cdot j, j) \cdot \left(\frac{2}{3}\right)^j \cdot \left(\frac{1}{3}\right)^{2 \cdot j} \\
&\geq \frac{c}{\sqrt{j}} \cdot \left(\frac{27}{4}\right)^j \cdot \left(\frac{2}{3}\right)^j \cdot \left(\frac{1}{3}\right)^{2 \cdot j} \\
&\geq \frac{c}{\sqrt{j}} \cdot \frac{1}{2^j}
\end{aligned}
$$

Having found a lower bound for $q_j$ we can derive a lower bound for $q$, the probability that the algorithm reaches $S$ in $3 \cdot n$ steps. Let $MM$ be the event that a random assignment has $j$ mismatches with $S$. So $q$ is the sum of $P(MM)q_j$. So $q \geq \frac{c}{\sqrt{n}} \cdot \left(\frac{3}{4}\right)^n$ is a geometric random variable. The expected number of assignments is $1/q$ and hence the algorithm is bounded by $O(n^{3/2} \cdot (4/3)^n)$

# Classes

## Definiton

State $i$ is **accessible** from state $j$ if, for some integer $n \geq 0$, $P_{i,j}^n > 0$ If two states $i$ and $j$ are accessible from each other, we say they **communicate** and write $i \leftrightarrow j$.

## Graph Representation

What does this mean in graphical terms? $i \leftrightarrow j$ if and only if there are directed paths connecting $i$ to $j$ and $j$ to $i$.

# Classes cont.

## Communicating Relation

The communicating relation defines an equivalence relation.

1. reflexive - for any state $i$, $i \leftrightarrow i$
2. symmetric - if $i \leftrightarrow j$ then $j \leftrightarrow i$
3. transitive - if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

## Communicating Classes

The communicating relation partitions the states into disjoint equivalence classes, which are referred to as **communicating classes**. Is it possible to move from one class to another? Yes. Is it possible to move back? There is no way back.

# Irreducible

**Definition**

A Markov chain is **irreducible** if all states belong to one communicating class. This means that every pair of states has a non-zero probability that the first state can reach the second state.

**Lemma**

A finite Markov chain is irreducible if and only if its graph representation is a strongly connected graph.

# Transient and Recurrent States

## Definition

Let: $r_{i,j}^t = P(X_t = j \text{ and, for } 1 \leq s \leq t-1, X_s \neq j \mid X_0 = i)$ which is the probability that starting in state $i$ the first transition to $j$ is at time $t$.
A state is **recurrent** if $\sum_{t \geq 1} r_{i,i}^t = 1$ and it is **transient** if $\sum_{t \geq 1} r_{i,i}^t < 1$. A Markov chain is recurrent if every state in the chain is recurrent.

## Question

- What does this mean intuitively for a recurrent state? If state $i$ is recurrent, once the chain visits state $i$, then it will eventually return to the state.
- If state $i$ is transient, then starting at state $i$, the chain will return to $i$ with some fixed probability $p = \sum_{t \geq 1} r_{i,i}^t$
- If one state is recurrent then all states in the communicating class of the state are recurrent.

# Transient and Recurrent States cont.

## Expected time

The expected time to return to state $i$ after starting at $i$ is $h_{i,i} = \sum_{t \geq 1} t \cdot r_{i,i}^t$ and the expected time to reach $j$ from $i$ is $h_{i,j} = \sum_{t \geq 1} t \cdot r_{i,j}^t$.

## Definition

A recurrent state $i$ is **positive recurrent** if $h_{i,i} < \infty$. Otherwise it is **null recurrent**.

## Lemma

In a finite Markov chain:

1. at least one state is recurrent; and
2. all recurrent states are positive recurrent.

# Periodic, Aperiodic and Ergodic

### Definition

A state $j$ in a discrete time Markov chain is **periodic** if there exists an integer $\delta > 1$ such that $P(X_{t+s} = j \mid X_t = j) = 0$ unless $s$ is divisible by $\delta$. A discrete time Markov chain is periodic if any state in the chain is periodic. A state or chain that is not periodic is **aperiodic**. Example (Think even integers!)

### Definition

An aperiodic, positive recurrent state is an **ergodic** state. A Markov chain is ergodic if all its states are ergodic.

### Corollary

Any finite, irreducible and aperiodic Markov chain is an ergodic chain.

### Corollary Proof

A finite chain has at least one recurrent state and if it is irreducible then all its states are recurrent. In a finite chain all recurrent states are positive recurrent.

# The Gambler's Ruin

## The Gambler's Ruin

Consider a sequence of independent, fair gambling games between two players. In each round a player can win 1 dollar or lose 1 dollar. Winning a dollar has a probability of $\frac{1}{2}$ as does losing a dollar. Also let $\ell_1$ be the max amount that player 1 can lose and $\ell_2$ be the max amount that player can win. If the state $-\ell_1$ is reached then player 1 is ruined and likewise if $\ell_2$ is reached then player 2 is ruined. Let the state of the system be the amount of money player 1 has won at time $t$. At the start player 1 has not won any money so the first state is 0.

Is the Gambler's Ruin a Markov Chain? If we are in state $\ell_2$ at time $t$ what states can we move to? Make states $-\ell_1$ and $\ell_2$ recurrent and absorbing.

# The Gambler's Ruin cont.

## Question

What is the probability that player 1 wins $\ell_2$ dollars before losing $\ell_1$ dollars? If $\ell_1 = \ell_2$ $1/2$. $\ell_1 \neq \ell_2$?

## Solution

What do we know about the states $-\ell_1$ and $\ell_2$? They are recurrent states.
Let $P_i^t$ be the probability that after $t$ steps the chain is in state $i$. For states other than $-\ell_1$ and $\ell_2$ what is $\lim_{t\to\infty} P_i^t =$? 0.
Let $q$ be the probability that the game ends with player 1 winning $\ell_2$.

$$\lim_{t\to\infty} P_{\ell_2}^t = q$$

What is the expected gain of player 1 in each step? 0! Let $W^t$ be the gain of player 1 after $t$ steps so $\mathbf{E}[W^t] = 0$, and $\mathbf{E}[W^t] = \sum_{i=-\ell_1}^{\ell_2} i \cdot P_i^t = 0$
So $\lim_{t\to\infty} \mathbf{E}[W^t] = \ell_2 \cdot q - \ell_1 \cdot (1-q) = 0$ Hence

$$q = \frac{\ell_1}{\ell_1 + \ell_2}$$

What does this mean? The probability of winning (or losing) is proportional to the amount of money a player is willing to lose (or win).