Markov Chains and Stationary Distributions

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Outline

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1. Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue
2. Random Walks on Undirected Graphs
   - Application: An *s*-*t* Connectivity Algorithm

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3. Parrondo's Paradox

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Probability Matrix

Recall

Let **P** be a one-step probability matrix of a Markov chain such that

$$\mathbf{P} = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\ P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

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Let $\bar{p}(t) = (p_0(t), p_1(t), p_2(t), \dots)$ be the vector giving the probability distribution of the state of the chain at time t.

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We are now interested in state probability distributions that **do not change after a transition**.

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- A stationary distribution represents a steady state (or an equilibrium) in the chain's behavior.
- Stationary distributions play a key role in analyzing Markov chains.

Stationary Distributions

Example

Suppose we have a Markov chain having state space $S = \{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}.$$

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- $\frac{6}{25} = 24\%$ of the time is spent in state 0.
- $\frac{10}{25} = 40\%$ of the time is spent in state 1.
- $\frac{9}{25} = 36\%$ of the time is spent in state 2.

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2. For all j and i, the limit $\lim_{t \to \infty} P^t_{j,i}$ exists, and it is independent of j.
3. $\pi_i = \lim_{t \to \infty} P^t_{j,i} = \frac{1}{h_{i,i}}$, where $h_{i,i}$ is the expected time to return to state i when starting at state i.

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2. π_i is the inverse of $h_{i,i} = \sum_{t=1}^{\infty} t \cdot r_{i,i}^t$.
   - If the average time to return to state i from i is $h_{i,i}$, then we expect to be in state i for $\frac{1}{h_{i,i}}$ of the time and thus, in the limit, we must have $\pi_i = \frac{1}{h_{i,i}}$.

Proof of Fundamental Theorem

Lemma

For any irreducible, ergodic Markov chain and for any state i, the limit $\lim_{t\to\infty} P^t_{i,i}$ exists and

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Intuition

- Since the expected time between visits to i is $h_{i,i}$, state i is visited $1/h_{i,i}$ of the time.
- Thus, $\lim_{t\to\infty} P^t_{i,i}$, which represents the probability a state chosen far in the future is at state i when the chain starts at state i, must be $1/h_{i,i}$.

Proof of Fundamental Theorem

Showing Limits Exist and are Independent of Starting State j

Using the fact that $\lim_{t\to\infty} P^t_{i,i}$ exists, for any j and i,

$$\lim_{t\to\infty} P^t_{j,i} = \lim_{t\to\infty} P^t_{i,i} = \frac{1}{h_{i,i}}.$$

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Proof

Read pages 168-169 in the book!

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Proving $\bar{\pi}$ Forms a Stationary Distribution

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- For any $t \geq 0$, $\sum_{i=0}^{n} P^t_{j,i} = 1$ and thus

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- $\bar{\pi}$ is a proper distribution.

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- Now,

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- Therefore, $\bar{\pi}$ is a stationary distribution.

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System of Linear Equations

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we have the following system
$$\pi_0 \quad = \quad \pi_0 \cdot P_{0,0} + \pi_1 \cdot P_{1,0} + \cdots + \pi_i \cdot P_{i,0} + \cdots$$

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Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck.

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Solution (Contd.)

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Solving the first equation gives us

$$\begin{aligned} \frac{3}{4} \cdot \pi_0 &= \frac{1}{5} \cdot \pi_1 \\ \pi_0 &= \frac{4}{15} \cdot \pi_1. \end{aligned}$$

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Therefore, $\pi_0 = \frac{4}{19}$. Thus, as we sit by the road, $\frac{4}{19}$ of all vehicles passing by will be trucks.

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If C is a cut-set in the graph representation of the chain, then in the stationary distribution, the probability of crossing the cut-set in one direction is equal to the probability of crossing the cut-set in the other direction.

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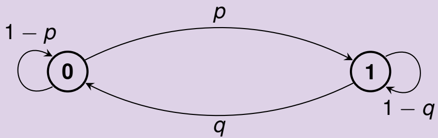

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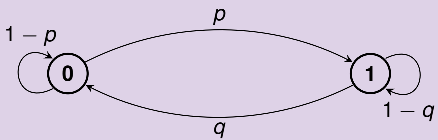

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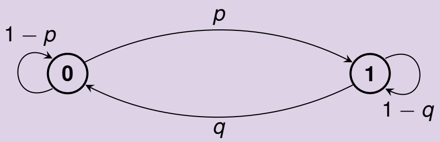

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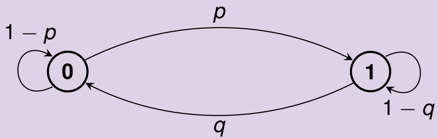

- From state 0, we move to state 1 with probability p and stay at state 0 with probability $1 - p$.
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- If p and q are small, state changes are rare.

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Using $\pi_0 + \pi_1 = 1$ yields $\pi_0 = \frac{q}{p+q}$ and $\pi_1 = \frac{p}{p+q}$.

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Consider a finite, irreducible, and ergodic Markov chain with transition matrix **P**. If there are nonnegative numbers $\bar{\pi} = (\pi_0, \ldots, \pi_n)$ such that $\sum_{i=0}^{n} \pi_i = 1$ and if, for any pair of states i, j,

$$\pi_i \cdot P_{i,j} = \pi_j \cdot P_{j,i},$$

then $\bar{\pi}$ is the stationary distribution corresponding to **P**.

Proof

Consider the j^{th} entry of $\bar{\pi} \cdot \mathbf{P}$. Using the assumption of the theorem, we find that it equals

$$\sum_{i=0}^{n} \pi_i \cdot P_{i,j} = \sum_{i=0}^{n} \pi_j \cdot P_{j,i} = \pi_j.$$

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Definition

Chains that satisfy the condition $\pi_i \cdot P_{i,j} = \pi_j \cdot P_{j,i}$ are called **time reversible**.

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Proof

Same as the proof of the Fundamental Theorem.

Outline

1. Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue
2. Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm
3. Parrondo's Paradox

Queue Example

Queues

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- If $\lambda = \mu$, there is still no stationary distribution, but the states are null recurrent.

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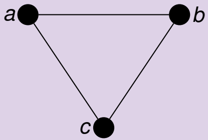
Random Walks

Example



Undirected Graph G

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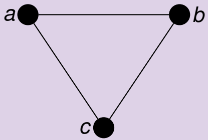
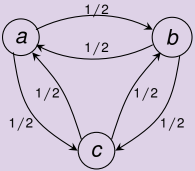


Undirected Graph G



Markov Chain

Criterion for Aperiodicity

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- By traversing that cycle we have an odd-length path from any vertex to itself.

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Note

A random walk on a finite, undirected, connected, and non-bipartite graph G satisfies the conditions of our Fundamental Theorem which means the random walk converges to a stationary distribution.

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Theorem

A random walk on G converges to a stationary distribution $\bar{\pi}$, where

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- Let $\mathbf{P}$ be the transition probability matrix of the Markov chain.
- Let $N(v)$ represent the neighbors of v.

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Proof

The relation $\bar{\pi} = \bar{\pi} \cdot \mathbf{P}$ is equivalent to

$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2 \cdot |E|} \cdot \frac{1}{d(u)}$$

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Recall that $h_{v,u}$ denotes the expected number of steps to reach u from v.

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Corollary

For any vertex u in G,

$$h_{u,u} = \frac{2 \cdot |E|}{d(u)}.$$

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Definition

The **cover time** of a graph $G = (V, E)$ is the maximum expected time to visit all of the vertices in the graph by a random walk starting from v, for all vertices $v \in V$.

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Connectivity Algorithm

Problem Description

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- Assume G is non-bipartite.

Connectivity Algorithm

Function s-t CONNECTIVITY(s, t)

1: Start a random walk from s.
2: **if** (walk reaches t within $4 \cdot n^3$ steps) **then**
3: **return** ("There is a path")
4: **else**
5: **return** ("There is no path")
6: **end if**

Algorithm 3.1: s-t Connectivity Algorithm

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- As long as there is some mechanism for choosing a random neighbor from each vertex, that is all the memory required.

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Main Idea

Given two games, each with a higher probability of losing than winning, it is possible to construct a winning strategy by playing the games alternately.

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- Let A and B be the two games.
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- Combine games A and B into a new game C, where you alternate between games A and B with a provided probability.

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- Let A and B be the two games.
- Use Markov Chains to show that both A and B are losing games by analyzing absorbing states or using stationary distributions.
- Combine games A and B into a new game C, where you alternate between games A and B with a provided probability. Game C ends up being a winning game.