The Probabilistic Method

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Outline

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Definition

What is Probabilistic Method?

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A way of proving the existence of objects.

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A way of proving the existence of objects.

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To prove the existence of an object with certain properties, demonstrate a sample space of objects in which the probability is positive that a randomly selected object has the required properties.
If the probability of selecting an object with the required properties is positive, then the sample space must contain such an object and hence the object exists.

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If there is a positive probability of getting an even number when a fair die is rolled, then there must be at least one face on the die having an even number.

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If there is a positive probability of winning a million-dollar prize in a raffle, then there must be at least one raffle ticket that wins that prize.

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Techniques

Techniques for Constructing proofs based on the probabilistic method

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(i) Simple Counting

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(ii) Averaging Arguments

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(i) Simple Counting
(ii) Averaging Arguments
(iii) Lovasz local Lemma

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(i) Simple Counting
(ii) Averaging Arguments
(iii) Lovasz local Lemma
(iv) Second Moment Method

Outline

Basic Counting Argument

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To prove the existence of an object with specific properties, construct an appropriate probability space S of objects and then show that the probability that an object in S with the required properties is selected is strictly greater than 0.

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Coloring edges of a graph with two colors so that there are no large cliques with all edges having same color.

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Let K_n be a complete graph having $C(n, 2)$ edges on n vertices. A clique of k vertices in K_n is a complete subgraph K_k.

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Let K_n be a complete graph having $C(n, 2)$ edges on n vertices. A clique of k vertices in K_n is a complete subgraph K_k.

Theorem

If $C(n, k)2^{-C(k,2)+1} < 1$, then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

Basic Counting Argument

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Define a sample space having all possible colorings of the edges of K_n using two colors.

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Assume that the first edge in the clique i is colored.

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Hence,

$$P(A_i) = 2^{-C(k,2)+1}$$

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Proof(Cont.)

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$$P\left(\cup_{i=1}^{C(n,k)} A_i\right) \leq \sum_{i=1}^{C(n,k)} P(A_i)$$

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The probability of choosing a coloring with no monochromatic k-vertex clique from sample space is greater than 0.

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Hence there is a coloring with no monochromatic k-vertex clique.

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Example

If the expected values of a raffle ticket is at least \$3, then there must be at least one ticket that ends up being worth no more than \$3 and at least one that ends up being worth no less than \$3.

The Expectation Argument

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This is a contradiction.

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There must be at least one instance in the sample space S for which the value of X is at least μ and at least one instance for which the value of X is no greater than μ

Finding a Large Cut

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Given an undirected graph G with m edges, there is a partition of V into two disjoint sets A and B such that at least $\frac{m}{2}$ edges connect a vertex in A to a vertex in B. That is, there is a cut with value at least $\frac{m}{2}$.

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Let $e_1 \ldots e_m$ be the edges of graph G.

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Let $e_1 \ldots e_m$ be the edges of graph G.

For $i = 1 \ldots m$,

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects } A \text{ to } B, \\ 0 & \text{otherwise.} \end{cases}$$

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$$E[X_i] = \frac{1}{2}$$

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Let $C(A, B)$ be some random variable denoting the value of the cut corresponding to the sets A and B. Then,

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Let $C(A, B)$ be some random variable denoting the value of the cut corresponding to the sets A and B. Then,

$$E[C(A,B)] = E\left[\sum_{i=1}^{m} X_i\right] = \sum_{i=1}^{m} E[X_i] = m \cdot \frac{1}{2} = \frac{m}{2}$$

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The expectation of the random variable $C(A,B)$ is $\frac{m}{2}$.

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The expectation of the random variable $C(A, B)$ is $\frac{m}{2}$.
Hence there exist a partition A and B with at least $\frac{m}{2}$ edges connecting sets A and B.

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As, $C(A, B) \leq$ m. We have,

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$$\frac{m}{2} = \sum_{i \leq \frac{m}{2}-1} i \cdot P(C(A,B)=i) + \sum_{i \geq \frac{m}{2}} i \cdot P(C(A,B)=i)$$

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$$\leq (1 - p) \cdot \left(\frac{m}{2} - 1\right) + p \cdot m$$

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The expected number of samples before finding a cut with value at least $\frac{m}{2}$ is $\frac{m}{2} + 1$.

Maximum Satisfiability: MAXSAT

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$(x_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_3) \wedge (x_1 \vee x2 \vee x_4) \wedge (x_4 \vee \overline{x}_3) \wedge (x_4 \vee \overline{x}_1)$

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Solution

Assignment of the variables to the values TRUE and FALSE so that all clauses are satisfied.

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$$\sum_{i=1}^{m}(1 - 2^{-k_i}) \geq m \cdot (1 - 2^{-k})$$

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The probability that the i^{th} clause with k_i literals is satisfied is at least $(1 - 2^{-k_i})$.

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The probability that the i^{th} clause with k_i literals is satisfied is at least $(1 - 2^{-k_i})$. Hence the expected number of satisfied clauses is at least,

Maximum Satisfiability: MAXSAT

Theorem

Given a set of m clauses, let k_i be the number of literals in the i^{th} clause for $i = 1 \dots m$. Let $k = min_{i=1}^{m} k_i$. Then there is a truth assignment that satisfies at least

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Derandomization Using Conditional Expectations

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By linearity of expectation, $E[C(A,B)\ |x_1, x_2, x_3 \dots x_k, Y_{k+1} = A]$ is the number of edges crossing the cut whose end points are among the first $k+1$ vertices, plus half the remaining edges.

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Similarly, $E[C(A,B)\ |x_1, x_2, x_3 \ldots x_k, Y_{k+1} = B]$

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All edges that do not have v_{k+1} as an endpoint contribute the same amount to the two expectations.

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Take the vertices in some order.

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Equivalently, place each vertex on the side with fewer neighbors, breaking ties arbitrarily.

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This guarantees a cut with at least $\frac{m}{2}$ edges.

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Theorem(Cont.)

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$$= \sum_{i=1}^{n} E[Y \,|X_i = 1] \cdot P(X_i = 1)$$

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