The Probabilistic Method

Haritha Eruvuru¹

¹ Lane Department of Computer Science and Electrical Engineering West Virginia University

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- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument
- Derandomization Using Conditional Expectations

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- 2 Techniques
 - Basic Counting Argument
 - The Expectation Argument
- Derandomization Using Conditional Expectations
- Conditional Expectation Inequality

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument
- 3 Derandomization Using Conditional Expectation
- Conditional Expectation Inequality

Techniques
Derandomization Using Conditional Expectations
Conditional Expectation Inequality

Probabilistic Method Definition Examples Techniques

Definition

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A way of proving the existence of objects.

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A way of proving the existence of objects.

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To prove the existence of an object with certain properties, demonstrate a sample space of objects in which the probability is positive that a randomly selected object has the required properties.

If the probability of selecting an object with the required properties is positive, then the sample space must contain such an object and hence the object exists.

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument
- 3 Derandomization Using Conditional Expectations
- Conditional Expectation Inequality

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If there is a positive probability of winning a million-dollar prize in a raffle, then there must be at least one raffle ticket that wins that prize.

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument
- Derandomization Using Conditional Expectations
- Conditional Expectation Inequality

Techniques

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Techniques for Constructing proofs based on the probabilistic method

(i) Simple Counting

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- (ii) Averaging Arguments

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- (iv) Second Moment Method

- Introduction
 - Probabilistic Method Definition
 - Examples
 - Techniques
- Techniques
 - Basic Counting Argument
 - The Expectation Argument
- Derandomization Using Conditional Expectations
- Conditional Expectation Inequality

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To prove the existence of an object with specific properties, construct an appropriate probability space S of objects and then show that the probability that an object in S with the required properties is selected is strictly greater than 0.

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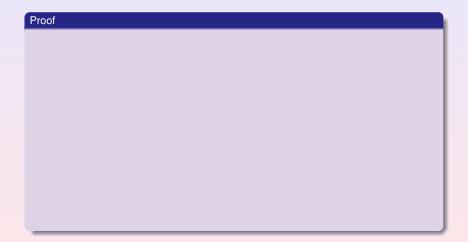
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Let K_n be a complete graph having C(n,2) edges on n vertices. A clique of k vertices in K_n is a complete subgraph K_k .

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If $C(n,k)2^{-C(k,2)+1} < 1$, then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.



Proof

Define a sample space having all possible colorings of the edges of K_n using two colors.

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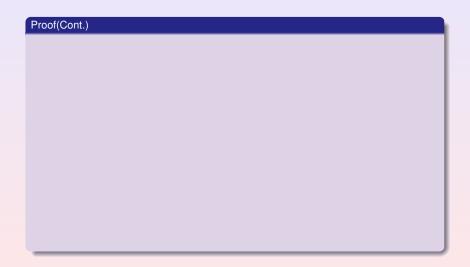
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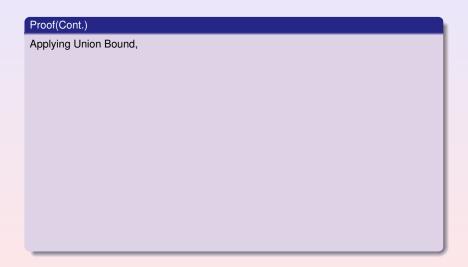
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$$P(A_i) = 2^{-C(k,2)+1}$$





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Hence there is a coloring with no monochromatic *k*-vertex clique.

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- Introduction
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 - Examples
 - Techniques
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 - Basic Counting Argument
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Example

If the expected values of a raffle ticket is at least \$3, then there must be at least one ticket that ends up being worth no more than \$3 and at least one that ends up being worth no less than \$3.

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There must be at least one instance in the sample space S for which the value of X is at least μ and at least one instance for which the value of X is no greater than μ

Finding a Large Cut

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Given an undirected graph G with m edges, there is a partition of V into two disjoint sets A and B such that at least $\frac{m}{2}$ edges connect a vertex in A to a vertex in B. That is, there is a cut with value at least $\frac{m}{2}$.

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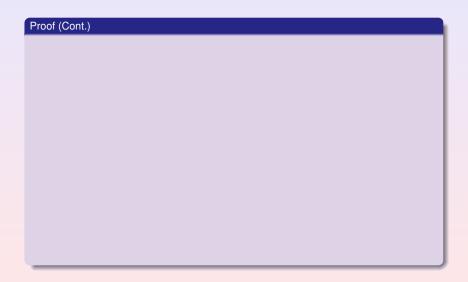
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Hence there exist a partition A and B with at least $\frac{m}{2}$ edges connecting sets A and B.



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$$\leq (1-p)\cdot \left(\frac{m}{2}-1\right)+p\cdot m$$

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The expected number of samples before finding a cut with value at least $\frac{m}{2}$ is $\frac{m}{2} + 1$.

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$$(x_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_4 \vee \overline{x}_3) \wedge (x_4 \vee \overline{x}_1)$$

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Solution

Assignment of the variables to the values TRUE and FALSE so that all clauses are satisfied.



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The probability that the i^{th} clause with k_i literals is satisfied is at least $(1 - 2^{-k_i})$.

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$$\sum_{i=1}^{m} (1-2^{-k_i}) \geq m \cdot (1-2^{-k})$$

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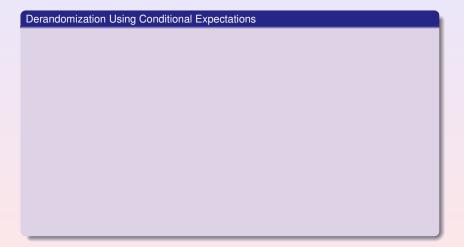
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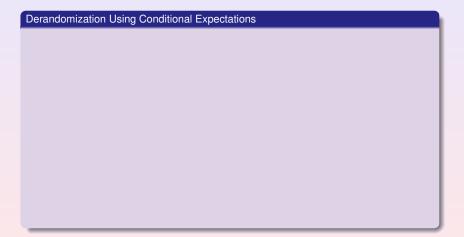
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Similarly,
$$E[C(A, B) | x_1, x_2, x_3 ... x_k, Y_{k+1} = B]$$

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All edges that do not have v_{k+1} as an endpoint contribute the same amount to the two expectations.

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This guarantees a cut with at least $\frac{m}{2}$ edges.

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